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Necessary geometric and analytic conditions for general estimates in the $\bar\partial$-Neumann problem

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Abstract We show the geometric and analytic consequences of a general estimate in the $\bar\partial$-Neumann problem: a "gain" in the estimate yields a bound in the "type" of the boundary, that is, in its order of contact with an analytic curve as well as in the rate of the Bergman metric. We also discuss the potential-theoretical consequence: a gain implies a lower bound for the Levi form of a bounded weight.

1 Introduction

In a smooth pseudoconvex domain $\Omega \subset \mathbb{C}^n$ whose boundary $b\Omega$ has finite type M (in the sense that the order of contact of any complex analytic variety is at most M, cf. [4, 5]) the $\bar\partial$-Neumann problem shows an ϵ-subelliptic estimate for some ϵ (Folland–Kohn [7] for $M = 2$, Kohn [12] for general M and real analytic boundary, Catlin [3] for smooth boundary) and conversely, an ϵ-estimate implies $M \le \frac{1}{\epsilon}$ (Catlin [1]). Thus, index of estimate and order of contact are related as inverse one to another. Contact of infinite order has also been studied: α-exponential contact implies an $\frac{1}{\alpha}$-logarithmic estimate (cf. e.g. [11]). What is proved here serves to explain the inverse: an $\frac{1}{\alpha}$-logarithmic estimate, for $\alpha < 1$, implies exponential contact $\le \alpha$ (apart from an error α^2). More generally, the gain in the estimate, which is quantified by a function $f(t)$, $t \to \infty$, such as t^ϵ or $(\log t)^{\frac{1}{\alpha}}$, is here related to the "type" of $b\Omega$ described by a function $F(\delta)$ (for $\delta = t$), such as δ^M or $\exp(-\frac{1}{\delta^\alpha})$: the general result is that F is estimated from below by the inverse to f. In similar

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way, the rate of the Bergman metric B_Ω at $b\Omega$ as well as the rate of the Levi form of a bounded weight are estimated. The latter is related to the celebrated "P-property" by Catlin [2].

We fix our formalism. Ω is a bounded pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary $b\Omega$ defined, in a neighborhood of a point $z_o = 0$, by $r = 0$ with $\partial r \neq 0$ and with $r < 0$ inside Ω. We introduce the notion of "type" of $b\Omega$ along a q-dimensional complex analytic variety $Z \subset \mathbb{C}^n$ as a quantitative description of the contact.

Definition 1.1 For a smooth increasing function F vanishing at 0, we say that the type of $b\Omega$ along Z is $\leq F$ when

$$|r(z)| \lesssim F(|z - z_o|), \quad z \in Z, \ z \to z_o. \tag{1.1}$$

Here and in what follows, $\lesssim$ or $\gtrsim$ denote inequality up to a positive constant. We choose local real coordinates $(a, r) \in \mathbb{R}^{2n-1} \times \mathbb{R} \simeq \mathbb{C}^n$ at z_o and denote by ξ the dual variables to the a's. We denote by $\Lambda_\xi := (1 + |\xi|^2)^{\frac{1}{2}}$ the standard elliptic symbol of order 1 and by $f(\Lambda_\xi)$ a general pseudodifferential symbol obtained by the aid of a smooth increasing function f. We associate to this symbol a pseudodifferential action defined by $f(\Lambda)u = \mathcal{F}^{-1}(f(\Lambda_\xi)\mathcal{F}u)$ for $u \in C^\infty_c$, where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^{2n-1}$. In our discussion, $f(\Lambda)$ ranges in the interval $\log(\Lambda) \ll f(\Lambda) \leq \Lambda^\epsilon$ (any $\epsilon \leq \frac{1}{2}$) where the symbol "$\ll$" means that $\frac{f}{\log} \to \infty$ at ∞ in a monotonic way. Moreover, we notice that $(\frac{f}{\log})^*(t) \geq t$ since $f(t) \leq t^{\frac{1}{2}}$, where the superscript * denotes the inverse function. By means of Λ^ϵ we can also define the tangential Sobolev ϵ-norm as $|||u|||_\epsilon := \|\Lambda^\epsilon u\|$. We set $\omega_n = \partial r$ and complete to an orthonormal basis of $(1, 0)$-forms $\omega_1, \dots, \omega_n$; we denote by $L_1, \dots, L_n$ the dual basis of vector fields. A q-form u is a combination of differentials $\bar\omega_J := \bar\omega_{j_1} \wedge \cdots \wedge \bar\omega_{j_q}$ over ordered indices $J = j_1 < j_2 < \cdots < j_q$ with smooth coefficients u_J, that is, an expression $\sum'_{|J|=q} u_J \bar\omega_J$. We decompose a form as $u = u^\tau + u^\nu$ where u^τ is obtained by collecting all coefficients u_J such that $n \notin J$ and u^ν is the complementary part; we have that $u \in D_{\bar\partial^*}$, the domain of $\bar\partial^*$, if and only if $u^\nu|_{b\Omega} \equiv 0$.

Definition 1.2 An f-estimate in degree q is said to hold for the $\bar\partial$-Neumann problem in a neighborhood U of z_o when

$$\|f(\Lambda)u\| \lesssim \|\bar\partial u\| + \|\bar\partial^* u\| + \|u\| \quad \text{for any } u \in C^\infty_c(\bar\Omega \cap U)^q \cap D_{\bar\partial^*}, \tag{1.2}$$

where the superscript ^{q} denotes forms of degree q. Since $u^\nu|_{b\Omega} \equiv 0$, then u^ν enjoys an elliptic estimate (for $f(\Lambda) = \Lambda$) on account of Garding Theorem; thus (1.2) for u^τ implies (1.2) for the full u. We will use the notation $Q(u, u)$ for the sum of the three terms in the right side of (1.2).

It has been proved by Catlin [1] that an ϵ-subelliptic estimate of index q implies that $b\Omega$ has finite type $M \leq \frac{1}{\epsilon}$ along any q-dimensional complex variety Z, that is, (1.2) holds for $F = |z - z_o|^M$ when $z \in Z$. Notice that $F = \delta^M$ is inverse to the reciprocal of $f = t^\epsilon$, $t = \delta^{-1}$. In full generality of f, with the only restraint $f \gg \log$, we define

$$G(\delta) := \left(\left(\frac{f}{\log}\right)^*\right)^{-1}(\delta^{-1}). \tag{1.3}$$

Up to a logarithmic loss, we get the generalization of Catlin's result, that is, we prove that $F \gtrsim G$.

