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Regularity of the $\bar\partial$-Neumann problem at point of infinite type

Tran Vu Khanh, Giuseppe Zampieri *

Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy

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Abstract

We introduce general estimates for "gain of regularity" of solutions of the $\bar\partial$-Neumann problem and relate it to the existence of weights with large Levi form at the boundary. This enables us to discuss in a unified framework the classical results on fractional ellipticity (= subellipticity), superlogarithmic ellipticity and compactness. For each case, we exhibit a corresponding class of domains.

Keywords: $\bar\partial$-Neumann problem

1. Introduction

Let D be a bounded domain of $\mathbb{C}^n$ defined by $r < 0$ for $\partial r \neq 0$. The $\bar\partial$-Neumann problem consists in finding the solution of $\bar\partial u = v$ which is orthogonal to $\ker\bar\partial$ under the compatibility condition $\bar\partial v = 0$. Here v and u are forms of degree k and $k-1$ respectively. Related to this, is the equation $\Box u = v$, with u and v of the same degree k, where $\Box := \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$. If $\Box$ is invertible, in a suitable Hilbert space, there is well-defined a Neumann operator $N := \Box^{-1}$ and the solution to the first problem is produced by $u := \bar\partial^* N v$. We discuss estimates for $(\bar\partial, \bar\partial^*)$ and for $\Box$ which assure continuity of N in the spaces H^s and C^∞ up to the boundary ∂D. We wish to recall the theory by Catlin of [2]. Assume that, in a neighborhood of a boundary point $z_o \in \partial D$, there is a

* Corresponding author.
E-mail addresses: khanh@math.unipd.it (T.V. Khanh), zampieri@math.unipd.it (G. Zampieri).

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family of weights $\varphi = \varphi_\delta$ for $\delta \to 0$, which are plurisubharmonic, have bound $|\varphi| \leqslant 1$ over the strip $\mathcal{S}_\delta := \{z \in D : \mathrm{dist}(z, \partial D) < \delta\}$ and whose Levi form φ_{ij} satisfies

$$\varphi_{ij}(z) \gtrsim \delta^{-2\epsilon} \quad \text{for any } z \in \mathcal{S}_\delta \tag{1.1}$$

(in the sense that the lowest eigenvalue of φ_{ij} is $\gtrsim \delta^{-2\epsilon}$). Here and in what follows, $\gtrsim$ or $\lesssim$ denote inequality up to a constant. Note that Catlin requires in addition $|\varphi| < 1$ on the whole D; it is clear from the proof of Theorem 1.4 that the φ's may be arranged so that this last condition is fulfilled. In [2] Catlin proves that finite type of ∂D in the sense of D'Angelo [4] yields a family of weights satisfying (1.1). In turn, he proves that these weights give subelliptic estimates for the $\bar\partial$-Neumann problem (which were already obtained by Kohn [13] for real analytic boundaries).

Theorem 1.1. *(See Catlin [2], Theorem 2.2.) Let D be pseudoconvex; then the existence of a family of weights $\{\varphi^\delta\}$ which satisfy (1.1) in a neighborhood of $z_o \in \partial D$ implies, for a smaller neighborhood V of z_o,*

$$|||u|||_\epsilon^2 \lesssim \|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2 + \|u\|_0^2 \quad \text{for } u \in C_c^\infty(\bar D \cap V) \cap Dom_{\bar\partial^*} \text{ of degree } k \geqslant 1. \tag{1.2}$$

Here $|||\cdot|||_\epsilon$ is the tangential ϵ-Sobolev norm. We want to generalize this result in two directions. The first consists in considering more general q-pseudoconvex (or q-pseudoconcave), instead of merely pseudoconvex, domains and prove (1.2) for forms of related degree $k \leqslant q$ (or $k \geqslant q$); this was already achieved in [10–12,20]. The second, consists in considering estimates with a weaker gain of regularity than the tangential fractional ϵ-Sobolev. This is the specific novelty of the present paper. To introduce q-pseudoconvexity/concavity we need to develop some notations and terminology: $L_{\partial D} = (r_{ij})|_{T^{\mathbb{C}}\partial D}$ is the Levi form of the boundary, $s^+_{\partial D}$, $s^-_{\partial D}$, $s^0_{\partial D}$ are the numbers of eigenvalues of $L_{\partial D}$ which are > 0, < 0, $= 0$ respectively and finally $\lambda_1^{\partial D} \leqslant \lambda_2^{\partial D} \leqslant \cdots \leqslant \lambda_{n-1}^{\partial D}$ are its ordered eigenvalues. We take a pair of indices $1 \leqslant q \leqslant n-1$ and $0 \leqslant q_o \leqslant n-1$ such that $q \neq q_o$. We assume that there is a bundle $\mathcal{V}^{q_o} \subset T^{1,0}(\partial D)$ of rank q_o with smooth coefficients in a neighborhood V of z_o, say the bundle of the first q_o coordinate tangential vector fields $\partial_{\omega_1}, \dots \partial_{\omega_{q_o}}$, such that

$$\sum_{j=1}^{q} \lambda_j^{\partial D} - \sum_{j=1}^{q_o} r_{jj} \geqslant 0 \quad \text{on } \partial D \cap V. \tag{1.3}$$

Definition 1.2.

(i) If $q > q_o$ we say that D is q-pseudoconvex at z_o.
(ii) If $q < q_o$ we say that D is q-pseudoconcave at z_o.

This condition contains, as a particular case, the classical q-pseudoconvexity (resp. q-pseudoconcavity) "by compensation" which corresponds to the choice $q_o = 0$ (resp. $q_o = n-1$), that is, $\sum_{j=1}^{q} \lambda_j^{\partial D} \geqslant 0$ (resp. $-\sum_{j=q+1}^{n-1} \lambda_j^{\partial D} \geqslant 0$) (cf. [7] and more recent developments by [19] and [18]). We write k-forms as $u = (u_J)_J$ where $J = j_1 < j_2 < \cdots < j_k$ are ordered multiindices. When the multiindices are not ordered, the coefficients are assumed to be alternating. Thus, if J decomposes as $J = jK$, then $u_{jK} = \mathrm{sign}\binom{J}{jK} u_J$. We take an orthonormal basis of $(1,0)$ forms $\omega_1,\dots,\omega_n=\partial r$ and the dual basis of $(1,0)$ vector fields $\partial_{\omega_1},\dots,\partial_{\omega_n}$; thus $\partial_{\omega_1},\dots,\partial_{\omega_{n-1}}$ generate $T^{1,0}(\partial D)$. Under the choice of such basis, we check readily that $u\in Dom_{\bar\partial^*}$ if and only if $u_{nK}|_{\partial D}\equiv 0$ for any K. We use the notation $r_j:=\partial_{\omega_j}r$. Integration by parts and use of the tangentiality conditions $u_{nK}|_{\partial D}\equiv 0$, as well of the vanishing $r_j|_{\partial D}\equiv 0$ for $j\leqslant n-1$ which follows from the choice of the orthonormal basis adapted to the boundary, yields the "basic" estimates [8,9,21]

