Resolution and Reconstruction issues

in CT and SPECT

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Axial Tomography machines provide result of some circular acquisitions. We discuss algorithms of reconstruction from projections and their efficiency in a few cases

- **transmission tomography** (e.g. CT): an electromagnetic ray passes through the patient and is detected at the exit in order to get a *morphological* analysis of its interior.

- **emission tomography** (e.g. PET, SPECT): a radioactive tracer is injected into the patient and detected by the machine in order to make an internal *functional* analysis of the organs.

- **hybrid tomography** (e.g. SPECT/CT, SPECT/MRI): two simultaneous analysis.

We will focus on CT, SPECT and SPECT/CT problems.
Transmission tomography
Emission tomography
If a ray passes through a body, it will be subject to attenuation. The Beer’s law says that if $I(x)$ is the intensity of a ray and $A(x)$ the attenuation coefficient of the point $x$, then

$$\frac{\Delta I}{\Delta x} = -A(x)I(x)$$

that is, by integrating:

$$\int_{x_0}^{x_1} A(x)dx = - \int_{x_0}^{x_1} \frac{dl}{l} = -\ln(I(x_1) - I(x_0)).$$
In transmission tomography, in 2 dimensions, the Beer’s law corresponds to the **Radon transform** (RT), defined as the integral over a line

\[
\mathcal{R}f(t, \theta) = \int_{\ell(t, \theta)} f = \int_{\mathbb{R}^2} f(\bar{x})\delta(t - \bar{x} \cdot \bar{\theta}) \, d\bar{x}
\]

with \( \bar{x} = (x, y) \), \( \bar{\theta} = (\cos \theta, \sin \theta) \)
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\]

with \( \bar{x} = (x, y) \), \( \bar{\theta} = (\cos \theta, \sin \theta) \) or equivalently:

\[
\mathcal{R} f(t, \theta) = \int_{\mathbb{R}} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) \, ds
\]
\[ \mathcal{R}f(t, \theta) = \int_{\mathbb{R}} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) \, ds \]
Now we ask for every point \((x, y)\) which is the average of the rays that pass through that point. This question is answered by the adjoint operator to \(\mathcal{R}\), called **Backprojection operator**

\[
\mathcal{R}^* g(x, y) = \frac{1}{|S^1|} \int_{\mathbb{R} \times S^1} g(s, \theta) \delta(s - \bar{x} \cdot \bar{\theta}) \, ds \, d\theta = \\
\frac{1}{|S^1|} \int_{S^1} g(x \cos \theta + y \sin \theta, \theta) \, d\theta
\]

where \(S^1 = [0, \pi]\) or \(S^1 = [0, 2\pi]\).
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**Warning**

\(\mathcal{R}^*\) is not the inverse transform of \(\mathcal{R}\). In fact \(\mathcal{R}^* \mathcal{R} f(\bar{x}) = \frac{2}{|\bar{x}|} * f\).
In the case of emission tomography things are more complicated. From Beer’s law we obtain that

\[ I(x_1) = I(x_0) \exp \left( - \int_{x_0}^{x_1} A(x) \, dx \right) \]

Suppose to know the attenuation coefficient, say \( a(\bar{x}) \), we want to obtain the radioactivity \( f(\bar{x}) \) by its angular projections.
We define the **attenuated Radon transform** (briefly AtRT)

\[
\mathcal{R}_a f(t, \theta) = \int_{\ell(t, \theta)} e^{-D_a(\bar{x}, \theta + \pi)} f(\bar{x})
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We define the **attenuated Radon transform** (briefly AtRT)

\[ R_a f(t, \theta) = \int_{\ell(t,\theta)} e^{-D_a(\bar{x},\theta+\pi)} f(\bar{x}) \]

where \( D \) is the **Divergent beam transform** defined as follows

\[ D h(\bar{x}, \theta) = \int_0^{+\infty} h(x + t \cos \theta, y + t \sin \theta) \ dt = \int_0^{+\infty} h(\bar{x} + t\bar{\theta}) \ dt \]
Then in the CT case we have to solve the following problem