Another goal of this work consists in describing the effect of an f-estimate on the growth at the boundary of the Bergman metric. The Bergman kernel $K_\Omega : \Omega \times \Omega \to \mathbb{C}$ provides the integral representation of the orthogonal projection $P : L^2(\Omega) \to \text{hol}(\Omega) \cap L^2(\Omega)$, $f \mapsto P(f) := \int_\Omega f(\zeta)K(z,\zeta)\,dV_\zeta$ where dV_ζ is the element of volume in the ζ-space. On a bounded smooth pseudoconvex domain, the projection P is related to the $\bar\partial$-Neumann operator N, the inverse of $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, by Kohn's formula $P = \text{Id} - \bar\partial^* N \bar\partial$. The Bergman metric is defined by $B_\Omega = \sqrt{\partial\bar\partial(\log K_\Omega(z,z))}$. It has been proved by McNeal in [16] that an ϵ-subelliptic estimate for $q = 1$ implies $B_\Omega(z, X) \gtrsim \delta^{\epsilon-\eta}(z)|X|$, $X \in T^{1,0}\mathbb{C}^n|_\Omega$, for any fixed $\eta > 0$ where $\delta(z)$ denotes the distance of z to $b\Omega$. We extend this conclusion to a general f-estimate and get a bound from below with $\delta^{\epsilon-\eta}(z)$ replaced by $G(\delta^{-1+\eta}(z))$. This behavior has relevant potential theoretical consequences. Historically, the equivalence of a subelliptic estimate with a finite type has been achieved by triangulating through a quantitative version of Catlin's "P-Property". This consists in the existence of a family of uniformly bounded weights $\{\varphi^\delta\}$ on the δ-strips $S_\delta := \{z \in \Omega : \delta(z) < \delta\}$, whose Levi-form have a lower bound $\delta^{-\epsilon}$ for some ϵ. We extend this notion for general f.

Definition 1.3 We say that Ω satisfies Property (f-P) over a neighborhood U of z_o, if there exists a family of weights $\varphi = \varphi^\delta$ which are absolutely bounded in $S_\delta \cap U$ and satisfy

$$i\partial\bar\partial\varphi^\delta \gtrsim f^2(\delta^{-1})\,\text{Id} \quad \text{for any } z \in S_\delta \cap U. \tag{1.4}$$

As it has already been recalled from [1], f-estimate ($f = t^\epsilon$) implies F-type ($F = \delta^M$). In turn, this implies ($\tilde f$-P)-Property ($\tilde f = t^{\tilde\epsilon}$ for $\tilde\epsilon$ (much) smaller than $\frac{1}{M}$ [3]), and this yields $\tilde f$-estimate [3] (cf. also [8–10]). So the cycle is closed but in going around, ϵ has decreased to $\tilde\epsilon$. In this process, the critical point is the rough relation between the type M and the exponent $\tilde\epsilon$ and this cannot be improved significantly: one must expect that $\tilde\epsilon$ is much smaller than $\frac{1}{M}$. The reason is that the type only describes the order of contact of a complex variety Z tangent to $b\Omega$, whereas what really matters is how big is the diameter of a Z_δ that can be inserted inside Ω at δ-distance from $b\Omega$. This can be bigger than δ^M as in the celebrated example by D'Angelo of the domain defined by $r = \mathrm{Re}\, z_3 + |z_1^2 - z_2^l z_2|^2 + |z_2^2|^2 + |z_3^m z_1|^2$ (cf. [1], p. 149). However, an estimate has effect over the families $Z_\delta \subset \Omega$ and not only over Z tangential to $b\Omega$. So the achievement of a direct proof of the implication from estimate to generalized P-property, which was envisaged by Straube, not only offers a shortcut in Catlin's theory, but also gains a good accuracy about indices. For a general $f \gg \log$ and for any η we define $\tilde{f} = \tilde{f}_\eta$ by

$$\tilde{f}(t) = \frac{f}{\log^{\frac{3}{2}+\eta}}(t^{1-\eta}); \tag{1.5}$$

then we prove the direct implication from f-estimate to $(\tilde{f}\text{-}P)$-Property. In particular, from an ϵ-subelliptic estimate, the $\tilde{\epsilon}$ we get is any index slightly smaller than ϵ. We collect the discussion in a single statement which is the main result of this paper.

Theorem 1.4 *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with smooth boundary in which the $\bar\partial$-Neumann problem has an f-estimate in degree q at $z_o \in b\Omega$ for $f \gg \log$. Let G, resp. $\tilde{f} = \tilde{f}_\eta$ for any $\eta > 0$, be the function associated to f by* (1.3), *resp.* (1.5), *and let $\delta(z)$ denote the distance from z to $b\Omega$. Then*

(i) *If $b\Omega$ has type $\le F$ along a q-dimensional complex analytic variety Z, then $F \gtrsim G$,*
(ii) *If $q = 1$, the Bergman metric satisfies $B_\Omega(z) \gtrsim \frac{f}{\log}(\delta^{-1+\eta}(z))\,\mathrm{Id}$, $z \in U$, for any η and for suitable $U = U_\eta$,*
(iii) *If $q = 1$, Property $(\tilde{f}\text{-}P)$ holds for any η and for suitable $U = U_\eta$.*

We say a few words about the technique of the proof. The main tool is an accurate localization estimate. By localization estimate, we mean an estimate which involves a fundamental system of cut-off functions χ_0, χ_1, χ_2 in a neighborhood of z_o with $\chi_0 \prec \chi_1 \prec \chi_2$ (in the sense that $\chi_{j+1}|_{\mathrm{supp}\,\chi_j} \equiv 1$) of the kind

$$\|\chi_0 u\|_s \lesssim \|\chi_1 \Box u\|_s + c_s \|\chi_2 u\|_0 \quad \text{for any } u \in (C^\infty)^q \cap D_\Box. \tag{1.6}$$

If (1.6) holds for a fundamental system of cut-off functions as above, then $\Box$ is "exactly" H^s-hypoelliptic or, with equivalent terminology, its inverse N is exactly H^s-regular in degree q. If this holds for any s, then $\Box$ and N are C^∞-hypoelliptic and regular respectively. To control commutators with the cut-off functions, Kohn introduced in [13] a pseudodifferential modification R^s of Λ^s

(cf. Sect. 2 below) which is equivalent to Λ^s over $\chi_0 u$ but has the advantage that $\dot{\chi}_1 R^s$ is of order 0. This yields quite easily (1.6) for some c_s. However, the precise description of c_s is a hard challenge; it is in the achievement of this task that consists this paper. Now, if the system of cut-off χ_j, $j=0,1,2$ shrinks to 0 depending on a parameter $t\to\infty$ as $\chi_j^t(z):=\chi_j(tz)$, then we are able to show that

$$c_s=\left(\left(\frac{f}{\log}\right)^*(t)\right)^{2s+1}. \tag{1.7}$$

In particular, when $\chi_1\Box u=0$, (1.6), with the constant c_s specified by (1.7), yields a constraint to the geometry of $b\Omega$ which produces all the above listed three consequences about type, lower bound for B_Ω and P-property.

