Resolution and Reconstruction issues

in CT and SPECT

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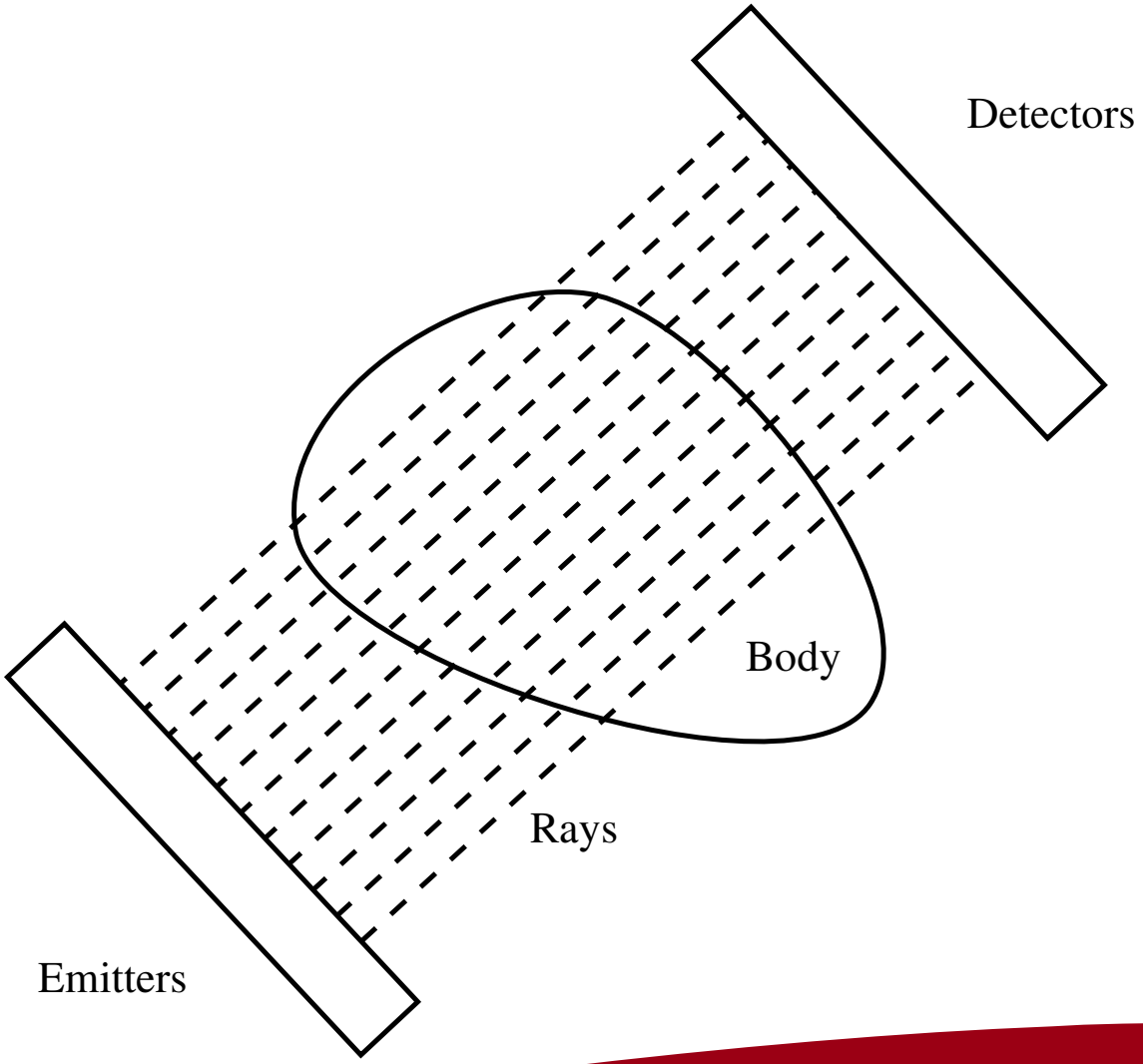
Axial Tomography machines provide result of some circular acquisitions. We discuss algorithms of reconstruction from projections and their efficiency in a few cases

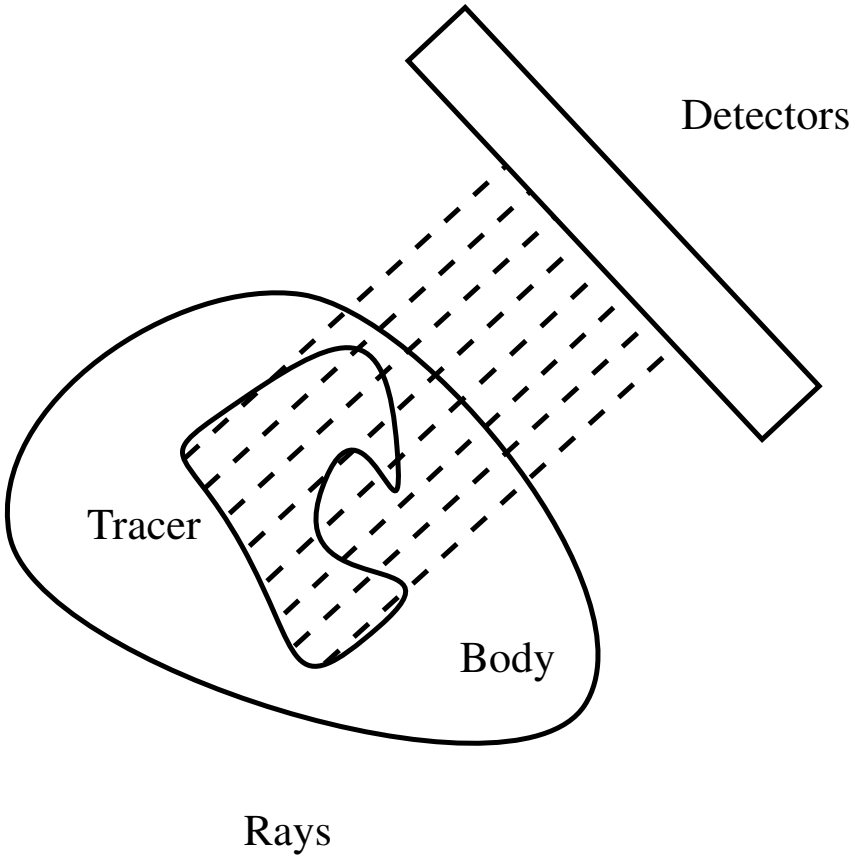
- transmission tomography (e.g. CT): an electromagnetic ray passes trough the patient and is detected at the exit in order to get a morphological analysis of its interior.
- emission tomography (e.g. PET, SPECT): a radioactive tracer is injected into the patient and detected by the machine in order to make an internal functional analysis of the organs.
- hybrid tomography (e.g. SPECT/CT, SPECT/MRI): two simultaneous analysis.