$$\begin{aligned}
&\|\bar\partial u\|^2_{H^0_\varphi}+\big\|\bar\partial^*_\varphi u\big\|^2_{H^0_\varphi}+C\|u\|^2_{H^0_\varphi}\\
&\quad\geqslant \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n}\int_D e^{-\varphi}\varphi_{ij}u_{iK}\bar u_{jK}\,dv-\sum_{|J|=k}{}'\sum_{j=1}^{q_o}\int_D e^{-\varphi}\varphi_{jj}|u_J|^2\,dv\\
&\qquad+\sum_{|K|=k-1}{}'\sum_{i,j=1}^{n-1}\int_{\partial D} e^{-\varphi}r_{ij}u_{iK}\bar u_{jK}\,ds-\sum_{|J|=q}{}'\sum_{j=1}^{q_o}\int_{\partial D}e^{-\varphi}r_{jj}|u_J|^2\,ds\\
&\qquad+\frac12\left(\sum_{j=1}^{q_0}\big\|\delta^\varphi_{\omega_j}u\big\|^2_{H^0_\varphi}+\sum_{j=q_o+1}^{n}\|\partial_{\bar\omega_j}u\|^2_{H^0_\varphi}\right)\quad \text{for } u\in C^\infty_c(\bar D\cap V)^k\cap Dom_{\bar\partial^*}. \qquad (1.4)
\end{aligned}$$

Here the $\delta^\varphi_{\omega_j}$'s are the adjoints to the $-\partial_{\bar\omega_j}$'s and dv and ds are the elements of volume in D and of area on ∂D respectively. We refer for instance to [21] for the proof (1.4). By choosing φ so that $e^{-\varphi}$ is bounded, we may remove the weight functions in (1.4). We note that there is no relation between k and q_o in the above inequality and that C is independent of φ (and u). However, if we assume that D is q-pseudoconvex (resp. q-pseudoconcave) and restrain the degree k of u to $k\geqslant q$ (resp. $k\leqslant q$), then the third line of (1.4) can be discarded since it is positive and we get an estimate which does not involve boundary integrals. Now the crucial point has become to make the right choice of the weight φ in order to get full advantage of the second line of (1.4). Also, we wish to treat lower bounds for the Levi form, smaller than $\delta^{-2\epsilon}$. Let f be a smooth monontonic increasing function $f:\mathbb{R}^+\to\mathbb{R}^+$ with $f(t)\lesssim t^{\frac12}$. We consider weights $\varphi=\varphi^\delta$ in $C^2(\bar D\cap V)$, absolutely bounded in $\bar{\mathcal S}_\delta\cap V$ and with the property that, if $\lambda^\varphi_1(z)\leqslant\lambda^\varphi_2(z)\leqslant\cdots$ are the ordered eigenvalues of the form $\varphi^\delta_{ij}(z)$, we have

$$\sum_{j=1}^{q}\lambda^\varphi_j(z)-\sum_{j=1}^{q_o}\varphi_{jj}(z)\gtrsim f\left(\frac1\delta\right)^2+\sum_{j=1}^{q_o}\big|\varphi_j(z)\big|^2,\quad \text{for any } z\in\bar{\mathcal S}_\delta\cap V. \qquad (1.5)$$

According to the point (a) of the proof of Theorem 1.4 which follows, we can modify φ to a new weight for which (1.5) holds in the whole $D\cap V$, instead of the only $\mathcal S_\delta\cap V$, but with the term in the right reduced to $\sum_{j=1}^{q_o}|\varphi_j(z)|^2$. In the same way as (1.3) says that the second line of (1.4) is positive, we can see that this modified version of (1.5) gives a good lower bound for the first line over forms in degree $k\geqslant q$.

Definition 1.3. If ∂D is q-pseudoconvex/concave and there is a family of weights $\{\varphi\}=\{\varphi^\delta\}$ which are absolutely bounded on $\bar{\mathcal S}_\delta\cap V$ and satisfy (1.5), we say that D satisfies $(f\text{-}P\text{-}q)$.

We introduce special coordinates $(a,r)\in\mathbb{R}^{2n-1}\times\mathbb{R}$, denote by ξ the dual coordinates to a and by $\mathcal F_\tau$ the tangential Fourier transform, that is, the partial Fourier transform with respect to a.

We denote by $\Lambda_\xi = (1+|\xi|^2)^{\frac{1}{2}}$ the standard elliptic symbol of order 1 and by Λ_∂ the operator with symbol Λ_ξ. For a smooth monotonic increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(t) \leqslant t^{\frac{1}{2}}$, we consider the symbol $f(\Lambda_\xi)$ and the associated tangential pseudodifferential operator $f(\Lambda_\partial)$. This is defined by

$$f(\Lambda_\partial)u = \mathcal{F}_\tau^{-1}\big(f(\Lambda_\xi)\mathcal{F}_\tau(u)\big).$$

Here is the main result of the present section.

Theorem 1.4. *Let D be q-pseudoconvex (resp. q-pseudoconcave), assume that ∂D satisfies $(f\text{-}P\text{-}q)$ for $q > q_o$ (resp. $q < q_o$) and let $k \geqslant q$ (resp. $k \leqslant q$). Then*

$$\big\| f(\Lambda_\partial)u\big\|^2 \lesssim \|\bar\partial u\|^2 + \big\|\bar\partial^* u\big\|^2 + \|u\|^2 \quad \textit{for } u \in C^\infty(\bar D \cap V) \cap Dom_{\bar\partial^*} \textit{ of degree } k. \tag{1.6}$$

Before the proof, some remarks are in order.

Remark 1.5. We point our attention to the rate of f as $t \to \infty$ in three relevant cases:

(i) $f \gtrsim t^\epsilon$,
(ii) $f \geqslant k \log t$ for any k,
(iii) $f \geqslant k$ for any k.

It is obvious that (i) implies (ii) and (ii) implies (iii). The estimates (1.6) are said subelliptic, superlogarithmic and of compactness, when f satisfies (i), (ii) and (iii) respectively. For the case of pseudoconvex domains, the first are discussed, as it has already been said, by Catlin in [2], the second by Kohn in [16] and the third by Catlin [1], Straube [19], Mc Neal [17], Harrington [6] and others.

Remark 1.6. Classically, superlogarithmic estimates are defined by

$$\big\|\log(\Lambda_\partial)u\big\|_0^2 \lesssim \epsilon\big(\|\bar\partial u\|_0^2 + \big\|\bar\partial^* u\big\|_0^2\big) + C_\epsilon \|u\|_{-1}^2 \quad \text{for any } \epsilon > 0. \tag{1.7}$$

But (1.6) for f satisfying (ii), that is, $f \geqslant k\log t$ for any k, implies (1.7). In fact, under the substitution $t = |\xi_a|$, we have $f \geqslant \epsilon^{-1}\log(|\xi_a|)$ for any ξ_a outside a suitable compact $K_\epsilon \Subset \mathbb{R}^{2n-1}$. It follows

$$\begin{aligned}\big\|\log(\Lambda_\partial)u\big\|^2 &\leqslant \epsilon \int\limits_{(\mathbb{R}^{2n-1}\setminus K_\epsilon)\times\mathbb{R}} f^2\big(\Lambda(\xi)\big)|\mathcal{F}_\tau u|^2\, d\xi\, dr + \sup_{\xi\in K_\epsilon} f^2(\Lambda_\xi) \int\limits_{K_\epsilon\times\mathbb{R}} |\mathcal{F}_\tau u|^2\, d\xi\, dr \\ &\leqslant \epsilon \int\limits_{(\mathbb{R}^{2n-1}\setminus K_\epsilon)\times\mathbb{R}} f^2\big(\Lambda(\xi)\big)|\mathcal{F}_\tau u|^2\, d\xi\, dr + C_\epsilon \|u\|_{-1}^2.\end{aligned}$$

This, combined with (1.6), yields (1.7).