**RT problem**

Given $g$ projection data (or sinogram) find $f$ such that $\mathcal{R}f = g$

while in SPECT case we can approximate $f$ with the solution of the previous problem
Then in the CT case we have to solve the following problem

**RT problem**

Given $g$ projection data (or sinogram) find $f$ such that $Rf = g$

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Given $g$ projection data and $a$ attenuation map find $f$ such that $Ra f = g$
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Given $g$ projection data and $a$ attenuation map find $f$ such that $\mathcal{R}_af = g$

to estimate the attenuation map we may need a simultaneous CT tomography $\rightarrow$ SPECT/CT.
We use two different approaches:

**First approach**

**Analytical methods:** discretization of exact solution.
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First approach

**Analytical methods**: discretization of exact solution.

Second approach

**Iterative methods**: by interpolation, we solve the linear system iteratively.
Theorem (Inversion of the Radon transform)

\[ f = \frac{1}{2} R^* \left[ \mathcal{F}^{-1}(|\nu|\mathcal{F}(Rf)) \right] \]

where we mean that the direct and inverse Fourier transform is applied only to the variable \( t \).
Theorem (Inversion of the Radon transform)

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where we mean that the direct and inverse Fourier transform is applied only to the variable \( t \).

As known the previous formula is numerically inaccurate. Then we use \( w(\nu) = p(\nu)|\nu| \) instead of \(|\nu|\), with \( p \) a low-pass filter, getting the approximated

**Filtered Back Projection formula (FBP)**

\[ f \approx \frac{1}{2} \mathcal{R}^* \left[ \mathcal{F}^{-1}(w(\nu)\mathcal{F}(\mathcal{R}f)) \right] \]
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Filtered Backprojection
Error bound
Novikov-Natterer formula

Original phantom \( f(x,y) \)

Filtered backprojection

Unfiltered backprojection

Sinegram: Radon transform \( R(f,\theta) \)
Theorem (Error estimate)

Let $f \in C_0^\infty(B(0,1))$ be a $b$-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let $\tilde{f}$ be the FBP reconstruction, then

$$\|f - \tilde{f}\|_{L^\infty(\mathbb{R}^2)} \leq 2|S^1| \|w_b\|_{L^1(\mathbb{R})}\|g - \tilde{g}\|_{L^\infty(\mathbb{R} \times S^1)} + |e_3|$$

with $e_3$ the quadrature error of the backprojection integral.
Also the attenuated transform has an inversion formula. Let by the following definitions:

**Definition**

Let \( g(t) \) be a suitable function, then its **Hilbert transform** is the function

\[
\mathcal{H}g(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{s-t} \, dt
\]

where the integral a Cauchy principal value.

**Definition**

Let us define the function

\[
h := \frac{1}{2}(I + i\mathcal{H})Ra
\]
Theorem (Novikov-Natterer formula)

Let $f$ be a transformable function $g = \mathcal{R}_a f$, and $h$ as in the previous slide. Assume $a(\bar{x})$ known, then $f$ is uniquely determined by the following formula

$$f(\bar{x}) = \frac{1}{4\pi} \Re e \ div \int_{S^1} \theta e^{D_a(\bar{x},\theta+\frac{\pi}{2})} (e^{-h\mathcal{H}e^h g})_{(\bar{x},\bar{\theta},\theta)} d\theta$$

where $S^1 = [0, 2\pi]$. 
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Attenuation phantom
Activity phantom
Reconstruction of the attenuation map
Reconstruction of the activity
Given a basis of functions \( \{b_i(\vec{x})\}_{i=1}^n \) that interpolates the function \( f \), i.e. such that

\[
f(\vec{x}) = \sum_{i=0}^{n} c_i b_i(\vec{x}) \quad \forall \vec{x} \in X
\]

where \( X \) is properly chosen, then for the linearity of the Radon transform

\[
\mathcal{R}f(\vec{y}) = \sum_{i=0}^{n} c_i \mathcal{R}b_i(\vec{y}) \quad \forall \vec{y} \in Y = \{(t_j, \theta_j)\}.
\]
This is equivalent to the solution of

\[ Af = c \]

where \( A(i,j) = Rb_i(t_j, \theta_j) \) is a matrix \( N^2 \times lp \), \( f \) is the unknown vector such that \( f_i = f(x_i) \) and \( c \) is the vector of the projection data \( c_j = Rf(t_j, \theta_j) \).