We will focus on CT, SPECT and SPECT/CT problems.

Transmission tomography



Emission tomography



If a ray passes through a body, it will be subject to attenuation. The Beer's law says that if $I(x)$ is the intensity of a ray and $A(x)$ the attenuation coefficient of the point x, then

$$\frac{\Delta I}{\Delta x} = -A(x)I(x)$$

that is, by integrating:

$$\int_{x_0}^{x_1} A(x)dx = -\int_{x_0}^{x_1} \frac{dI}{I} = -ln(I(x_1) - I(x_0)).$$

In transmission tomography, in 2 dimensions, the Beer's law corresponds to the Radon transform (RT), defined as the integral over a line

$$\mathcal{R}f(t,\theta) = \int_{\ell_{(t,\theta)}} f = \int_{\mathbb{R}^2} f(\bar{x})\delta(t - \bar{x}\cdot\bar{\theta})\, d\bar{x}$$

with $\bar{x} = (x, y)$, $\bar{\theta} = (\cos\theta, \sin\theta)$

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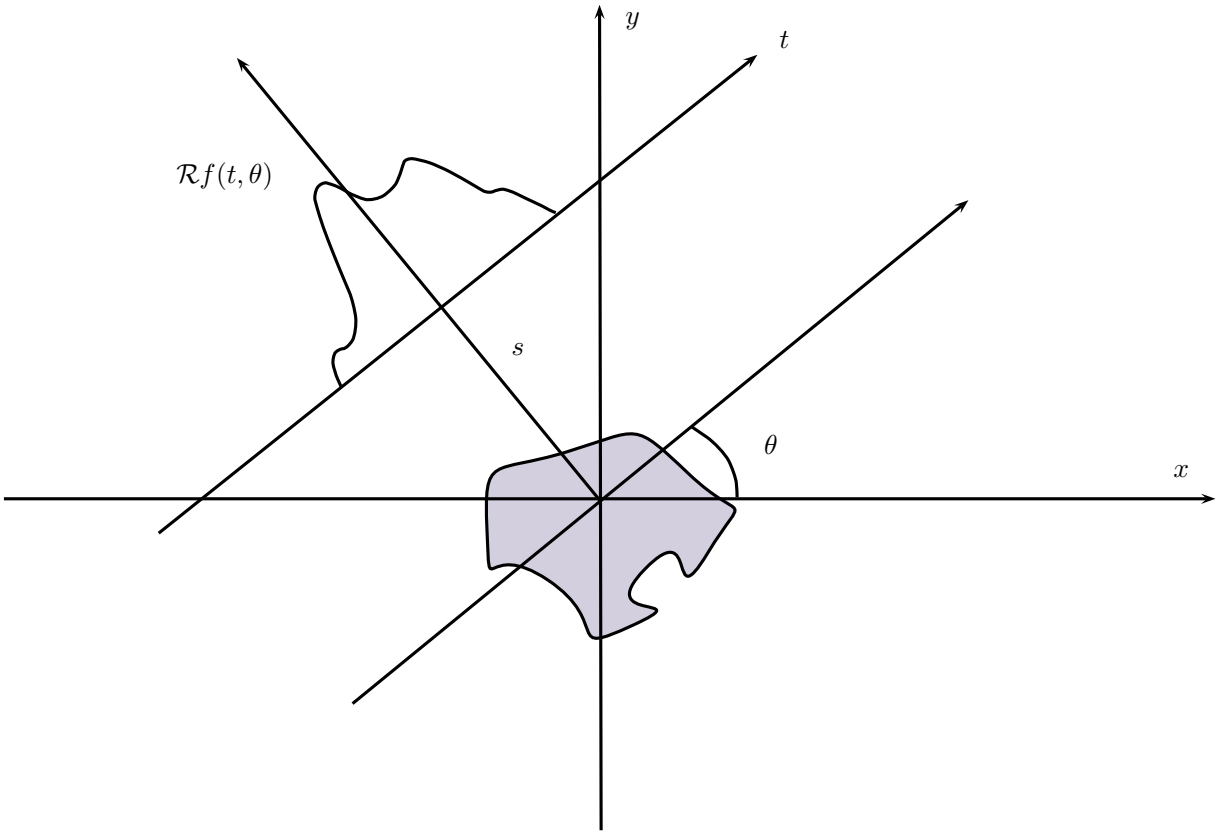
with $\bar{x} = (x, y)$, $\bar{\theta} = (\cos\theta, \sin\theta)$ or equivalently:

$$\mathcal{R}f(t,\theta) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds$$



$$\mathcal{R}f(t,\theta) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds$$

Now we ask for every point (x, y) which is the average of the rays that pass through that point. This question is answered by the adjoint operator to $\mathcal{R}$, called Backprojection operator

$$\mathcal{R}^* g(x, y) = \frac{1}{|S^1|} \int_{\mathbb{R} \times S^1} g(s, \theta) \delta(s - \bar{x} \cdot \bar{\theta}) \, ds \, d\theta =$$

$$\frac{1}{|S^1|} \int_{S^1} g(x \cos\theta + y \sin\theta, \theta) \, d\theta$$

where $S^1 = [0, \pi]$ or $S^1 = [0, 2\pi]$.

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Warning

$\mathcal{R}^*$ is not the inverse transform of $\mathcal{R}$. In fact $\mathcal{R}^* \mathcal{R} f(\bar{x}) = \frac{2}{|\bar{x}|} * f$.