Similarly, compactness is classically defined by $\|u\|_0^2 \lesssim \epsilon(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2) + C_\epsilon\|u\|_{-1}^2$ for any ϵ; again, this estimate is a consequence of (1.6) for f satisfying (iii).

Remark 1.7. Let $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ be the $\bar\partial$-Neumann Laplacian. It is well known (cf. [5] and [16]) that subelliptic and superlogarithmic estimates imply local hypoellipticity of $\Box$: $\Box u \in C^\infty(\bar D \cap V)$ implies $u \in C^\infty(\bar D \cap V)$. On the other hand, compactness over a covering $\{D\}$ of ∂D implies global hypoellipticity: $\Box u \in C^\infty(\bar D)$ implies $u \in C^\infty(\bar D)$. We have another version of these two statements. For the equation $\bar\partial u = v$ with $\bar\partial v = 0$, we define the "canonical" solution by $u := \bar\partial^* N v$ where N is the H^0 inverse to $\Box$. Thus local (global) hypoellipticity of $\Box$ implies that the canonical solution u inherits local (global) smoothness from v at ∂D (it surely does in the interior).

Proof of Theorem 1.4. (a) We "globalize" φ by multiplication for a cut-off χ and next deform by composition with a convex function ψ so that the resulting function $\psi \circ (\chi\varphi)$, that we still denote by φ, satisfies

$$\sum_{|K|=k-1}' \sum_{ij=1}^{n-1} \varphi_{ij} u^\tau_{iK} \bar u^\tau_{jK} - \sum_{j=1}^{q_o} \varphi_{jj} \left|u^\tau\right|^2 - 2 \sum_{|K|=k-1}' \left|\partial\varphi \cdot u^\tau_{\cdot K}\right|^2 \gtrsim \begin{cases} 0 & \text{in } D \cap V, \\ f(\delta^{-1})^2 & \text{in } \mathcal{S}_{\frac{\delta}{2}} \cap V, \end{cases} \tag{1.8}$$

for u^τ tangential, that is, satisfying $u^\tau_J \equiv 0$ if $n \in J$ even for $z \notin \partial D$, of degree $k \geqslant q$ (resp. $k \leqslant q$). For this, we take a smooth decreasing cut-off function satisfying $\chi \equiv 1$ on $[0, \frac{1}{2}]$ and $\chi \equiv 0$ on $[\frac{2}{3}, 1]$, and define $\tilde\varphi = \tilde\varphi^\delta$ by $\tilde\varphi^\delta := \chi(-\frac{r}{\delta})\varphi^\delta$. Recall that $r_j = 0$ for $j \leqslant n-1$ and $u^\tau_{nK} \equiv 0$. Then, over such forms we have

$$\left(\sum_{i,j=1}^{n-1} \tilde\varphi_{ij} u^\tau_{iK} \bar u^\tau_{jK} - \sum_{j=1}^{q_o} \tilde\varphi_{jj} \left|u^\tau_J\right|^2\right) \geqslant \chi \cdot \left(\sum_{i,j=1}^{n} \varphi_{ij} u^\tau_{iK} \bar u^\tau_{jK} - \sum_{j=1}^{q_o} \varphi_{jj} \left|u^\tau_J\right|^2\right). \tag{1.9}$$

In fact, if i and j denote derivation in ∂_{ω_i} and $\partial_{\bar\omega_j}$ respectively, we have

$$\begin{aligned} \left(\chi\left(-\frac{r}{\delta}\right)\varphi\right)_{ij} &= \ddot\chi \frac{r_i r_j}{\delta^2}\varphi - \dot\chi \frac{\varphi}{\delta} r_{ij} - \dot\chi \frac{r_j \varphi_i}{\delta} - \dot\chi \frac{r_i \varphi_j}{\delta} + \chi\varphi_{ij} \\ &= -\dot\chi \frac{\varphi}{\delta} r_{ij} + \chi\varphi_{ij} \quad \text{over tangential forms } u^\tau \end{aligned} \tag{1.10}$$

(where we have to remember that $r_j \equiv 0$ for any $j \leqslant n-1$). Since $-\dot\chi \geqslant 0$, then (1.10) implies (1.9). Note that $\partial\tilde\varphi = \dot\chi \partial r \varphi + \chi \partial\varphi$ and recall that $\partial r \cdot u^\tau \equiv 0$ (and that u has degree $\geqslant q$). It follows that $\tilde\varphi$ satisfies (1.8) with the constant 2 replaced by a more general $c > 0$ according to (f-P-q). To get (1.8) for the precise constant 2 we have to compose $\psi \circ \tilde\varphi$. From

$$\partial\bar\partial(\psi \circ \tilde\varphi) = \dot\psi \partial\bar\partial\tilde\varphi + \ddot\psi \partial\tilde\varphi \otimes \bar\partial\tilde\varphi,$$

we get that $\psi \circ \tilde\varphi$ satisfies (1.8) as soon as

$$\begin{cases} \ddot\psi \geqslant 2\dot\psi^2, \\ \ddot\psi \leqslant \frac{c}{2}\dot\psi. \end{cases} \tag{1.11}$$

A choice for such a function is $\psi = \frac{1}{2}e^{\frac{c}{2}(t-1)}$; we still denote by φ this new weight $\psi \circ \tilde{\varphi}$ which satisfies (1.8). We wish to remove now the weight from the adjunction $\bar\partial^*_\varphi$. We note that

$$\|\bar\partial^* u^\tau\|^2 \geqslant \frac{1}{2}\|\bar\partial^*_\varphi u^\tau\|^2 - \sum_{|K|=k-1}{}' \|\partial\varphi \cdot u^\tau_{\cdot K}\|^2. \tag{1.12}$$

It follows

$$\begin{aligned} Q_D(u^\tau, u^\tau) + C\|u^\tau\|^2_D &\gtrsim \frac{1}{2}\|\bar\partial^*_\varphi u^\tau\|^2_D + \|\bar\partial u^\tau\|^2_D - \sum_{|K|=k-1}{}' \|\partial\varphi \cdot u_{\cdot K}\|^2_D \\ &\geqslant \frac{1}{2}\int_D \left(\sum_{|K|=k-1}{}' \sum_{ij=1}^{n-1} \varphi_{ij} u^\tau_{iK} \bar u^\tau_{jK} - \sum_{j=1}^{q_o} \varphi_{jj} |u^\tau|^2 \right) dv \\ &\quad - \sum_{|K|=k-1}{}' \|\partial\varphi \cdot u_{\cdot K}\|^2_D \\ &\geqslant \int_{\mathcal{S}_{\frac{\delta}{2}}} \cdot - \sum_{|K|=k-1}{}' \|\cdot\|^2_{\mathcal{S}_{\frac{\delta}{2}}} \\ &\gtrsim f(\delta^{-1})^2 \|u^\tau\|^2_{\mathcal{S}_{\frac{\delta}{2}}}, \end{aligned} \tag{1.13}$$

where the first inequality follows from (1.12), the second from (1.4) in addition to q-pseudoconvexity/concavity, the third from the first occurrence of (1.8) and the fourth from the second of (1.8).