The methods using this approach are known as Algebraic Reconstruction Techniques or ART.
Using the ART approach we may have to face some problems:

- underdetermined $\rightarrow$ least squares
- ill-conditioned $\rightarrow$ regularization (not needed)
- huge $\rightarrow$ sparseness
We choose the *natural pixel basis*

\[ b_i(x, y) = \chi_{P_i}(x, y). \]

with \( P_i \) the \( i \)-th pixel of the reconstructed image. We know the Radon transform of each one of these items

\[ \mathcal{R}b_i(t, \theta) = \text{meas}(\ell_{t,\theta} \cap P_i) \]

where \( \ell_{t,\theta} = \{ t\bar{\theta} + s\bar{\theta}^\perp \mid s \in \mathbb{R} \} \).
Since the matrix $A$ is large and sparse, we can solve the system $Af = c$ by iterative methods.

- the initial vector $f^{(0)}$ is a *blank*.
- the image at step $k$, $f^{(k)}$, is projected and compared with the data.
- the image is modified considering the error found in the previous step.
The following methods are the most popular

- **The Kaczmarz method** projects the vector $f^{(k)}$ on $k$-th row of $A$ (or on a random row).
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**LSCG example**
Now instead of the natural pixel basis we can use another basis:

\[ b_i(\vec{x}) = B(\vec{x} - \vec{x}_i) = K(\vec{x}, \vec{x}_i) \]

with \( B \) a **(essentially)** compact supported and radial.

<table>
<thead>
<tr>
<th>Function name</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ball</td>
<td>( \chi_{B(\frac{1}{\varepsilon},0)}(r) )</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( e^{-\varepsilon^2 r^2} )</td>
</tr>
<tr>
<td>Wendland ( \varphi_{2,0} )</td>
<td>( (1 - \varepsilon r)^2_{+} )</td>
</tr>
<tr>
<td>Wu ( \psi_{1,1} )</td>
<td>( (1 - \varepsilon r)^2_{+}(\varepsilon r + 2) )</td>
</tr>
</tbody>
</table>

with \( r = \|x\|_2 \) and \( \varepsilon \) the shape parameter.

The ray of the support is \( 1/\varepsilon \).
Original phantom and kernel reconstruction with Gaussian kernel and shape parameter $\varepsilon = 1$, after 50 iterations of LSCG.
Lemma

If $\phi(x) = \varphi(\|x\|)$ is a radial function, then its Radon transform $Rf$ is radial, i.e. it depends only on $t$ and it is even.

Theorem

If $\phi(x - y) = K(x, y)$ is a radial function, $\phi \in L^1(\mathbb{R}^d)$, continuous, bounded and positive definite on $\mathbb{R}^2$, then its Radon transform $Rf(t)$ is bounded and positive definite on $\mathbb{R}^1$, provided $Rf \in L^1(\mathbb{R})$. 
Theorem (Interpolation error bound for Kernel method)

Let $f \in C_0^\infty(B(0, 1))$ a $b$-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let $K$ be a strictly positive definite kernel and its domain $\Omega$ be such that $\partial \Omega$ has regularity at least $C^1$. Then there exist positive constants $h_0$ and $\tilde{C}$ such that, if $h_{\chi, \Omega} \leq h_0$, then

$$\|f(\cdot) - \sum_{i=0}^{n} c_i K_i(\cdot)\|_{L^\infty} \leq 2 |S^1| b \sqrt{\frac{N}{18}} \tilde{C} h_{\chi, \Omega} \|\mathcal{R}f\|_{\mathcal{N}_{\mathcal{R}K}(\Omega)}$$

with $h_{\chi, \Omega}$ the meshsize.
Let us consider, as before, the basis $b_i(\bar{x}) = \chi_{P_i}$ and assume

$$f(\bar{x}) = \sum_{i=0}^{n} c_i b_i(\bar{x})$$

then for linearity

$$\mathcal{R}_a f(\bar{y}) = \sum_{i=0}^{n} c_i \mathcal{R}_a b_i(\bar{y}) \quad \forall \bar{y} \in Y = \{(t_j, \theta_j)\}$$

i.e.