In the case of emission tomography things are more complicated. From Beer's law we obtain that

$$I(x_1) = I(x_0) \exp\left(-\int_{x_0}^{x_1} A(x)dx\right)$$

Suppose to know the attenuation coefficient, say $a(\bar{x})$, we want to obtain the radioactivity $f(\bar{x})$ by its angular projections.

We define the attenuated Radon transform (briefly AtRT)

$$\mathcal{R}_a f(t,\theta) = \int_{\ell_{(t,\theta)}} e^{-\mathcal{D}a(\bar{x},\theta+\pi)} f(\bar{x})$$

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where $\mathcal{D}$ is the Divergent beam transform defined as follows

$$\mathcal{D}h(\bar{x},\theta) = \int_0^{+\infty} h(x + t\cos\theta, y + t\sin\theta)\, dt = \int_0^{+\infty} h(\bar{x} + t\bar{\theta})\, dt$$

Then in the CT case we have to solve the following problem

RT problem

$$\text{Given } g \text{ projection data (or sinogram)}$$
$$\text{find } f \text{ such that } \mathcal{R}f = g$$

while in SPECT case we can approximate f with the solution of the previous problem

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$$\text{Given } g \text{ projection data and } a \text{ attenuation map}$$
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to estimate the attenuation map we may need a simultaneous CT tomography $\longrightarrow$ SPECT/CT.

We use two different approaches:

First approach

Analytical methods: discretization of exact solution.

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Second approach

Iterative methods: by interpolation, we solve the linear system iteratively.

Theorem (Inversion of the Radon transform)

$$f = \frac{1}{2}\mathcal{R}^* \left[\mathcal{F}^{-1}(|\nu|\mathcal{F}(\mathcal{R}f))\right]$$

where we mean that the direct and inverse Fourier transform is applied only to the variable t.

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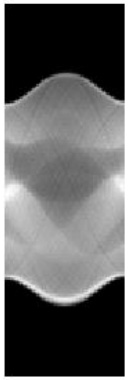
As known the previous formula is numerically inaccurate. Then we use $w(\nu) = p(\nu)|\nu|$ instead of $|\nu|$, with p a low-pass filter, getting the approximated

Filtered Back Projection formula (FBP)

$$f \cong \frac{1}{2}\mathcal{R}^* \left[\mathcal{F}^{-1}(w(\nu)\mathcal{F}(\mathcal{R}f))\right]$$

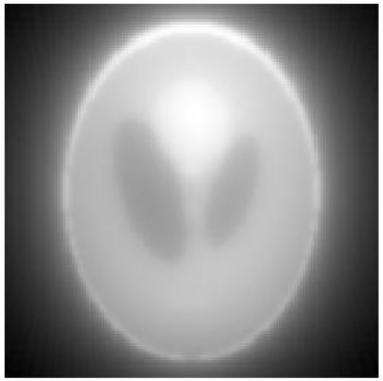
Original phantom f(x,y)

sinogram: Radon transorm Rf(s,theta)



Filtered backprojection

Unfiltered backprojection



Theorem (Error estimate)

Let $f \in C_0^\infty(B(0,1))$ be a b-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let $\tilde{f}$ be the FBP reconstruction, then

$$\|f - \tilde{f}\|_{L^\infty(\mathbb{R}^2)} \leq 2|S^1| \ \ \|w_b\|_{L^1(\mathbb{R})} \|g - \tilde{g}\|_{L^\infty(\mathbb{R}\times S^1)} + |e_3|$$

with e_3 the quadrature error of the backprojection integral.

Also the attenuated transform has an inversion formula. Let by the following definitions:

Definition

Let $g(t)$ be a suitable function, then its Hilbert transform is the function

$$\mathcal{H}g(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{s-t}\, dt$$

where the integral a Cauchy principal value.

Definition

Let us define the function

$$h := \frac{1}{2}(I + i\mathcal{H})\mathcal{R}a$$

Theorem (Novikov-Natterer formula)

Let f be a transformable function $g = \mathcal{R}_a f$, and h as in the previous slide. Assume $a(\bar{x})$ known, then f is uniquely determined by the following formula

$$f(\bar{x}) = \frac{1}{4\pi} \mathfrak{Re}\ div \int_{S^1} \theta e^{\mathcal{D}a(\bar{x},\theta+\frac{\pi}{2})} (e^{-h}\mathcal{H}e^{h}g)_{(\bar{x}\cdot\bar{\theta},\theta)}\ d\theta$$

where $S^1 = [0, 2\pi]$.

Attenuation phantom

Activity phantom

Reconstruction of the attenuation map

Reconstruction of the activity

Given a basis of functions $\{b_i(\bar{x})\}_{i=1\ldots n}$ that interpolates the function f, i.e. such that

$$f(\bar{x}) = \sum_{i=0}^{n} c_i b_i(\bar{x}) \qquad \forall \bar{x} \in X$$

where X is properly chosen, then for the linearity of the Radon transform

$$\mathcal{R}f(\bar{y}) = \sum_{i=0}^{n} c_i \mathcal{R}b_i(\bar{y}) \qquad \forall \bar{y} \in Y = \{(t_j, \theta_j)\}.$$

This is equivalent to the solution of

$$Af = c$$

where $A(i,j) = \mathcal{R}b_i(t_j, \theta_j)$ is a matrix $N^2 \times lp$, f is the unknown vector such that $f_i = f(x_i)$ and c is the vector of the projection data $c_j = \mathcal{R}f(t_j, \theta_j)$.
The methods using this approach are known as Algebraic Reconstruction Techniques or ART.

Using the ART approach we may have to face some problems:

- underdetermined $\longrightarrow$ least squares
- ill-conditioned $\longrightarrow$ regularization (not needed)
- huge $\longrightarrow$ sparseness

We choose the *natural pixel basis*

$$b_i(x, y) = \chi_{P_i}(x, y).$$

with P_i the i-th pixel of the reconstructed image. We know the Radon transform of each one of these items

$$\mathcal{R}b_i(t, \theta) = meas(\ell_{t,\theta} \cap P_i)$$

where $\ell_{t,\theta} = \{t\bar{\theta} + s\bar{\theta}^{\perp} \mid s \in \mathbb{R}\}$.

Since the matrix A is large and sparse, we can solve the system $Af = c$ by iterative methods.