(b) We will prove in (c) and (d) which follow that (1.13) implies (1.6) for tangential forms u^τ. We prove now that an estimate for u^τ entails an estimate for the full u:

Lemma 1.8. *The estimate $\|f(\Lambda_\partial)u^\tau\|^2 \lesssim Q(u^\tau, u^\tau)$ for any u^τ implies $\|f(\Lambda_\partial)u\|^2 \lesssim Q(u, u)$ for any u.*

Proof. We decompose u as $u = u^\tau + u^\nu$ where u^τ is the tangential part which collects the coefficients u_J of u with $n \notin J$ and u^ν is the normal part, that is, the complementary component. Since $u^\nu|_{\partial D} \equiv 0$, we then have by Garding inequality

$$\begin{cases} Q(u^\nu, u^\nu) \leqslant |u^\nu|^2_1 \lesssim Q(u, u) + \|u\|^2, \\ Q(u^\tau, u^\tau) \leqslant Q(u, u) + Q(u^\nu, u^\nu) \lesssim Q(u, u) + \|u\|^2. \end{cases} \tag{1.14}$$

We then have

$$\begin{aligned} \|f(\Lambda_\partial)u\|^2 &\lesssim \|f(\Lambda_\partial)u^\tau\|^2 + \|u^\nu\|^2_1 \\ &\lesssim Q(u^\tau, u^\tau) + \|u^\nu\|^2_1 \\ &\lesssim Q(u, u) + \|u\|^2, \end{aligned}$$

where the first inequality is obvious, the second follows from (1.6) for u^τ and the third from (1.14). □

(c) The rest of the proof is devoted to prove that (1.8) implies (1.6) for u^τ. To begin with, we need the following generalization of [5], Theorem 2.4.5. The generalization consists in passing from the system $\{\partial_{\bar\omega_j}\}_{j=1,\dots,n}$ to any elliptic system $\{M_j\}_{j=1,\dots,N}$ such as $\{\partial_{\omega_j}\}_{j=1,\dots,q_o} \cup \{\partial_{\bar\omega_j}\}_{j=q_o+1,\dots,n}$.

Proposition 1.9. *Let $\{M_j\}_{j=1,\dots,N}$ be a elliptic system of vector fields, that is, the symbols $\sigma(M_j)$ have no common zeroes in $\mathbb{R}^{2n} \setminus \{0\}$. We then have*

$$\sum_{i=1}^{2n} \left\| \Lambda_\partial^{-1} f(\Lambda_\partial) D_i u \right\|^2 \lesssim \sum_{j=1}^{N} \left\| \Lambda_\partial^{-1} f(\Lambda_\partial) M_j u \right\|^2 + \left\| f(\Lambda_\partial) u_b \right\|^2_{-\frac{1}{2}} \quad \text{for } u \in C_c^\infty(\bar D \cap V), \tag{1.15}$$

where D_i denote all coordinate derivatives and u_b the restriction of u to M.

Proof. (i) It is not restrictive to assume that the M_j's have constant coefficients, that is, $M_j = \sum_i a_{ij} D_i$ for $a_{ij} \equiv a_{ij}(z_o)$. In fact, if $|a_{ij}(z) - a_{ij}(z_o)| < \epsilon$ in a neighborhood of z_o, then, if u is supported by such neighborhood, each $\|\Lambda_\partial^{-1} f(\Lambda_\partial)(M_j - M_j(z_o))u\|^2$ can be absorbed in the left of (1.15).

(ii) We define

$$w := \mathcal{F}_\tau^{-1}\left(e^{(1+|\xi|^2)^{\frac{1}{2}} r} \mathcal{F}_\tau u(\xi, 0)\right),$$

and set $v := u - w$. Since $v|_{\partial D} \equiv 0$, then

$$\sum_{j=1}^{N} \|M_j v\|^2 \sim \sum_{j=1}^{N} \|M_j v\|^2 + \sum_{j=1}^{N} \|\bar M_j v\|^2 \sim \sum_{i=1}^{2n} \|D_i v\|^2.$$

Similarly, $\sum_{j=1}^N \|f(\Lambda_\partial) M_j v\|^2 \sim \sum_{i=1}^{2n} \|f(\Lambda_\partial) D_i v\|^2$. Combination of these estimates yields (1.15) for v without boundary integral. It is useful for the following to notice that it is not made any assumption compactness of the support of v.

(iii) To carry out the proof of the proposition, we need to prove that

$$\left\| \Lambda_\partial^{-1} f(\Lambda_\partial) D_i w \right\|^2 \leqslant \left\| \Lambda_\partial^{-\frac{1}{2}} f(\Lambda_\partial) w_b \right\|^2.$$

We distinguish now the case $D_i = D_{a_i}$ (tangential derivative) from the case $D_i = D_r$ (normal derivative). In the first case we have

$$\left\| \Lambda_\partial^{-1} f(\Lambda_\partial) D_{a_i} w \right\|^2 = \int_{\mathbb{R}^{2n-1}} \int_{-\infty}^{0} \left(1 + |\xi|^2\right)^{-1} |\xi_{a_i}|^2 \left(f\left(\left(1 + |\xi|^2\right)^{\frac{1}{2}}\right)\right)^2 \times \exp\left[2\left(1 + |\xi|^2\right)^{\frac{1}{2}} r\right] |\mathcal{F}_\tau u|^2(\xi, 0)\, dr\, d\xi$$

$$\leqslant \int_{\mathbb{R}^{2n-1}} (1+|\xi|^2)^{-\frac{1}{2}} \big(f\big((1+|\xi|^2)^{\frac{1}{2}}\big)\big)^2 \left(\int_{-\infty}^{0} e^{2\rho}\, d\rho\right) |\mathcal{F}_\tau u|^2(\xi,0)\, d\xi$$
$$= \frac{1}{2} \big\| f(\Lambda_\partial) u_b \big\|^2_{-\frac{1}{2}}.$$