2 Localization estimate with parameter

Let Ω be a bounded smooth pseudoconvex domain of $\mathbb{C}^n$, z_o a boundary point, $\chi_0\prec\chi_1\prec\chi_2$ a triplet of cut-off functions at z_o and $\chi_0^t\prec\chi_1^t\prec\chi_2^t$ a fundamental system of cut-off functions defined by $\chi_j^t(z)=\chi_j(tz)$, $j=0,1,2$ for $t\to\infty$. The content of this section is the following

Theorem 2.1 *Assume that an f-estimate holds in degree q at z_o with $f\gg\log$. Then, for any positive integer s, we have*

$$\|\chi_0^t u\|_s^2\lesssim t^{2s}\|\chi_1^t\Box u\|_s^2+\left(\left(\frac{f}{\log}\right)^*(t)\right)^{2(s+1)}\|\chi_2^t u\|^2, \tag{2.1}$$

for any $u\in(C^\infty)^q\cap\mathrm{Dom}(\Box)$, where "$*$" denotes the inverse.

Remark 2.2 In [1], Catlin proves the same statement for the particular choice $f=t^\epsilon$ ending up with f itself, instead of $\frac{f}{\log}$. In fact, starting from subelliptic estimates, (2.1) is obtained by induction over j such that $j\epsilon\geq s$. For us, who use Kohn method of [13], a logarithmic loss seems to be unavoidable.

Remark 2.3 A byproduct of Theorem 2.1 is the local H^s regularity of the Neumann operator $N=\Box^{-1}$. For this, the accuracy in the description of the constant in the last norm in (2.1) is needless and the conclusion is obtained from (2.1) by the method of the elliptic regularization. This method, which was first introduced for subelliptic estimates in [15], indeed also works for superlogarithmic estimates according to [13].

Proof of Theorem 2.1 Apart from the quantitative description of the constant in the error term of (2.1), the proof follows [13] Sect. 7. Let U be the neighborhood of z_o where the f-estimate holds; the whole discussion takes place on U. For each integer $s \geq 0$, we interpolate two families of cut-off functions $\{\zeta_m\}_{m=0}^s$ and $\{\sigma_m\}_{m=1}^s$ with support in U and such that $\zeta_j \prec \sigma_j \prec \zeta_{j-1}$. It is assumed that $\zeta_0 = \chi_1$ and $\zeta_s = \chi_0$. We define two new sequences $\{\zeta_m^t\}$ and $\{\sigma_m^t\}$ shrinking to z_o by $\zeta_m^t(z) = \zeta_m(tz)$ and $\sigma_m^t(z) = \sigma_m(tz)$.

We also need a pseudodifferential partition of the unity. Let $\lambda_1(|\xi|)$ and $\lambda_2(|\xi|)$ be real valued C^∞ functions such that $\lambda_1 + \lambda_2 \equiv 1$ and

$$\lambda_1(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2. \end{cases}$$

Recall that Λ^m is the tangential pseudodifferential operator of order m. Denote by Λ_t^m the pseudodifferential operator with symbol $\lambda_2(t^{-1}|\xi|)(1 + |\xi|^2)^{\frac{m}{2}}$ and by E_t the operator with symbol $\lambda_1(t^{-1}|\xi|)$. Note that

$$\|\Lambda^m \zeta_m^t u\|^2 \lesssim \|\Lambda_t^m \zeta_m^t u\|^2 + t^{2m} \|\zeta_m^t u\|^2. \tag{2.2}$$

In this estimate, it is understood that $t \leq (\frac{f}{\log})^*(t)$. From now on, to simplify notations, we write g instead of $\frac{f}{\log}$.

Following Kohn [13], we define for $m = 1, 2, \ldots,$ the pseudodifferential operator R_t^m by

$$\begin{aligned} &R_t^m \varphi(a, r) \\ &\quad = (2\pi)^{-2(n-1)} \int_{\mathbb{R}^{2n-1}} e^{ia\cdot\xi} \lambda_2(t^{-1}|\xi|)(1 + |\xi|^2)^{\frac{m\sigma_m^t(a,r)}{2}} \mathcal{F}(\varphi)(\xi, r)\, d\xi \end{aligned}$$

for $\varphi \in C_c^\infty(U \cap \bar{\Omega})$. Since $\zeta_m^t \prec \sigma_m^t$, the symbol of $(\Lambda_t^m - R_t^m)\zeta_m^t$ is of order zero and therefore

$$\begin{aligned} \|\Lambda_t^m \zeta_m^t u\|^2 &\lesssim \|R_t^m \zeta_m^t u\|^2 + \|\zeta_m^t u\|^2 \\ &\lesssim \|\zeta_m^t R_t^m \zeta_{m-1}^t u\|^2 + \|[R_t^m, \zeta_m^t]\zeta_{m-1}^t u\|^2 + \|\zeta_m^t u\|^2 \\ &\lesssim \|f(\Lambda)\zeta_{m-1}^t R_t^m \zeta_{m-1}^t u\|^2 + \|[R_t^m, \zeta_m^t]\zeta_{m-1}^t u\|^2 + \|\zeta_m^t u\|^2, \end{aligned} \tag{2.3}$$

(since $\zeta_m^t \prec \zeta_{m-1}^t$ and $f \geq 1$). By Proposition 2.4 below, the commutator term in the last line of (2.3) is dominated by $\sum_{j=1}^m t^{2j} ||| \zeta_{m-j}^t u |||_{m-j}^2$. From (2.2) and (2.3), we get the estimate for the tangential norm

$$|||\zeta_m^t u|||_m^2 \lesssim \|f(\Lambda)\zeta_{m-1}^t R_t^m \zeta_{m-1}^t u\|^2 + \sum_{j=1}^m t^{2j}\ |||\zeta_{m-j}^t u|||_{m-j}^2 . \tag{2.4}$$

As for the normal derivative D_r, we have

$$\begin{aligned}|||D_r\Lambda^{-1}\zeta_m^t u|||_m^2 &\lesssim \|D_r\Lambda^{-1}f(\Lambda)\zeta_{m-1}^t R_t^m \zeta_{m-1}^t u\|^2\\ &\quad + \sum_{j=1}^m t^{2j}\ |||D_r\Lambda^{-1}\zeta_{m-j}^t u|||_{m-j}^2 .\end{aligned} \tag{2.5}$$