- the initial vector $f^{(0)}$ is a *blank*.
- the image at step k, $f^{(k)}$, is projected and compared with the data.
- the image is modified considering the error found in the previous step.

The following methods are the most popular

- The Kaczmarz method projects the vector $f^{(k)}$ on k-th row of A (or on a random row).

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LSCG example

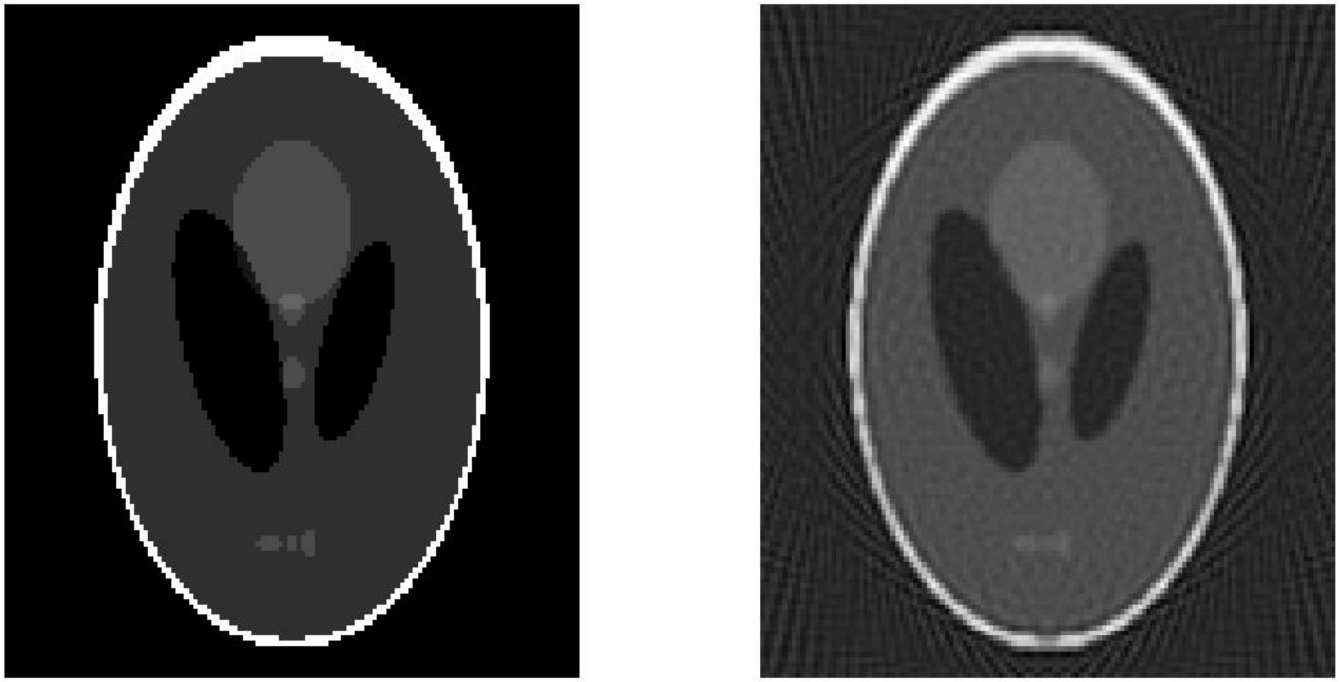
Now instead of the natural pixel basis we can use another basis:

$$b_i(\bar{x}) = B(\bar{x} - \bar{x}_i) = K(\bar{x}, \bar{x}_i)$$

with B a (essentially) compact supported and radial.

Function name	f
Ball	$\chi_{B(\frac{1}{\varepsilon},0)}(r)$
Gaussian	$e^{-\varepsilon^2 r^2}$
Wendland $\varphi_{2,0}$	$(1-\varepsilon r)_+^2$
Wu $\psi_{1,1}$	$(1-\varepsilon r)_+^2(\varepsilon r + 2)$

with $r = \|x\|_2$ and ε the shape parameter.
The ray of the support is $1/\varepsilon$.



Original phantom and kernel reconstruction with Gaussian kernel and shape parameter $\varepsilon = 1$, after 50 iterations of LSCG.

Lemma

If $\phi(x) = \varphi(\|x\|)$ is a radial function, then its Radon transform $\mathcal{R}f$ is radial, i.e. it depends only on t and it is even.

Theorem

If $\phi(x-y) = K(x,y)$ is a radial function, $\phi \in L^1(\mathbb{R}^d)$, continuous, bounded and positive definite on $\mathbb{R}^2$, then its Radon transform $\mathcal{R}f(t)$ is bounded and positive definite on $\mathbb{R}^1$, provided $\mathcal{R}f \in L^1(\mathbb{R})$.

Theorem (Interpolation error bound for Kernel method)

Let $f \in C_0^\infty(B(0,1))$ a b-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let K be a strictly positive definite kernel and its domain Ω be such that $\partial\Omega$ has regularity at least C^1.
Then there exist positive constants h_0 and $\tilde{C}$ such that, if $h_{X,\Omega} \leq h_0$, then

$$\|f(\cdot) - \sum_{i=0}^{n} c_i K_i(\cdot)\|_{L^\infty} \leq 2|S^1|\ b\sqrt{\frac{N}{18}}\ \tilde{C} h_{X,\Omega} \|\mathcal{R}f\|_{\mathcal{N}_{\mathcal{R}K}(\Omega)}$$

with $h_{X,\Omega}$ the meshsize.

Let us consider, as before, the basis $b_i(\bar{x}) = \chi_{P_i}$ and assume

$$f(\bar{x}) = \sum_{i=0}^{n} c_i b_i(\bar{x})$$

then for linearity

$$\mathcal{R}_a f(\bar{y}) = \sum_{i=0}^{n} c_i \mathcal{R}_a b_i(\bar{y}) \qquad \forall \bar{y} \in Y = \{(t_j, \theta_j)\}$$

i.e.

$$Bc = d$$

where $B_{i,j} = \mathcal{R}_a b_i(t_j, \theta_j)$, $c_i = f(\bar{x}_i)$ the unknown term and $d_j = \mathcal{R}_a f(t_j, \theta_j)$ the data.