In the second case

$$\big\| \Lambda_\partial^{-1} f(\Lambda_\partial) D_r w \big\|^2 = \int_{\mathbb{R}^{2n-1}} \int_{-\infty}^{0} (1+|\xi|^2)^{-1} \big(f\big((1+|\xi|^2)^{\frac{1}{2}}\big)\big)^2 \big|2(1+|\xi|^2)^{\frac{1}{2}}\big|^2$$
$$\times \exp\big[(2+|\xi|^2)^{\frac{1}{2}} r\big] |\mathcal{F}_\tau u|^2(\xi,0)\, dr\, d\xi$$
$$= \int_{\mathbb{R}^{2n-1}} \big(f\big((1+|\xi|^2)^{\frac{1}{2}}\big)\big)^2 (1+|\xi|^2)^{-\frac{1}{2}} \left(\int_{-\infty}^{0} e^{2\rho}\, d\rho\right) |\mathcal{F}_\tau u|^2(\xi,0)\, d\xi$$
$$= \frac{1}{2} \big\| f(\Lambda_\partial) u_b \big\|^2_{-\frac{1}{2}}.$$

This completes the proof of Proposition 1.9. □

(d) We complete the proof of (1.6) for u^τ. We begin by noticing that the term

$$\sum_{j=1}^{N} \big\| \Lambda_\partial^{-1} f(\Lambda_\partial) M_j u^\tau \big\|^2$$

of (1.15) can be estimated by $Q(u^\tau, u^\tau)$; thus what is left to prove is that

$$\big\| f(\Lambda_\partial) u_b \big\|^2_{-\frac{1}{2}} \lesssim Q\big(u^\tau, u^\tau\big) + \big\| u^\tau \big\|^2.$$

The first part of the discussion holds for general u, not necessarily tangential. We recall the microlocalization procedure of Catlin. Let $\{p_k\}_k$ be a sequence of C^∞ functions in $\mathbb{R}^+$ such that

$$\sum_k p_k^2 = 1, \qquad \operatorname{supp}(p_k) \subset \big(2^{k-1}, 2^{k+1}\big), \qquad \operatorname{supp}(p_0) \subset (0,2), \qquad \big|p_k'\big| \leqslant 2^{-k}.$$

By the aid of the p_k's we introduce, following Catlin, the pseudodifferential operators

$$P_k^\tau u = \mathcal{F}_\tau^{-1}\big(p_k(\Lambda_\xi) \mathcal{F}_\tau u\big).$$

We can then show that

$$\big\| \Lambda_\partial^s f(\Lambda_\partial) u \big\|_0^2 \sim \sum_k 2^{2ks} f\big(2^k\big)^2 \big\| P_k^\tau u \big\|_0^2.$$

This is a standard result granted that

$$2^{ks} f\big(2^k\big) \leqslant \big(1+|\xi|^2\big)^{\frac{1}{2}} f\big(|\xi|\big) \leqslant 2^{(k+1)s} f\big(2^{k+1}\big) \quad \text{for } |\xi| \in \big(2^{k-1}, 2^{k+1}\big).$$

We apply this result for $s = \frac{1}{2}$. We follow now step by step the procedure of the proof of Theorem 2.2 of [2]. We use the elementary inequality

$$\big|g(0)\big|^2 \leqslant \frac{2^k}{\eta} \int_{-2^{-k}}^{0} \big|g(r)\big|^2\,dr + 2^{-k}\eta \int_{-2^{-k}}^{0} \big|g'(r)\big|^2\,dr,$$

which holds for any g such that $g(-2^{-k}) = 0$. If we apply it for $g(r) = \chi_k(r) P_k u(\cdot, r)$, where $\chi_k \in C_c^\infty(-2^{-k}, 0]$ with $0 \leqslant \chi_k \leqslant 1$ and $\chi_k(0) = 1$, we get

$$\begin{aligned}
\big\| f(\Lambda_\partial) u_b \big\|^2_{-\frac{1}{2}} &\sim \sum_{k=0}^{\infty} f\big(2^k\big)^2 2^{-k} \big\| \chi_k(0) P_k u_b \big\|^2 \\
&\leqslant \eta^{-1} \sum_{k=0}^{\infty} f\big(2^k\big)^2 \int_{-2^{-k}}^{0} \big\| \chi_k P_k u(.,r) \big\|^2\,dr \\
&\quad + \eta \sum_{k=0}^{\infty} f\big(2^k\big)^2 2^{-2k} \int_{-2^{-k}}^{0} \big\| D_r\big(\chi_k P_k u(.,r)\big) \big\|^2\,dr.
\end{aligned}$$

We specify now $u = u^\tau$ and denote by (I) and (II) the two sums in the second line of the above estimate. Now,

$$\begin{aligned}
\text{(I)} &\leqslant \eta^{-1} \sum_{k=o}^{\infty} Q\big(P_k u^\tau, P_k u^\tau\big) \\
&\leqslant \eta^{-1}\big(Q\big(u^\tau, u^\tau\big) + \big\|u^\tau\big\|^2 + \big|\!\big|\!\big| D_r u^\tau \big|\!\big|\!\big|^2_{-1}\big),
\end{aligned} \tag{1.16}$$

where the first inequality follows from (1.13) and the second from the estimates of the commutators $[\bar\partial, P_k]$ and $[\bar\partial^*, P_k]$. We also have the estimate

$$\text{(II)} \lesssim \eta Q\big(u^\tau, u^\tau\big) + \eta \big\| f(\Lambda_\partial) u^\tau \big\|^2. \tag{1.17}$$

By combining (1.16) and (1.17) and by absorbing $\eta \| f(\Lambda_\partial) u \|^2$ in the left-hand side of the estimate we get

$$\big\| f(\Lambda_\partial) u_b \big\|^2_{-\frac{1}{2}} \lesssim Q\big(u^\tau, u^\tau\big) + \big\|u^\tau\big\|^2.$$

The proof of Theorem 1.4 is complete. □

2. A geometric criterion for $(f\text{-}P\text{-}q)$ property

When the Levi form of the boundary is nondegenerate one has the strongest estimates for the $\bar\partial$-Neumann problem, that is, $\frac{1}{2}$-subelliptic ones. When the Levi form decreases with a certain rate in correspondence to a submanifold $S \subset \partial D$, with $\dim_{CR} S \leqslant q-1$, then we can prove $(f\text{-}P\text{-}q)$ for f related to the inverse of the Levi vanishing rate. Let ∂D be q-pseudoconvex (resp. q-pseudoconcave) in a neighborhood of z_o and let $S \subset \partial D$, be a submanifold containing z_o and with the properties

$$\begin{cases} \text{(i)} \quad T^{\mathbb{C}}S \supset \mathcal{V}^{q_o}|_S \quad \big(\text{resp. } T^{\mathbb{C}}S \subset \mathcal{V}^{q_o}|_S\big), \\ \text{(ii)} \quad \dim\big(T_z^{\mathbb{C}}S\big) \leqslant q-1 \quad \big(\text{resp. } \dim\big(T_z^{\mathbb{C}}S\big) \geqslant q-1\big) \text{ for any } z \text{ close to } z_o. \end{cases} \tag{2.1}$$

We denote by d_S the distance-function to S, consider a real function $F = F(\delta)$, $\delta \in R^+$ such that $\frac{F(\delta)}{\delta^2} \nearrow +\infty$ as $\delta \searrow 0$, denote by F^* the inverse to F and define $f(t) := (F^*(t^{-1}))^{-1}$. With these notations we have

Theorem 2.1. *Let ∂D be q-pseudoconvex (resp. q-pseudoconcave) and let $S \subset \partial D$ be a submanifold satisfying* (2.1). *Suppose that*

$$\sum_{j=1}^{q} \lambda_j^{\partial D} - \sum_{j=1}^{q_o} r_{jj} \gtrsim \frac{F(d_S)}{d_S^2}. \tag{2.2}$$

Then $(f\text{-}P\text{-}q)$ property holds for $q > q_o$ (resp. $q < q_o$) where $f(t) := (F^*(t^{-1}))^{-1}$.