We define the operator $A_t^m := \zeta_{m-1}^t R_t^m \zeta_{m-1}^t$ and remark that A_t^m is self-adjoint; also, we have $A_t^m u \in (C_c^\infty)^q \cap \mathrm{Dom}(\bar\partial^*)$ if $u \in (C^\infty)^q \cap \mathrm{Dom}(\bar\partial^*)$. In particular, the f-estimate can be applied to $A_t^m u$; this can further strengthened to

$$\|f(\Lambda)A_t^m u\|^2 + \|D_r\Lambda^{-1}f(\Lambda)A_t^m u\|^2 \lesssim Q(A_t^m u, A_t^m u). \tag{2.6}$$

In fact, without the term D_r, this is precisely the f-estimate. As for the term involving D_r, we write $D_r = \bar L_n + T$ for a tangential operator T and remark that, for $\tilde u = A_t^m u$, we have

$$\begin{aligned}\|D_r\Lambda^1 f(\Lambda)\tilde u\|^2 &\lesssim \|\bar L_n\tilde u\|^2 + \|T\Lambda^{-1}f(\Lambda)\tilde u\|^2\\ &\lesssim \|\bar L_n\tilde u\|^2 + \|f(\Lambda)\tilde u\|^2 \le Q(\tilde u, \tilde u).\end{aligned}$$

Next, we estimate $Q(A_t^m u, A_t^m u)$. We have

$$\begin{aligned}\|\bar\partial A_t^m u\|^2 &= (A_t^m\bar\partial u, \bar\partial A_t^m u) + ([\bar\partial, A_t^m]u, \bar\partial A_t^m u)\\ &= \big((A_t^m\bar\partial^*\bar\partial u, A_t^m u) - ([\bar\partial, A_t^m]^* u, \bar\partial^* A_t^m u)\\ &\quad - f(\Lambda)^{-1}[[A_t^m, \bar\partial^*], \bar\partial]u, f(\Lambda)A_t^m u)\big)\\ &\quad + ([\bar\partial, A_t^m]u, \bar\partial A_t^m u).\end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}\|\bar\partial^* A_t^m u\|^2 &= \big((A_t^m\bar\partial\bar\partial^* u, A_t^m u) - ([\bar\partial^*, A_t^m]^* u, \bar\partial A_t^m u)\\ &\quad - (f(\Lambda)^{-1}[[A_t^m, \bar\partial], \bar\partial^*]u, f(\Lambda)A_t^m u)\big)\\ &\quad + ([\bar\partial^*, A_t^m]u, \bar\partial^* A_t^m u)\end{aligned} \tag{2.8}$$

Taking summation of (2.7) and (2.8), we obtain

$$Q(A_t^m u, A_t^m u) \lesssim (A_t^m\Box u, A_t^m u) + error$$

$$\lesssim C_\epsilon \, |||\zeta^t_{m-1}\Box u|||^2_m + \epsilon\|A^m_t u\|^2 + error, \tag{2.9}$$

where

$$\begin{aligned} error = &\|[\bar\partial, A^m_t]u\|^2 + \|[\bar\partial^*, A^m_t]u\|^2 + \|[\bar\partial, A^m_t]^*u\| + \|[\bar\partial^*, A^m_t]^*u\| \\ &+ \|f(\Lambda)^{-1}[A^m_t, \bar\partial]^*, \bar\partial^*]u\|^2 + \|f(\Lambda)^{-1}[A^m_t, \bar\partial^*]^*, \bar\partial]u\|^2. \end{aligned} \tag{2.10}$$

The error term should also contain ϵQ where ϵ comes from the small/large argument, but this can be absorbed in the left of (2.9). Using Proposition 2.4 below, the error is dominated by

$$\epsilon Q(A^m_t u, A^m_t u) + C_\epsilon (g^*(t))^{2(m+1)}\|\chi^t_2 u\|^2 + \sum_{j=1}^m t^{2j}\|\zeta^t_{m-j}u\|^2_{m-j}. \tag{2.11}$$

Therefore

$$\begin{aligned} Q(A^m_t u, A^m_t u) \lesssim\ & C_\epsilon \, |||\zeta^t_{m-1}\Box u|||^2_m + \sum_{j=1}^m t^{2j}\|\zeta^t_{m-j}u\|^2_{m-j} \\ &+ C_\epsilon (g^*(t))^{2(m+1)}\|\chi^t_2 u\|^2 + \epsilon\|A^m_t u\|^2. \end{aligned} \tag{2.12}$$

Combining (2.4), (2.5), (2.6) and (2.12), and absorbing $\epsilon\|A^m_t u\|^2$ in the left side of (2.6), we obtain

$$\begin{aligned} &|||\zeta^t_m u|||^2_m + |||D_r\Lambda^{-1}\zeta^t_m u|||^2_m \\ &\quad\lesssim |||\zeta^t_{m-1}\Box u|||^2_m + \sum_{j=1}^m t^{2j}\|\zeta^t_{m-j}u\|^2_{m-j} + (g^*(t))^{2(m+1)}\|\chi^t_2 u\|^2. \end{aligned} \tag{2.13}$$

Since the operator $\Box$ is elliptic, and therefore non-characteristic with respect to the boundary, we have for $m \geq 2$

$$\|\zeta^t_m u\|^2_m \lesssim \|\Box\zeta^t_m u\|^2_{m-2} + |||\zeta^t_m u|||^2_m + |||D_r\zeta^t_m u|||^2_{m-1}. \tag{2.14}$$

Replace the first term in the right of (2.14) by $\|\zeta^t_m\Box u\|^2_{m-2} + \|[\Box, \zeta^t_m]u\|^2_{m-2}$ and observe that the commutator is estimated by $t^2\|\zeta^t_{m-1}u\|^2_{m-1} + t^4\|\zeta^t_{m-1}u\|^2_{m-2}$. Application of (2.13) to the last two terms of (2.14), yields

$$\begin{aligned} \|\zeta^t_m u\|^2_m \lesssim\ & \|\zeta^t_{m-1}\Box u\|^2_m + \sum_{j=1}^m t^{2j}\|\zeta^t_{m-j}u\|^2_{m-j} \\ &+ (g^*(t))^{2(m+1)}\|\chi^t_2 u\|^2, \quad m = 1, \ldots, s. \end{aligned} \tag{2.15}$$

Iterated use of (2.15) to estimate the terms of type $\zeta^t_{m-j}u$ by those of type $\zeta^t_{m-1}\Box u$ in the right side yields

$$\begin{aligned}\|\zeta^t_s u\|^2_s &\lesssim \sum_{m=0}^{s} t^{2m}\|\zeta^t_{s-m}\Box u\|^2_{s-m} + (g^*(t))^{2(s+1)}\|\chi^t_2 u\|^2\\ &\lesssim t^{2s}\|\zeta^t_0\Box u\|^2_s + (g^*(t))^{2(s+1)}\|\chi^t_2 u\|^2.\end{aligned} \tag{2.16}$$

Choose $\chi^t_0 = \zeta^t_s$ and $\chi^t_1 = \zeta^t_0$; we then conclude

$$\|\chi^t_0 u\|^2_s \lesssim t^{2s}\|\chi^t_1\Box u\|^2_s + (g^*(t))^{2(s+1)}\|\chi^t_2 u\|^2, \tag{2.17}$$

for any $u \in (C^\infty)^q \cap D_\Box$. □

The proof of the theorem is complete but we have skipped a crucial technical point that we face now.