In order to compute the matrix B we have to consider the attenuation coefficients in the natural pixel basis

$$a(\bar{x}) = \sum_{k=1}^{N^2} g_k \chi_{P_k}(\bar{x}).$$

According to Beer's law

$$I_{out} = I_{in} \ \exp\left(-\sum_{k=1}^{N^2} g_k \ \ meas(P_k \cap \ell^+_{\bar{x},\theta}) \right).$$

Now, if we consider the matrix A used in CT tomography we can compute the outgoing rays from the pixel P_i in (t_j, θ_j) as

$$B_{i,j} = A_{i,j} \ \exp\left(-\sum_{(k_1,k_2)\in K_{(i,j)}} g_k \ \ meas(P_k \cap \ell_{t_j,\theta_j})\right) =$$

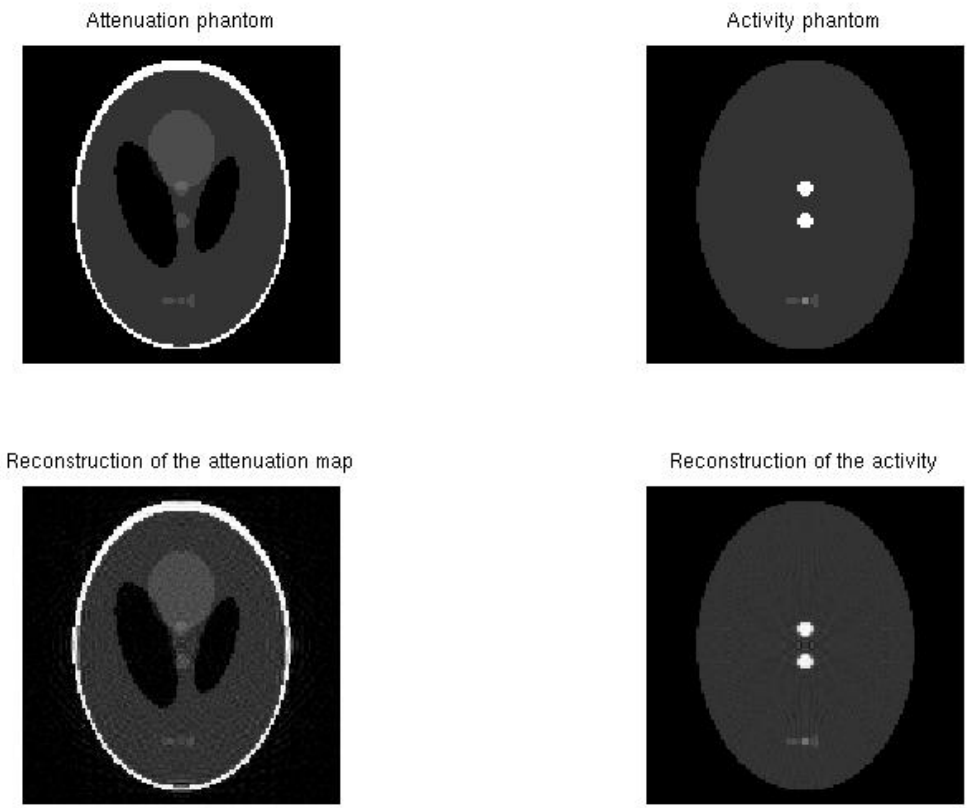
$$A_{i,j} \ \exp\left(\sum_{(k_1,k_2)\in K_{(i,j)}} g_k \ A_{k_1,k_2}\right)$$

where $K_{(i,j)}$ is the set of the indexes of the pixels covered by the line $\ell^{+}_{\bar{x}_i,\theta_j}$.

We can introduce a relaxation parameter $\lambda \in [0, 1]$ to weight the effect of the attenuation

$$B_{i,j}^{(\lambda)} = A_{i,j} \ \exp\left(-\lambda \sum_{(k_1,k_2)\in K_{(i,j)}} g_k \ A_{k_1,k_2}\right)$$

Note that $B^{(0)} = A$ and $B^{(1)} = B$.
We observe that with this little change the linear system is more accurate.



Analytical reconstruction of a SPECT/CT phantom data with $\lambda = 0.1$.

From the *in silico* experiments we see that:

- The iterative methods are more fast and precise than the analytical ones both in CT and in SPECT cases.
- The kernel methods can be in some cases more fast or more precise than its classic iterative versions.