Proof. We first consider the case q-pseudoconvex. We take $\chi = \chi(t)$ in C^∞ with $\chi \equiv 1$ for $0 \leqslant t \leqslant 1$ and $\chi \equiv 0$ for $t \geqslant 2$ and define our family of weights $\varphi = \varphi^\delta$ by

$$\varphi^\delta = -\log\left(\frac{-r}{\delta} + 1\right) + c\chi\left(\frac{d_S^2}{2f^{-2}(\delta^{-1})}\right)\log\left(\frac{d_S^2}{2f^{-2}(\delta^{-1})} + 1\right), \tag{2.3}$$

where c is a small constant to be specified later. Note that the φ's are absolutely bounded on $\bar{\mathcal{S}}_\delta \cap V$. We observe that

$$\partial\bar\partial d_S^2 = 2\partial d_S \otimes \bar\partial d_S + 2d_S\partial\bar\partial d_S. \tag{2.4}$$

When we compose with log we get

$$\begin{aligned} \partial\bar\partial \log\big(d_S^2 + 2f^{-2}\big) &= \frac{2\partial d_S \otimes \bar\partial d_S + 2d_S\partial\bar\partial d_S}{(d_S^2 + 2f^{-2})} - 4\frac{d_S^2\partial d_S \otimes \bar\partial d_S}{(d_S^2 + 2f^{-2})^2} \\ &= \frac{\partial d_S \otimes \bar\partial d_S(2d_S^2 + 4f^{-2} - 4d_S^2) + 2d_S\partial\bar\partial d_S(d_S^2 + 2f^{-2})}{(d_S^2 + 2f^{-2})^2}. \end{aligned} \tag{2.5}$$

Now, if $d_S^2 < f^{-2}$, then (2.5) can be continued by

$$\geqslant \frac{2\partial d_S \otimes \bar\partial d_S f^{-2} + 2d_S\partial\bar\partial d_S(d_S^2+2f^{-2})}{(d_S^2+2f^{-2})^2} \geqslant \frac{\frac{2}{3}\partial d_S \otimes \bar\partial d_S + 2d_S\partial\bar\partial d_S}{(d_S^2+2f^{-2})}$$
$$\gtrsim \frac{\frac{2}{3}\partial d_S \otimes \bar\partial d_S + 2d_S\partial\bar\partial d_S}{3f^{-2}}. \tag{2.6}$$

We denote by $(d_S)_j$ and $(d_S)_{ij}$ the components of ∂d_S and $\partial\bar\partial d_S$ respectively in the basis of forms $\{\omega_j\}$. We notice that by (2.4) implies for forms u of degree $k \geqslant q$,

$$\sum_{|K|=k-1}' \left|\sum_j (d_S)_j u_{jK}\right|^2 \gtrsim |u|^2. \tag{2.7}$$

We also notice that

$$d_S \sum_{|K|=k-1}' \sum_{ij} (d_S)_{ij} u_{iK}\bar u_{jK} \geqslant -\epsilon|u|^2. \tag{2.8}$$

We introduce the notation $(B_{ij}) := \partial\bar\partial(\log(\frac{d_S^2}{2f^{-2}}+1))$. In conclusion, if $d_S^2 < f^{-2}$, combination of (2.5), (2.6), (2.7) and (2.8) yields

$$\sum_{ij=1}^{n} B_{ij}u_{iK}\bar u_{jK} - \sum_{j=1}^{q_o} B_{jj}|u_J|^2 \gtrsim f^{-2}|u|^2. \tag{2.9}$$

Because of (1.3) for $\partial\bar\partial r$ and the similar property for $(A_{ij}) := \partial\bar\partial(-\log(\frac{-r}{\delta}+1))$, we have that (2.9) is true not only for (B_{ij}) but also for $\partial\bar\partial\varphi^\delta$.

We suppose now $d_S^2 \geqslant f^{-2}$; then

$$\frac{F(d_S)}{d_S^2} \geqslant \frac{F(f^{-1}(\delta^{-1}))}{f^{-2}(\delta^{-1})} = \frac{\delta}{f^{-2}(\delta^{-1})}. \tag{2.10}$$

It follows that

$$\sum_{ij=1}^{n} A_{ij}u_{iK}\bar u_{jK} - \sum_{j=1}^{q_o} A_{jj}|u_J|^2 \gtrsim f^2(\delta^{-1})|u|^2. \tag{2.11}$$

Now, because of the cut-off χ, the contribution of (B_{ij}) can get negative when $d_S^2 \geqslant f^{-2}$ and therefore $\dot\chi \neq 0$ or $\ddot\chi \neq 0$. However, $(B_{ij}) \geqslant -cf^2(\delta^{-1})$ and hence (A_{ij}) controls this negative term by a suitable choice of c; thus (2.11) implies the similar estimate for $\partial\bar\partial\varphi^\delta$.

The family of weights $\{\varphi^\delta\}$ satisfies (1.5) on $\bar{\mathcal{S}}_\delta \cap V$ without the term $\sum_{j=1}^{q_o} |\varphi_j(z)|^2$ in the right-hand side. As for this term, in the q-pseudoconvex case, it vanishes and so there is nothing else to prove. Instead, in the q-pseudoconcave case, it is not 0; also, the definition of φ needs to be modified by multiplying the second term in the right of (2.3) by -1. The proof goes through with a slight modification such as in [10] formulas (5.9)–(5.13). □

3. Domains which have subelliptic, superlogarithmic and compactness estimates

We introduce fairly general classes of domains D for which we are able to prove the hypotheses of Theorem 2.1; this implies $(f\text{-}P\text{-}q)$ property according to Section 2 and then $(f\text{-}q)$ estimates by Section 1. First, we treat the case q-decoupled-pseudoconvex domains; these are defined near $z_o = 0$ by $r < 0$ for r in the form

$$r = 2\mathrm{Re}\, z_n - h(z_1, \dots, z_{q_o}) + \sum_{j=q}^{n-1} h_j(z_j) \quad \text{for } q \geqslant q_o + 1, \tag{3.1}$$

where $\partial\bar\partial h \geqslant 0$ and the h_j's are subharmonic, non-harmonic, functions vanishing at $z_j = 0$. Decoupled domains are treated, among others, by Mc Neal [17]. Similarly, we consider q-pseudoconcave domains whose defining function r is of the type

$$r = 2\mathrm{Re}\, z_n + h(z_{q_o+1}, \dots, z_{n-1}) - \sum_{j=1}^{q+1} h_j(z_j) \quad \text{for } q \leqslant q_o - 1 \tag{3.2}$$

with $\partial\bar\partial h \geqslant 0$ and the h_j's subharmonic, non-harmonic and vanishing at $z_j = 0$. It is obvious that a domain D endowed with such a defining function r is q-pseudoconvex or q-pseudoconcave in the two respective cases of (3.1) and (3.2). When the h_j's have finite vanishing order $2m_j$, these domains are treated in [10]. This leads to subelliptic estimates, that is (1.6) for f satisfying (i) for $\epsilon < \frac{1}{2\max_j m_j}$. We recall briefly the argument of the proof. We choose the weights

$$\varphi := -\log(-r + \delta) + \sum_j \log\bigl(|z_j|^2 + \delta^{\frac{1}{m_j}}\bigr),$$

and normalize them by a factor $c|\log\delta|^{-1}$. Thus they are absolutely bounded and their Levi form φ_{ij} satisfies (1.5) for $f(\frac{1}{\delta})^2 = \delta^{-2\epsilon}$ for any $\epsilon < \frac{1}{2\max_j m_j}$. Thus the conclusion is a consequence of Theorem 1.1.