$$Bc = d$$

where $B_{i,j} = \mathcal{R}_a b_i(t_j, \theta_j)$, $c_i = f(\bar{x}_i)$ the unknown term and $d_j = \mathcal{R}_a f(t_j, \theta_j)$ the data.
In order to compute the matrix $B$ we have to consider the attenuation coefficients in the natural pixel basis

$$a(\bar{x}) = \sum_{k=1}^{N^2} g_k \chi_{P_k}(\bar{x}).$$

According to Beer’s law

$$l_{out} = l_{in} \exp \left( - \sum_{k=1}^{N^2} g_k \ meas(P_k \cap \ell_{\bar{x},\theta}^+) \right).$$
Now, if we consider the matrix $A$ used in CT tomography we can compute the outgoing rays from the pixel $P_i$ in $(t_j, \theta_j)$ as

$$B_{i,j} = A_{i,j} \exp \left( - \sum_{(k_1,k_2) \in K(i,j)} g_k \ meas(P_k \cap \ell_{t_j,\theta_j}) \right) =$$

$$A_{i,j} \exp \left( \sum_{(k_1,k_2) \in K(i,j)} g_k A_{k_1,k_2} \right)$$

where $K(i,j)$ is the set of the indexes of the pixels covered by the line $\ell_{x_i,\theta_j}^+$. 


We can introduce a relaxation parameter $\lambda \in [0, 1]$ to weight the effect of the attenuation

$$B_{i,j}^{(\lambda)} = A_{i,j} \exp \left( -\lambda \sum_{(k_1,k_2) \in K_{(i,j)}} g_k A_{k_1,k_2} \right)$$

Note that $B^{(0)} = A$ and $B^{(1)} = B$. We observe that with this little change the linear system is more accurate.
Analytical reconstruction of a SPECT/CT phantom data with $\lambda = 0.1$. 
From the *in silico* experiments we see that:

- The iterative methods are more fast and precise than the analytical ones both in CT and in SPECT cases.
- The kernel methods can be in some cases more fast or more precise than its classic iterative versions.
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**Conclusions:**

The new methods introduced in this work (kernel and iterative SPECT/CT) are promising, but they need many *in vitro* tests before they can be used on patients.
The resolution is the smallest distance between two objects such that the machine can recognize them as two separate objects and is a crucial parameter that measures the reliability of the machine. We will find the resolution in two ways:

1. As a function of the parameters of the machine, declared by the producers (Analytical resolution).
2. Experimentally as the FWHM of the radiation profile of a point source (Experimental resolution).

and compare the results to see if the first one is a reasonable estimate of the resolution.
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Analytical resolution
Experimental FWHM
Comparison

\[ h^2 + c^2 + (L + c)^2 = x^2 + t^2 \]

Radiation profile

Collimators

Point source
By the similitude of the triangles $\triangle HDB$ and $\triangle HFG$

$$L_{\text{eff}} : D = (L_{\text{eff}} + x + c) : R_c$$

where $L_{\text{eff}} = L - \frac{2}{\mu}$ is a weighted length used to take into account the septal penetration. Hence

$$R_c = D \left(1 + \frac{x + c}{L_{\text{eff}}} \right)$$
The resolution of the system $R_s$ then depends on the *collimator resolution* $R_c$ and on the *intrinsic resolution* $R_i$, by

$$R_s = \sqrt{R_c^2 + R_i^2}$$

since $R_c$ is linear with $x$ and $R_i$ constant the generic formula is

$$R_s(x) = \sqrt{ax^2 + bx + c}$$
Table 1: Description of parameters

<table>
<thead>
<tr>
<th>Description</th>
<th>L (mm)</th>
<th>D (mm)</th>
<th>c (mm)</th>
<th>t (mm)</th>
<th>R_i (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UHR Irix</td>
<td>58.4</td>
<td>1.78</td>
<td>19</td>
<td>0.152</td>
<td>4.1</td>
</tr>
</tbody>
</table>