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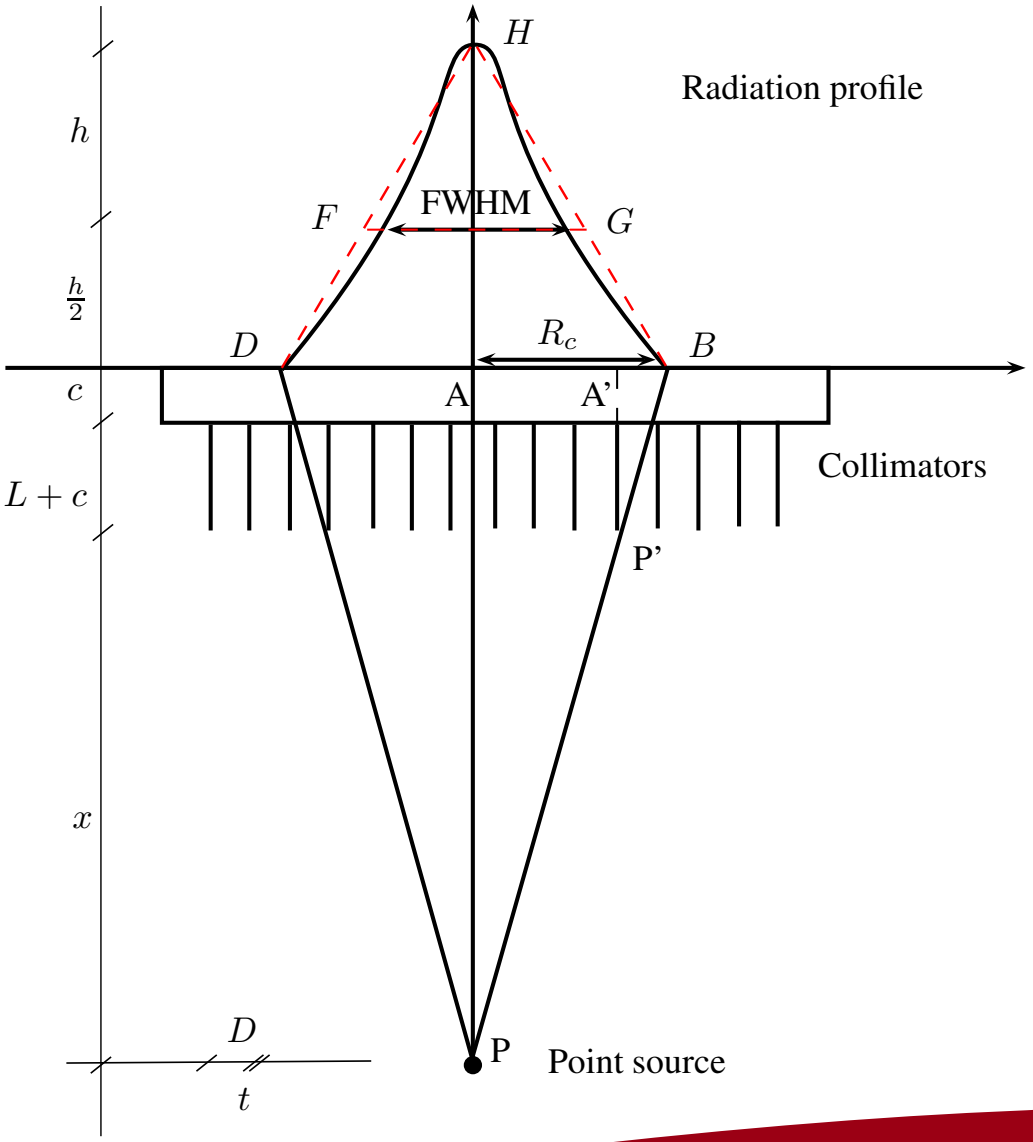
Conclusions:

The new methods introduced in this work (kernel and iterative SPECT/CT) are promising, but they need many *in vitro* tests before they can be used on patients.

The resolution is the smallest distance between two objects such that the machine can recognize them as two separate objects and is a crucial parameter that measures the reliability of the machine. We will find the resolution in two ways:

1. As a function of the parameters of the machine, declared by the producers (Analytical resolution).
2. Experimentally as the FWHM of the radiation profile of a point source (Experimental resolution).

and compare the results to see if the first one is a reasonable estimate of the resolution.


H
Radiation profile
h
F
FWHM
G
$\frac{h}{2}$
D
R_c
B
c
A
A'
Collimators
$L + c$
P'
x
D
t
P
Point source

By the similitude of the triangles $\overset{\triangle}{HDB}$ and $\overset{\triangle}{HFG}$

$$L_{eff} : D = (L_{eff} + x + c) : R_c$$

where $L_{eff} = L - \frac{2}{\mu}$ is a weighted length used to take into account the septal penetration. Hence

$$R_c = D\left(1 + \frac{x + c}{L_{eff}}\right)$$

The resolution of the system R_s then depends on the *collimator resolution* R_c and on the *intrinsic resolution* R_i, by

$$R_s = \sqrt{R_c^2 + R_i^2}$$

since R_c is linear with x and R_i constant the generic formula is

$$R_s(x) = \sqrt{ax^2 + bx + c}$$

Description	L (mm)	D (mm)	c (mm)	t (mm)	R_i (mm)
UHR Irix	58.4	1.78	19	0.152	4.1

From these parameters we can estimate the parameters a, b, c of the resolution formula

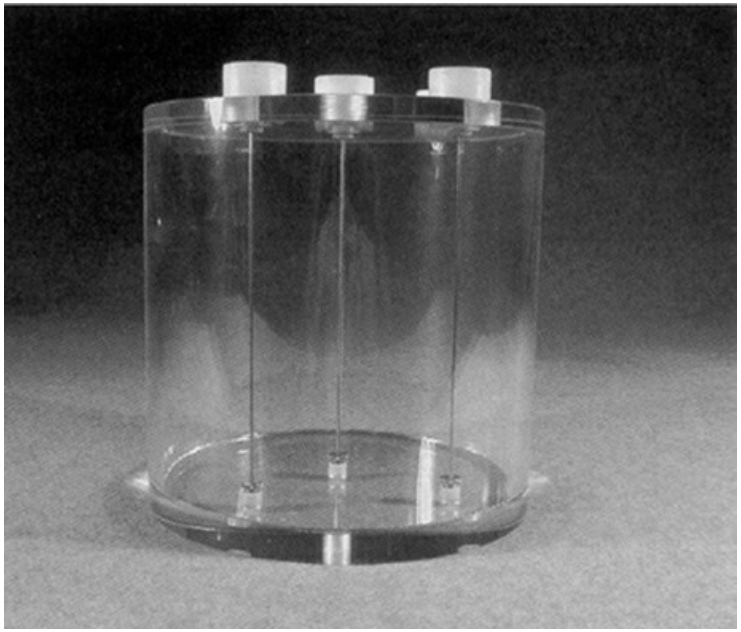
$$R_s(x) = \sqrt{ax^2 + bx + c}$$

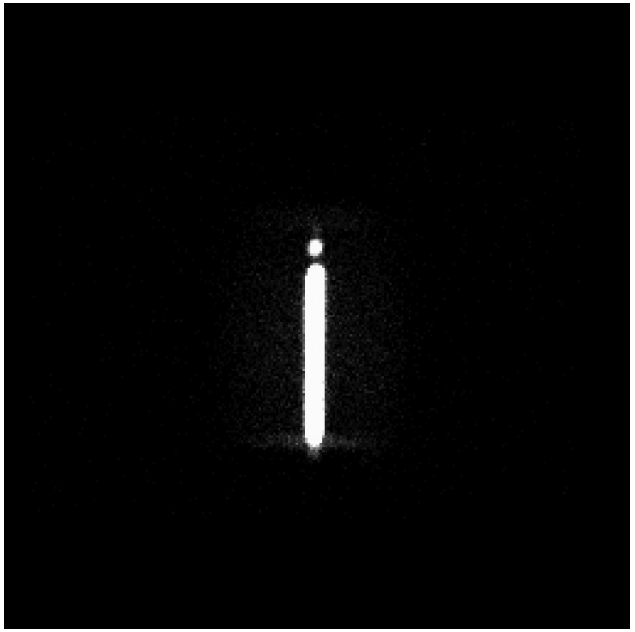
$$a = (\frac{D}{L_{eff}})^2 \cong 9.6 \cdot 10^{-4}$$