We introduce two new cases. Before, we notice that it is not restrictive to assume

$$\bigl|\log\bigl(\partial^2_{z_j\bar z_j} h_j\bigr)\bigr| \nearrow +\infty \quad \text{as } |z_j| \searrow 0. \tag{3.3}$$

Otherwise, we would have $\partial^2_{\bar z_j \bar z_j} h_j(z_o) \neq 0$ for some j and therefore $-\log(-r+\delta)$ would have a Levi form that, applied to u_{jK} would be $\gtrsim \delta^{-1}|u_{jK}|^2$. But this is the $\frac{1}{2}$-subelliptic estimate which is the best we can expect in a neighborhood of the boundary. Thus we assume (3.3) in what follows.

Proposition 3.1. *Let D be a q-pseudoconvex* (*resp. q-pseudoconcave*) *domain defined by* (3.1) (*resp.* (3.2)) *and suppose that*

$$|z_j|^\alpha \big|\log\big(\partial^2_{z_j\bar z_j} h_j(z_j)\big)\big| \searrow 0 \quad \text{as } |z_j| \searrow 0. \tag{3.4}$$

(a) *If $0 < \alpha \leqslant 1$ then we have* (1.6) *with f satisfying* (ii)*, that is, superlogarithmic estimates for the $\bar\partial$-Neumann problem over forms of degree $k \geqslant q$ (resp. $k \leqslant q$).*
(b) *If $\alpha > 1$ then we have* (1.6) *with f satisfying* (iii)*, that is, compactness of the $\bar\partial$-Neumann problem for forms of degree $k \geqslant q$ (resp. $k \leqslant q$).*

Before the proof, an example is in order.

Example 3.2. For the case (a), we can choose $h_j = e^{-\frac{1}{|z_j|^\alpha}}$ for $\alpha < 1$ or else $h_j = e^{-\frac{1}{|z_j||\log|z_j||}}$. For (b) we take $h_j = e^{-\frac{1}{|z_j|^\alpha}}$ for $\alpha \geqslant 1$. Now, there is no doubt that the two above choices of h_j fulfill (3.3) for α satisfying (a) and (b) respectively. We then consider the domains defined by

$$2\mathrm{Re}\, z_n - h + \sum_j h_j < 0,$$

or

$$2\mathrm{Re}\, z_n + h - \sum_j h_j < 0,$$

A few words are maybe needed to show that the above domains are q-pseudoconvex and q-pseudoconcave respectively. In fact, if we write the exponentials as $e^{-\frac{1}{g}}$ it suffices to prove that these are subharmonic. Here $g = |z_j||\log|z_j||$ or $g = |z_j|^\alpha$; thus g is subharmonic for $z_j \neq 0$. But then $e^{-\frac{1}{g}}$ itself is subharmonic, including at $z_j = 0$ because of the identity

$$\partial\bar\partial\big(e^{-\frac{1}{g}}\big) = e^{-\frac{1}{g}}\left(\frac{\partial\bar\partial g}{g^2} - 2\frac{|\partial g|^2}{g^3} + \frac{|\partial g|^2}{g^4}\right).$$

Thus, the above domains have superlogarithmic or compactness estimates according to the cases (a) and (b) (and in degree $k \geqslant q$ and $k \leqslant q$ in the case q-pseudoconvex and q-pseudoconcave respectively).

Proof of Proposition 3.1. We define $2a_j^{-1}(|z_j|) = |z_j|^\alpha |\log(\partial^2_{z_j\bar z_j}(h_j(z_j))|z_j|^2)|$; we note that we have $a_j^{-1} \searrow 0$ as $|z_j| \searrow 0$. Referring to the terminology of Theorem 2.1, we denote by S_j the origin in the z_j-plane; thus $d_{S_j} = |z_j|$. We have

$$\partial^2_{z_j\bar z_j} h_j = \frac{e^{-\frac{2}{|z_j|^\alpha a_j(z_j)}}}{|z_j|^2} = \frac{F_j(|z_j|)}{|z_j|^2},$$

for $F_j = e^{-\frac{1}{\delta^\alpha a_j(\delta)}}$. Setting $f_j^{-1}(\delta) = c(|\log\delta| a_j(|\log\delta|^{-\frac{1}{\alpha}}))^{-\frac{1}{\alpha}}$ and choosing a cut-off χ with $\chi \equiv 1$ for $0 \leqslant t \leqslant 1$ and $\chi \equiv 0$ for $t \geqslant 2$, we define

$$\varphi^\delta = -\log\left(\frac{-r}{\delta} + 1\right) + \sum_{j=q}^{n-1} \chi\left(\frac{|z_j|^2}{2f_j^{-2}(\delta)}\right) \log\left(\frac{|z_j|^2}{2f_j^{-2}(\delta)} + 1\right). \tag{3.5}$$

We also set $f := \min_j f_j$. When D is q-pseudoconvex, the family of weights φ satisfies (1.5) for the above defined f on $\bar{S}_\delta \cap V$ without the term $\sum_{j=1}^{q_o} |\varphi_j(z)|^2$ in the right-hand side. However, this term vanishes and so there is nothing else to prove. The variant for the q-pseudoconcave case follows the lines of the similar variant in Theorem 2.1 (in particular by inserting a crucial factor -1 in the second log of (3.5)). $\square$