Proposition 2.4 *We have*

(i) $\|[R^m_t, \zeta^t_m]\zeta^t_{m-1}u\|^2 \lesssim \sum_{j=1}^m t^{2j}\, |||\zeta^t_{m-j}u|||^2_{m-j}$
(ii) *Assume that an f-estimate holds with $f \gg$ log, then for any ϵ and for suitable C_ϵ, the error term in (2.9) is dominated by (2.11).*

Proof (i) It is well known that the principal symbol $\sigma_P([A,B])$ of the commutator of two operators A and B is the Poisson bracket $\{\sigma_P(A), \sigma_P(B)\}$. For the full symbol, and with tangential variables a and dual variables ξ, we have the formula

$$\sigma([A,B]) = \sum_{|\kappa|>0} \frac{D^\kappa_\xi\sigma(A)D^\kappa_a\sigma(B) - D^\kappa_\xi\sigma(B)D^\kappa_a\sigma(A)}{\kappa!}. \tag{2.18}$$

We apply this formula to $[R^m_t, \zeta^t_m]$ and obtain

$$\begin{aligned}\sigma([R^m_t, \zeta^t_m]) &= \sum_{|\kappa|>0}\frac{1}{\kappa!}D^\kappa_\xi\Big(\lambda_2(t^{-1}|\xi|)(1+|\xi|^2)^{\frac{m\sigma^t_m(a,r)}{2}}\Big)D^\kappa_a\zeta^t_m(a,r)\\ &= \sum_{j=1}^{m}\alpha_j(a,r,t,|\xi|)t^j(1+|\xi|^2)^{\frac{m-j}{2}},\end{aligned}$$

where the α_j's are functions uniformly bounded with respect to t and $|\xi|$.

(ii) First, we show

$$\|[\bar\partial, A_t^m]u\| \le \epsilon Q(A_t^m u, A_t^m u) + C_\epsilon (g^*(t))^{2(m+1)}\|\chi_2^t u\|^2 + \sum_{j=1}^m t^{2j}\|\zeta_{m-j}^t u\|_{m-j}^2. \tag{2.19}$$

By Jacobi identity,

$$\begin{aligned}[\bar\partial, A_t^m] &= [\bar\partial, \zeta_{m-1}^t R_t^m \zeta_{m-1}^t] \\ &= [\bar\partial, \zeta_{m-1}^t] R_t^m \zeta_{m-1}^t + \zeta_{m-1}^t [\bar\partial, R_t^m]\zeta_{m-1}^t \\ &\quad + \zeta_{m-1}^t R_t^m [\bar\partial, \zeta_{m-1}^t].\end{aligned} \tag{2.20}$$

Since the support of the derivative of ζ_{m-1}^t is disjoint from the support of σ_m^t, the first and third terms in the second line of (2.20) are bounded by $|\dot\zeta_m^t| \sim t$ in L^2. The middle term in (2.20) is treated as follows. Let b be a function which belongs to the Schwartz space $\mathcal{S}$ and D be D_{a_j} or D_r; we have

$$[bD, R_t^m] = [b, R_t^m]D + b[D, R_t^m]. \tag{2.21}$$

As for the second term of (2.21), we note that $[D, R_t^m] = mD(\sigma_m^t)\log(\Lambda)R_t^m$; in particular, $[D, R_t^m]$ is bounded by $t\log(\Lambda)R_t^m$. For the same reason, when $D = D_{a_j}$, the first term is bounded by $t\log(\Lambda)R_t^m$ and, when $D = D_r$ and thus $D = \bar L_n + T$, by $t\log(\Lambda)R_t^m + t\log(\Lambda)\Lambda^{-1}\bar L_n R_t^m$. It follows

$$\|[\bar\partial, A_t^m]u\|^2 \lesssim t\|\log(\Lambda)A_t^m u\|^2 + \epsilon\|\bar L_n A_t^m u\|^2 + C_\epsilon\|\chi_2^t u\|^2 + \sum_{j=1}^m t^{2j}\|\zeta_{m-j}^t u\|_{m-j}^2. \tag{2.22}$$

To estimate the first term in (2.22), we check that

$$t\log\Lambda_\xi \le \epsilon f(\Lambda_\xi) \quad \text{ in the set } \{\xi : \lambda_1(g^{*-1}(\epsilon^{-1}t)\Lambda_\xi) \ne 1\}$$

and hence

$$t\log\Lambda_\xi \lesssim \epsilon f(\Lambda_\xi) + t\lambda_1(g^{*-1}(\epsilon^{-1}t)\Lambda_\xi)\log\Lambda_\xi. \tag{2.23}$$

It follows

$$\begin{aligned} t^2\|\log\Lambda A_t^m u\|^2 &\le \epsilon^2\|f(\Lambda)A_t^m u\|^2 + t^2(g^*(\epsilon^{-1}t))^{2m}\log^2(g^*(\epsilon^{-1}t))\|\chi_2^t u\|^2 \\ &\le \epsilon^2\|f(\Lambda)A_t^m u\|^2 + C_\epsilon(g^*(t))^{2(m+1)}\|\chi_2^t u\|^2. \end{aligned} \tag{2.24}$$

Since we are supposing that an f-estimate holds, we get the proof of the inequality (2.19). By a similar argument, we can estimate all subsequent error terms in (2.10) and obtain the conclusion of the proof of Theorem 2.1. □

3 From estimate to type—proof of Theorem 1.4(i)

Proof of Theorem 1.4(i) We follow the guidelines of [1] and begin by recalling two results therein. The first is stated in [1] Theorem 2 for domains of finite type, that is for $F = \delta^M$, but it holds in full generality of F.

(a) Let Ω be a domain in $\mathbb{C}^n$ with smooth boundary and assume that there is a function F and a q-dimensional complex-analytic variety Z passing through z_o such that (1.1) is satisfied for $z \in Z$. Then, in any neighborhood U of z_o, there is a family $\{Z_\delta\}$ of q-dimensional complex manifolds $Z_\delta \subset \Omega$ of diameter comparable to δ such that

$$\sup_{z\in Z_\delta} |r(z)| \lesssim F(\delta).$$

The proof is just a technicality for passing from variety to manifold. The second result, consists in exhibiting, as a consequence of pseudoconvexity, holomorphic functions bounded in L^2 norm which blow up approaching the boundary.