$$b = 2\frac{D^2}{L_{eff}}(1 + \frac{c}{L_{eff}}) \cong 1.5 \cdot 10^{-1}$$

$$c = D^2\left(1 + \frac{c}{L_{eff}}\right)^2 + R_i^2 \cong 2.2. \cdot 10^1$$

A quick look on the data





The phantom used in our experiments (left) and a graphic view on the data coming from this phantom, with the central capillary posed at a distance of 280 mm from the collimators. The color refers to the value of the data in the relative pixel of the $N \times N$ =256x256 matrix.

Direct interpolation method: we think our data as they come from a deterministic function plus a small level of noise. The linear least-squares approximation consists in finding a function

$$f_{\bar{a}}(x) = \sum_{i=1}^{n} a_i \phi_i(x)$$

depending on some parameters $\bar{a} = (a_1, \ldots, a_n)$, with $\phi_i,\ i = 1, \ldots, n$ a set of known (basis) functions. Let $(x_i, y_i)_{i=1,\ldots,N}$ be the experimental data at a given height, then we look for

$$\bar{a}^* = \arg\min_{\bar{a}\in\mathbb{R}^n} J(\bar{a}) = \arg\min_{\bar{a}\in\mathbb{R}^n} \|y_i - f_{\bar{a}}(x_i)\|^2 .$$

We choose three different functions $f_{\bar{a}}$:

- The *Gaussian*

$$f_{\bar{a}}(x) = a_1 e^{-a_2^2 x^2}.$$

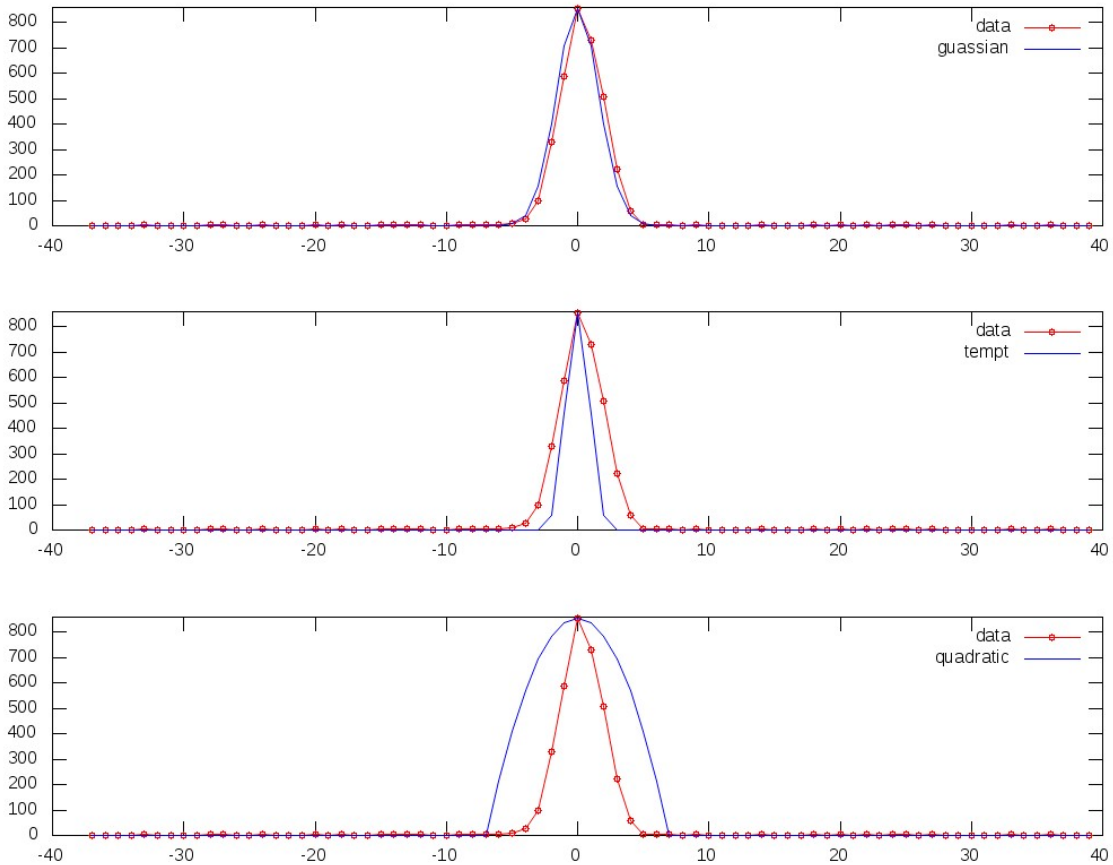
- The *Tempt*

$$f_{\bar{a}}(x) = \max\left((-a_1 x + a_2)\chi_{x>0} + (a_1 x + a_2)\chi_{x\leq 0}, 0\right)$$

  which is a piecewise differentiable function.

- The *Truncated quadratic*

$$f_{\bar{a}}(x) = (a_1 x^2 + a_2)_+ = \max(a_1 x^2 + a_2, 0) \quad \text{with } a_1 < 0.$$



The Gaussian function gives the best fit.

Local interpolation method: Now we want to approximate the data $(x_i, y_i)_{i=1,\dots,N}$ by spline functions, then evaluate the spline on a huge number of points in order to find

$$X_1 < b, X_2 > b s.t.\ argmin|a/2 - f(X_j)| \quad j = 1, 2$$

where the data have a maximum in (a, b), and compute simply $FWHM = X_2 - X_1$.

We have assigned a cost function on the two methods, and found the better method by testing them with our data.
According to our tests, the second method is the best choice for finding the resolution.
Since our data came from a line source object, we apply the method to several heights $j \in J \subseteq \{1, \ldots, N\}$ we obtain a mean value estimate for $FWHM$, and a standard deviation value σ.