4. The tangential system

We consider a hypersurface $M \subset \mathbb{C}^n$ and denote by $D^\pm$ the two sides of M. We suppose all through this section that D^+ is pseudoconvex. We parametrize M over $\mathbb{R}^{2n-1}$ with variable a by a diffeomorphism Φ, so that $\partial_{a_{2n-1}}$ corresponds to the totally real vector field tangential to M that we also denote by T. We denote by $\bar\partial_b = \Phi_*^{-1}\bar\partial$ the induced complex, by $\bar\partial_b^*$ the adjoint to $\bar\partial_b$, set $\Box_b := \bar\partial_b^*\bar\partial_b + \bar\partial_b\bar\partial_b^*$ and put $Q_b(u_b, u_b) = \|\bar\partial_b u_b\|^2 + \|\bar\partial_b^* u_b\|^2$ for any form u_b on M. We denote by ξ the coordinates dual to the a's. We consider a conic partition of the unity $1 \equiv \psi^+ + \psi^- + \psi^0$ on $\mathbb{R}^{2n-1}$ for $|\xi| \geqslant 1$ such that $\psi^+ \equiv 1$ for $\xi_{a_{2n-1}} > \epsilon|\xi|$, $\psi^- \equiv 1$ for $\xi_{a_{2n-1}} < -\epsilon|\xi|$ and $\psi^0 \equiv 1$ in a conic neighborhood of the plane $\xi_{a_{2n-1}} = 0$. We introduce briefly the conclusions of the microlocalization method by Kohn. Let $u_b{}^+ = \psi^+(\Lambda_\partial)u_b$ where $\psi^+(\Lambda_\partial)$ is the pseudodifferential operator with symbol $\psi^+(\Lambda_\xi)$ and similarly define u_b^- and u_b^0. This yields a microlocal decomposition $u_b = u_b^+ + u_b^- + u_b^0$. We then say that u_b is C^∞ in direction $+da_{2n-1}$ (resp. $-da_{2n-1}$) when u_b^+ (resp. u_b^-) is C^∞. For $u \in Dom_{\bar\partial^*}(D^\pm)$, we set $Q_{D^\pm}(u,u) := \|\bar\partial u\|_{D^\pm}^2 + \|\bar\partial^* u\|_{D^\pm}^2$. The operation of restriction of forms of $Dom_{\bar\partial^*}$ from $D^\pm$ to M and that of (harmonic) extension from M to $D^\pm$ yields

Theorem 4.1. *(See Kohn [16].) We have for forms of degree k with* $1 \leqslant k \leqslant n-2$,

$$\left\| f(\Lambda_\partial)u \right\|_{D^\pm}^2 \lesssim Q_{D^\pm}(u,u) + \|u\|_{D^\pm}^2 \quad \text{for any } u \in Dom_{\bar\partial^*}\left(D^\pm\right),$$

if and only if

$$\left\| f(\Lambda_\partial)\zeta u_b^\pm \right\|_b^2 \lesssim Q_b\left(u_b^\pm, u_b^\pm\right) + \left\| u_b^\pm \right\|_{D^\pm}^2 \quad \text{for any } u_b^\pm,$$

where ζ *denotes a cut-off function.*

The result of Kohn is stated only for $f(\Lambda_\partial) = \Lambda_\tau^\epsilon$; but the extension to general f is straightforward. We observe now that $\|Tu_b^0\|_b^2 \lesssim \sum_{j=1}^{n-1}(\|\partial_{\omega_j} u_b^0\|_b^2 + \|\partial_{\bar\omega_j} u_b^0\|_b^2)$ because the characteristic variety of T is transversal to supp(ψ^0). We also have

$$\sum_{j=1}^{n-1} \left\| \partial_{\omega_j} u_b^0 \right\|_b^2 \lesssim \sum_{j=1}^{n-1} \left\| \partial_{\bar\omega_j} u_b^0 \right\|_b^2 + \epsilon \left\| T u_b^0 \right\|_b^2 + C_\epsilon \left\| u_b^0 \right\|_b^2,$$

which is readily proved by integration by parts. It follows

$$\begin{aligned}\|u_b^0\|^2_{H^1(M)} &\lesssim \sum_{j=1}^{n-1}\|\partial_{\bar\omega_j}u_b^0\|_b^2+\|u_b^0\|_b^2\\ &\lesssim Q_b\big(u_b^0,u_b^0\big)+\|u_b^0\|_b^2.\end{aligned}\tag{4.1}$$

By combining (4.1) with Theorem 4.1 we get

Corollary 4.2. *We have for forms of degree k with* $1\leqslant k\leqslant n-2$,

$$\|f(\Lambda_\partial)u\|_D^2\lesssim Q_D(u,u)+\|u\|_D^2\quad \text{for both } D=D^+ \text{ and } D=D^-\tag{4.2}$$

if and only if

$$\|f(\Lambda_\partial)u_b\|_b^2\lesssim Q_b(u_b,u_b)+\|u_b\|^2.\tag{4.3}$$

Corollary 4.2 provides a tool for transferring estimates from D^+ and D^- to ∂D and conversely; in this way, when $f(t)\geqslant k\log t$ for any k, for a suitable c_k and for any $t\geqslant c_k$, then the related hypoellipticity, is also transferred. In particular, let M be graphed as $x_n=g$ where

$$g=\sum_{j=1}^{n-1}e^{-\frac{1}{|z_j|^\alpha}}\quad\text{or}\quad g=\sum_{j=1}^{n-1}e^{-\frac{1}{|x_j|^\alpha}};\tag{4.4}$$

note that D^+ is pseudoconvex (and D^- is $(n-1)$-pseudoconcave). If $\alpha<1$ or, for $\alpha=1$, if we replace $|z_j|$ by $a_j(z_j)|z_j|$ for $a_j\nearrow+\infty$ when $z_j\searrow 0$, then we have superlogarithmic estimates in $D^\pm$; in particular $\Box$ is hypoelliptic. Using Corollary 4.2 we have, for forms in degree k with $1\leqslant k\leqslant n-2$,

Proposition 4.3.

(i) *If* $\alpha<1$, *then* $\Box_b$ *has superlogarithmic estimates*; *in particular, it is hypoelliptic.*
(ii) *Is* $\alpha=1$ *and we replace* $|z_j|$ *by* $a_j(z_j)|z_j|$ (*or* $|x_j|$ *by* $a_j(x_j)|x_j|$) *for* $a_j\nearrow+\infty$, *then the same conclusion as in* (i) *holds.*

If $\bar\partial_b$ has closed range over functions (resp. $\bar\partial_b^*$ has closed range over $(n-1)$-forms), Kohn has a result also in the critical degree $k=0$ (resp. $k=n-1$) [16], Theorem 1.6. There are superlogarithmic estimates which imply that $\bar\partial_b$ (resp. $\bar\partial_b^*$) is hypoelliptic on the orthogonal complement of $\operatorname{Ker}\bar\partial_b$ over functions (resp. $\operatorname{Ker}\bar\partial_b^*$ over $(n-1)$-forms).

We still keep the structure (4.4) for the equation of the domain but restrict to dimension $n=2$. We also point our attention to the action of $\bar\partial_b$ over functions and disregard $\bar\partial_b^*$ over $(n-1)$-forms. Now, when $\alpha\geqslant 1$ the domain defined by the first occurrence of (4.4), in which g depends on $|z_1|$, stays hypoelliptic. Instead, in the "tube domain", in which g depends on the only $|x_1|$, is not. The first follows from Kohn [15] combined with the argument of Kohn [14] Theorem 2.6, whereas the second is proved by Christ in [3]. Here is the geometric explanation. In the first case, the set of the points where the system of complex tangential vector fields fails to have finite type is confined to the real curve $\{0\} \times \mathbb{R}_{y_2}$ transversal to $T^{\mathbb{C}}M$. Instead, for the tube, the points of non-finite type are the two-dimensional plane $\mathbb{R}^2_y$.

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