(b) Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain in a neighborhood of $z_o \in b\Omega$. For any point $z \in \Omega$ near z_o there is $G \in \text{hol}\,(\Omega) \cap L^2(\Omega)$ such that
   (1) $\quad \|G\|_0^2 \lesssim 1$
   (2) $\quad |\partial_{z_n}^m G(z)| \gtrsim \delta^{-(m+\frac{1}{2})}(z)$ for all $m \geq 0$.

(We always denote by $\delta(z)$ the distance of z to $b\Omega$ and assume that $\frac{\partial}{\partial_{z_n}}$ is a normal derivative.) By (a), for any δ there is a point $\gamma_\delta \in Z_\delta$, which satisfies $\delta(\gamma_\delta) \lesssim F(\delta)$ and by (b) there is a function $G_\delta \in \text{hol}(\Omega) \cap L^2(\Omega)$ such that

$$\|G_\delta\| \leq 1$$

and

$$\left|\frac{\partial^m G_\delta}{\partial z_n^m}(\gamma_\delta)\right| \gtrsim F^{-(m+\frac{1}{2})}(\delta(\gamma_\delta)).$$

We parametrize Z_δ over $\mathbb{C}^q \times \{0\}$ by

$$z' \mapsto (z', h_\delta(z')) \quad \text{for } z' = (z_1, \dots, z_q).$$

We observe that it is not restrictive to assume that γ_δ is the "center" of Z_δ, that is, the image of $z'=0$ (by the properties of uniformity of the parametrization with respect to δ). Let φ be a cut-off function on $\mathbb{R}^+$ such that $\varphi=1$ on $[0,1)$ and $\varphi=0$ on $[2,+\infty)$. We use our standard relation $t=\delta^{-1}$ and define, for some c to be chosen later

$$\psi_t(z')=\varphi\left(\frac{8t|z'|}{c}\right).$$

Choose the datum α_t as

$$\alpha_t=\psi_t(z')G_t(z)\,d\bar z_1\wedge\cdots\wedge d\bar z_q.$$

Clearly the form α_t is $\bar\partial$-closed and its coefficient belongs to L^2. Let P_t be the q-polydisc with center $z'=0$ and radius ct^{-1}, let w_t be the q-form

$$w_t=\varphi\left(\frac{8t|z'|}{3c}\right)d\bar z_1\wedge\cdots\wedge d\bar z_q,$$

and define

$$\mathcal{K}_t^m:=\int_{P_t}\left\langle\frac{\partial^m}{\partial z_n^m}\alpha_t(z',h_t(z')),w_t\right\rangle dV. \tag{3.1}$$

Using the mean value property for $\frac{\partial^m}{\partial z_n^m}G_t(z',h_t(z'))$ over the spheres $|z'|=s$ and integrating over s with $0\le s\le t$, we get, by Property (2) of G

$$\mathcal{K}_t^m\gtrsim t^{-2q}F(t^{-1})^{-(m+\frac{1}{2})}. \tag{3.2}$$

Let v_t be the canonical solution of $\bar\partial v_t=\alpha_t$, that is, $v_t=\bar\partial^*u_t$ for $u_t=N\alpha_t$ where $N=\Box^{-1}$. If ϑ is the adjoint of $\bar\partial$, then integration by parts yields

$$\mathcal{K}_t^m=\int_{P_t}\left\langle\bar\partial\frac{\partial^m}{\partial z_n^m}v_t(h_t),w_t\right\rangle dV=\int_{P_t}\left\langle\frac{\partial^m}{\partial z_n^m}v_t(h_t),\vartheta w_t\right\rangle dV.$$

We define a set $S_t=\{z'\in\mathbb{C}^k:\frac{3c}{8t}\le|z'|\lesssim\frac{6c}{8t}\}$. Since ϑw_t is supported in S_t and $|\vartheta w_t|\lesssim t$, then (for $\delta=t^{-1}$)

$$\begin{aligned}\mathcal{K}_t^m&\lesssim t^{-2q+1}\sup_{Z_\delta}\left|\frac{\partial^m}{\partial z_n^m}v_t(h_t)\right|\lesssim t^{-2q+1}\sup_{Z_\delta}\left|\frac{\partial^m}{\partial z_n^m}\bar\partial^*u_t\right|\\&\lesssim t^{-2q+1}\sup_{Z_\delta}\left|\underset{|\beta|=m+1}{D^\beta}u_t\right|.\end{aligned} \tag{3.3}$$

Recall the notation $g := \frac{f}{\log}$; before completing the proof of Theorem 1.4(i), we need to state an upper bound for $\mathcal{K}_t^m$, which follows from

$$\sup_{Z_\delta}\Big|\underset{|\beta|=m+1}{D^\beta} u_t\Big| \lesssim g^*(t)^{m+n+3}. \tag{3.4}$$

To prove (3.4), we start by noticing that, since the set S_t has diameter $0(t^{-1})$ and the function h_t satisfies $|dh_t(z')| \le C$ for $z' \in P_t$, then the set $Z_\delta = (\mathrm{id} \times h_t)(S_t)$ (for $\delta = t^{-1}$) has diameter of size $0(\delta)$. Moreover, by construction, there exists a constant d such that

$$\inf\{|z_1 - z_2| : z_1 \in \operatorname{supp} \alpha_t,\ z_2 \in Z_\delta\} > 2dt^{-1}.$$

Therefore, we may choose χ_0 and χ_1 such that if we set $\chi_k^t(z) = \chi_k(\frac{tz}{d})$ for $k = 0, 1$, we have the properties

(1) $\chi_0^t = 1$ on Z_δ
(2) $\alpha_t = 0$ on $\operatorname{supp} \chi_1^t$.