Last step, we use data we have taken placing the phantom at some known distances from the collimators to compute the value of FWHM at the corresponding distance and we extrapolate a curve from those values. As said before the model is

$$R_s(x) = \sqrt{ax^2 + bx + c}.$$

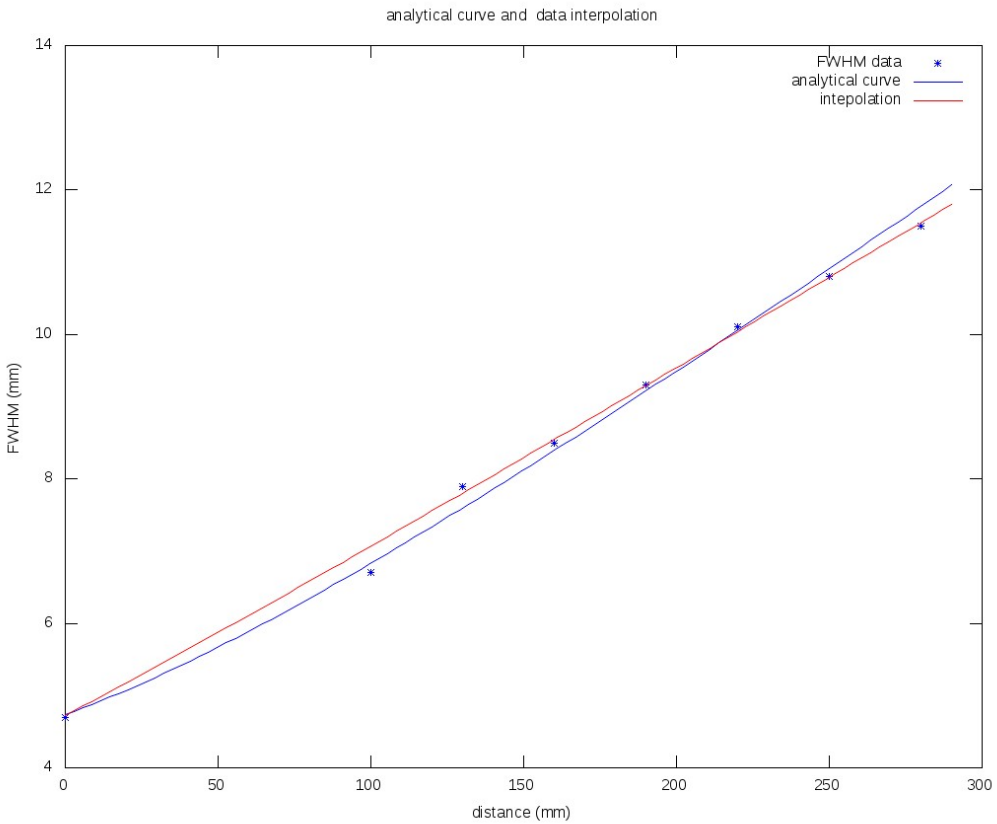
For our purpose we will use the weighted least squares method for finding the parameters a, b, c. From the previous formula we get

$$y^2 = ax^2 + bx + c$$

with y the vector of FWHMs and x the vector of distances.

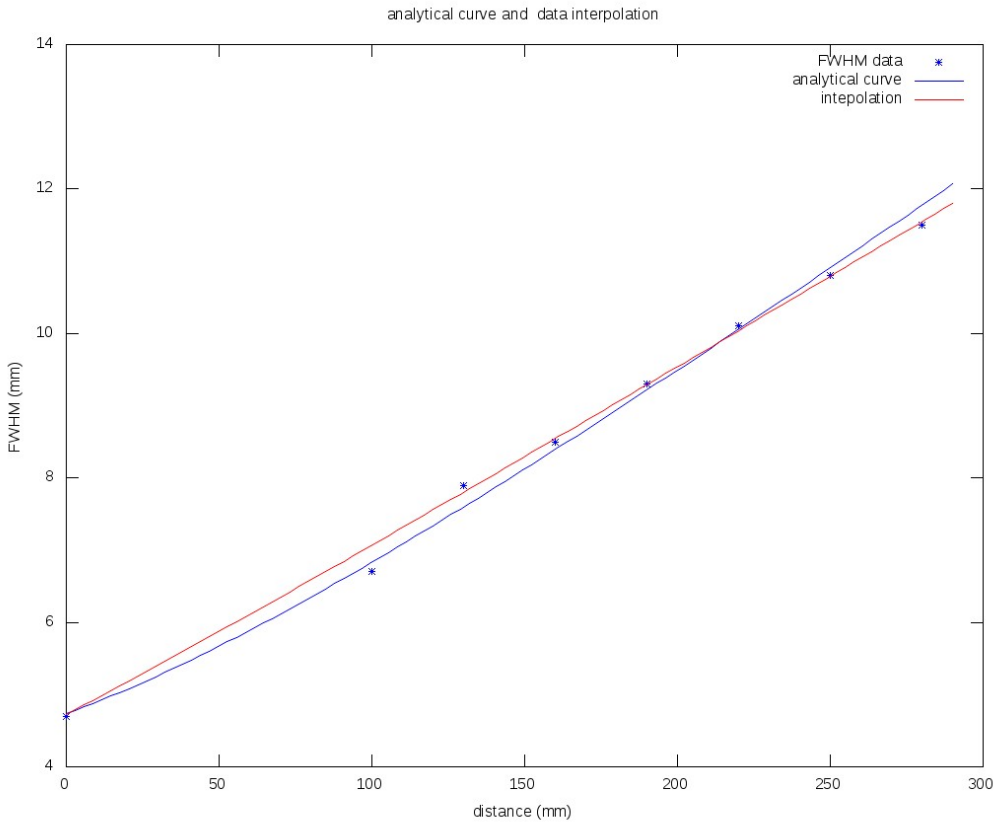
The system is linear on its parameters. We find the parameters $p = (a, b, c)$ by solving the normal equations

$$(V^T \Sigma V)p = V^T \Sigma b$$



where V is the

Vandermonde matrix, $b = y^2$



This is an example of extrapolation. As we can see the analytical curve is really close to the experimental one.

Essential bibliography

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