From these parameters we can estimate the parameters $a, b, c$ of the resolution formula

$$R_s(x) = \sqrt{ax^2 + bx + c}$$

$$a = \left(\frac{D}{L_{\text{eff}}}\right)^2 \approx 9.6 \cdot 10^{-4}$$

$$b = 2\frac{D^2}{L_{\text{eff}}}(1 + \frac{c}{L_{\text{eff}}}) \approx 1.5 \cdot 10^{-1}$$

$$c = D^2 \left(1 + \frac{c}{L_{\text{eff}}}\right)^2 + R_i^2 \approx 2.2 \cdot 10^1$$
A quick look on the data

The phantom used in our experiments (left) and a graphic view on the data coming from this phantom, with the central capillary posed at a distance of 280 mm from the collimators. The color refers to the value of the data in the relative pixel of the $N \times N = 256 \times 256$ matrix.
Direct interpolation method: we think our data as they come from a deterministic function plus a small level of noise. The linear least-squares approximation consists in finding a function

\[ f_{\bar{a}}(x) = \sum_{i=1}^{n} a_i \phi_i(x) \]

depending on some parameters \( \bar{a} = (a_1, \ldots, a_n) \), with \( \phi_i, \ i = 1, \ldots, n \) a set of known (basis) functions. Let \( (x_i, y_i)_{i=1,\ldots,N} \) be the experimental data at a given height, then we look for

\[
\bar{a}^* = \arg \min_{\bar{a} \in \mathbb{R}^n} J(\bar{a}) = \arg \min_{\bar{a} \in \mathbb{R}^n} \| y_i - f_{\bar{a}}(x_i) \|^2.
\]
We choose three different functions $f_{\tilde{a}}$:

- The *Gaussian*
  
  $$f_{\tilde{a}}(x) = a_1 e^{-a_2^2 x^2}.$$  

- The *Tempt*
  
  $$f_{\tilde{a}}(x) = \max((-a_1 x + a_2)\chi_{x>0} + (a_1 x + a_2)\chi_{x\leq 0}, 0)$$  

  which is a piecewise differentiable function.

- The *Truncated quadratic*
  
  $$f_{\tilde{a}}(x) = (a_1 x^2 + a_2)_+ = \max(a_1 x^2 + a_2, 0) \quad \text{with} \quad a_1 < 0.$$  

The Gaussian function gives the best fit.
Local interpolation method: Now we want to approximate the data $(x_i, y_i)_{i=1,...,N}$ by spline functions, then evaluate the spline on a huge number of points in order to find

$$X_1 < b, X_2 > bs.t. \arg\min |a/2 - f(X_j)| \quad j = 1, 2$$

where the data have a maximum in $(a, b)$, and compute simply $FWHM = X_2 - X_1$. 
We have assigned a cost function on the two methods, and found the better method by testing them with our data. According to our tests, the second method is the best choice for finding the resolution. Since our data came from a line source object, we apply the method to several heights $j \in J \subseteq \{1, \ldots, N\}$ we obtain a mean value estimate for $FWHM$, and a standard deviation value $\sigma$. 
Last step, we use data we have taken placing the phantom at some known distances from the collimators to compute the value of FWHM at the corresponding distance and we extrapolate a curve from those values. As said before the model is

$$R_s(x) = \sqrt{ax^2 + bx + c}.$$  

For our purpose we will use the weighted least squares method for finding the parameters $a, b, c$. From the previous formula we get

$$y^2 = ax^2 + bx + c$$

with $y$ the vector of FWHMs and $x$ the vector of distances.
The system is linear on its parameters. We find the parameters $p = (a, b, c)$ by solving the normal equations

$$(V^T \Sigma V)p = V^T \Sigma b$$

where $V$ is the Vandermonde matrix, $b = y^2$, and $\Sigma$ is the weight matrix, with $\sigma_i$ the standard deviation (in mm) of the $i$-th resolution value.
This is an example of extrapolation. As we can see the analytical curve is really close to the experimental one.
Essential bibliography