Hence

$$\sup_{Z_\delta}\Big|\underset{|\beta|=m+1}{D^\beta} u_t\Big| \lesssim \sup_{\Omega\cap Z_\delta}\Big|\underset{|\beta|=m+1}{D^\beta} \chi_0^t u_t\Big| \lesssim \|\chi_0^t u_t\|_{m+n+1}, \tag{3.5}$$

where the last inequality follows from Sobolev Lemma since $\chi_0^t u_t$ is smooth by Remark 2.3. We use now Theorem 2.1 and observe that $\chi_1^t \Box u_t = 0$ (by Property (2) of χ_1^t). It follows

$$\begin{aligned} \|\chi_0^t u_t\|_{m+n+1}^2 &\lesssim g^*(t)^{2(m+n+2)} \|u_t\|^2 \\ &\lesssim g^*(t)^{2(m+n+2)}, \end{aligned}$$

where for the last inequality we have to observe that, Ω being bounded and pseudoconvex, then $\|u_t\|^2 \lesssim \|\Box u_t\|^2 = \|\alpha_t\|^2 \lesssim 1$. This completes the proof of (3.4). We return to the proof of Theorem 1.4(i). Combining (3.2) with (3.3) and (3.4), we get the estimate

$$t^{2k} F(t)^{-(m+\frac{1}{2})} \le C t^{2k-1} g^*(t)^{m+n+2}.$$

Taking m-th root and going to the limit for $m \to \infty$, yields

$$F(t)^{-1} \le g^*(t).$$

This concludes the proof of Theorem 1.4(i). ∎

4 From estimate to lower bound for the Bergman metric B_Ω—proof of Theorem 1.4(ii)

The Bergman kernel K_Ω has been introduced in Sect. 1: as already recalled, it provides the integral representation of the orthogonal projection $P : L^2(\Omega) \to \text{hol}(\Omega) \cap L^2(\Omega)$. From K_Ω one obtains the Bergman metric $B_\Omega := \sqrt{\partial\bar\partial \log(K_\Omega(z,z))}$. Let

$$b_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar z_j} \log K(z,z); \tag{4.1}$$

then the action of B_Ω over a $(1,0)$ vector field $X = \sum_j a_j \partial_{z_j}$ is expressed by

$$B_\Omega(z,X) = \left(\sum_{ij=1}^n b_{ij} a_i \bar a_j\right)^{\frac{1}{2}}. \tag{4.2}$$

This differential metric is primarily interesting because of its invariance under a biholomorphic transformation on Ω.

One can obtain the value of the Bergman kernel on the diagonal of $\Omega \times \Omega$ and the length of a tangent $(1,0)$-vector X in the Bergman metric by solving the following extremal problems:

$$\begin{aligned} K_\Omega(z,z) &= \inf\{\|\varphi\|^2 : \varphi \in \text{hol}(\Omega), \varphi(z) = 1\}^{-1} \\ &= \sup\{|\varphi(z)|^2 : \varphi \in \text{hol}(\Omega), \|\varphi\| \le 1\} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} B_\Omega(z,X) &= \frac{\inf\{\|\varphi\| : \varphi \in \text{hol}(\Omega), \varphi(z) = 0,\ X\varphi(z) = 1\}^{-1}}{\sqrt{K_\Omega(z,z)}} \\ &= \frac{\sup\{|X\varphi(z)| : \varphi \in \text{hol}(\Omega), \varphi(z) = 0, \|\varphi\| \le 1\}}{\sqrt{K_\Omega(z,z)}}. \end{aligned} \tag{4.4}$$

The purpose of this section is to study the boundary behavior of $B_\Omega(z,X)$ for z near a point $z_o \in b\Omega$, when a f-estimate for the $\bar\partial$-Neumann problem holds. We prove Theorem 1.4(ii) for a general f-estimate; this extends [16] which deals with subelliptic estimates. For the proof of Theorem 1.4(ii), we start from a result by [16] about locally comparable properties of the Bergman kernel and the Bergman metric, that is,

(a) Let Ω_1, Ω_2 be bounded pseudoconvex domains in $\mathbb{C}^n$ such that a portion of $b\Omega_1$ and $b\Omega_2$ coincide. Then

$$K_{\Omega_1}(z,z) \cong K_{\Omega_2}(z,z);$$
$$B_{\Omega_1}(z,X) \cong B_{\Omega_2}(z,X), \quad X \in T^{1,0}_z\mathbb{C}^n,$$

for z near the coincidental portion of the two boundaries (cf. [16] or [6]).

We need to modify Ω to a pseudoconvex domain $\tilde{\Omega} \subset \Omega$ which shares a piece of its boundary with $b\Omega$ near z_o over which there is exact, global regularity of the $\bar{\partial}$-Neumann operator. For this, we recall another result from [16]:

(b) Let Ω be a smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ and let $z_o \in b\Omega$. Then, there exist a neighborhood U of z_o and a smooth, bounded, pseudoconvex domain $\tilde{\Omega}$ satisfying the following properties:

- $\tilde{\Omega} \subset \Omega \cap U$,
- $b\tilde{\Omega} \cap b\Omega$ contains a neighborhood of z_o in $b\Omega$,
- all points in $b\tilde{\Omega} \setminus b\Omega$ are points of strong pseudoconvexity,
- the relative boundary S of $b\tilde{\Omega} \cap b\Omega$ and $b\tilde{\Omega} \setminus b\Omega$ is the intersection of $b\Omega$ with a sphere centered at z_o.

Next, we have the result below which is crucial in our application (cf. also [17] Prop. 4.4).

Proposition 4.1 *If Ω has compactness estimates for the $\bar{\partial}$-Neumann problem in a neighborhood of z_o, then, for suitable U, the domain $\tilde{\Omega}$ of* (b) *above has compactness estimates on the whole boundary. In particular, its Neumann operator N is globally, exactly, regular.*

Proof We show, in fact, a stronger statement. If $\tilde{\Omega}$ is a general domain and S a piece of its boundary obtained as the intersection of $b\tilde{\Omega}$ with a strongly pseudoconvex boundary (defined by $h = 0$ for $\partial h \neq 0$), and if at any point of $b\tilde{\Omega} \setminus S$ there are (local) compactness estimates, then there are compactness estimates on the whole $b\tilde{\Omega}$. Note that our specific $\tilde{\Omega}$ meets these requirements. To prove our claim, we first deal with the points in a neighborhood of S. We use the notation $S_\epsilon = \{z \in b\tilde{\Omega} : |h| < \epsilon\}$ and define $\varphi_\epsilon = \frac{h}{\epsilon}$. We have, over S_ϵ,

$$\begin{cases} |\varphi_\epsilon| < 1, \\ \partial\bar{\partial}\varphi_\epsilon \gtrsim \frac{1}{\epsilon}. \end{cases} \tag{4.5}$$

Let ζ_ϵ be a cut-off such that $\zeta_\epsilon \equiv 1$ on $S_{\frac{\epsilon}{2}}$ and $\text{supp}\zeta_\epsilon \subset S_\epsilon$; we can also assume that $|\dot{\zeta}_\epsilon| \lesssim \frac{1}{\epsilon}$ and $\ddot{\zeta}_\epsilon \lesssim \frac{1}{\epsilon^2}$. We recall the basic estimate with weight φ

$$\sum_{ij} \sum_{|K|=k-1}' \int_{\tilde{\Omega}} e^{-\varphi}\varphi_{ij} u_{iK} \bar{u}_{jK}\, dV \lesssim \|\bar{\partial}u\|^2_\varphi + \|\bar{\partial}^* u\|^2_\varphi, \tag{4.6}$$

where $\|\cdot\|_\varphi$ is the L^2 norm weighted by $e^{-\varphi}$ and $\bar\partial^*_\varphi$ is the adjoint of $\bar\partial$ in this norm. We use an auxiliary function $\chi : \mathbb{R}^+ \to \mathbb{R}^+$, $\chi = \chi(t)$, increasing, convex, and such that, for $t < 1$, we have

$$\begin{cases} \ddot\chi \geq 2\dot\chi^2, \\ \dot\chi e^{-\chi} > C_1, \\ e^{-\chi} < C_2, \end{cases} \tag{4.7}$$

for suitable $C_1,\ C_2 > 0$. For instance, a good choice is $\chi = \frac{1}{2}e^{t-1}$. We apply (4.6) for $\varphi = \chi \circ \varphi_\epsilon$ and with u replaced by $\zeta_\epsilon u$. Because of (4.7) combined with the first of (4.5), we can remove φ both from the norms and the adjunction. On the other hand, the second of (4.5) yields a lower bound for the left side of (4.6). Altogether, we have got

$$\frac{1}{\epsilon}\|\zeta_\epsilon u\|^2 \lesssim Q(\zeta_\epsilon u, \zeta_\epsilon u), \quad \text{for any } u. \tag{4.8}$$

Combining (4.8) with compactness estimates on $\operatorname{supp}(1-\zeta_\epsilon) \subset b\tilde\Omega \setminus S$, we get

$$\begin{aligned} \|u\|^2 &\lesssim \|\zeta_\epsilon u\|^2 + \|(1-\zeta_\epsilon)u\|^2 \\ &\lesssim \epsilon\big(Q(\zeta_\epsilon u, \zeta_\epsilon u) + Q((1-\zeta_\epsilon)u, (1-\zeta_\epsilon)u)\big) + c_\epsilon\|u\|^2_{-1} \\ &\lesssim \epsilon\big(Q(u,u) + \|\dot\zeta_\epsilon u\|^2\big) + c_\epsilon\|u\|^2_{-1}. \end{aligned} \tag{4.9}$$

Using compactness estimates on $\operatorname{supp}\dot\zeta_\epsilon \subset b\tilde\Omega \setminus S$, we get

$$\begin{aligned} \|\dot\zeta_\epsilon u\|^2 &\leq \epsilon^4 Q(\dot\zeta_\epsilon u, \dot\zeta_\epsilon u) + c_\epsilon\|u\|^2_{-1} \\ &\lesssim \epsilon^4\left(\frac{1}{\epsilon^2}Q(u,u) + \|\ddot\zeta_\epsilon u\|^2\right) + c_\epsilon\|u\|^2_{-1} \\ &\lesssim \epsilon^2 Q(u,u) + \|u\|^2 + c_\epsilon\|u\|^2_{-1}. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10) and absorbing $\|u\|^2$ in the last line, we get the conclusion of the proof. □

Our domain Ω satisfies the assumptions of Proposition 4.1; in fact, superlogarithmic estimates imply compactness estimates. It follows that the modified domain $\tilde\Omega$ also has compactness estimates and, in particular, its Neumann operator $N = N_{\tilde\Omega}$ is regular; from now on, we change our notation and write Ω instead of $\tilde\Omega$.

Let ψ be a cut-off function such that

$$\psi(z) = \begin{cases} 0 & \text{if } z \in \mathbb{B}_1(z_o), \\ 1 & \text{if } z \in \mathbb{C}^n \setminus \mathbb{B}_2(z_o), \end{cases}$$

where $\mathbb{B}_c(z_o)$ is the ball in $\mathbb{C}^n$ with center z_o and radius c; we also set $\psi^t = \psi(tz)$.

Proposition 4.2 *Let an f-estimate in degree q hold at z_o and N be exactly globally regular on Ω. Then if $\alpha \in C_c^\infty(\mathbb{B}_{\frac{1}{8t}}(z_o) \cap \bar{\Omega})^q$, for any nonnegative integer s_1, s_2, we have*

$$\|\psi^t N\alpha\|^2_{s_1} \lesssim g^*(t)^{2(s_1+s_2+4)} \|\alpha\|^2_{-s_2}. \tag{4.11}$$

Proof We choose a triplet of cut-off functions χ_0^t, χ_1^t and χ_2^t in Theorem 2.1, such that $\chi_0^t \equiv 1$ on a neighborhood of the support of the derivative of ψ^t and supp $\chi_2^t \subset \mathbb{B}_{3t^{-1}}(z_o) \setminus \mathbb{B}_{\frac{1}{2}t^{-1}}(z_o)$; hence $\chi_1^t \alpha = 0$. We notice that for t sufficiently small, supp$\chi_j^t \Subset U$ for $j = 0, 1, 2$, so that we can apply Theorem 2.1 to this triplet of cut-off functions. Using the global regularity estimate and Theorem 2.1 for an arbitrary q-form $u \in (C^\infty)^q \cap \mathrm{Dom}(\Box)$, and for $t \le g^*(t)$, we have

$$\begin{aligned} \|\psi^t u\|^2_{s_1} &\lesssim \|\Box \psi^t u\|^2_{s_1} \\ &\lesssim \|\psi^t \Box u\|^2_{s_1} + \|[\Box, \psi^t]u\|^2_{s_1} \\ &\lesssim \|\psi^t \Box u\|^2_{s_1} + t^2 \|\chi_0^t u\|^2_{s_1+1} + t^4 \|\chi_0^t u\|^2_{s_1} \\ &\lesssim \|\psi^t \Box u\|^2_{s_1} + t^{2(s_1+2)} \|\chi_1^t \Box u\|^2_{s_1+1} + g^*(t)^{2(s_1+3)} \|\chi_2^t u\|^2. \end{aligned} \tag{4.12}$$

Recall that we are supposing that the $\bar{\partial}$-Neumann operator is globally regular. If $\alpha \in C^\infty(\bar{\Omega})^q$, then $N\alpha \in C^\infty(\bar{\Omega})^q \cap \mathrm{Dom}(\Box)$. Substituting $u = N\alpha$ in (4.12) for $\alpha \in C_c^\infty(\mathbb{B}_{\frac{1}{8t}}(z_o) \cap \bar{\Omega})^q$, we obtain

$$\|\psi^t N\alpha\|^2_{s_1} \lesssim g^*(t)^{2(s_1+3)} \|\chi_2^t N\alpha\|^2. \tag{4.13}$$

However,

$$\|\chi_2^t N\alpha\| = \sup\{|(\chi_2^t N\alpha, \beta)| : \|\beta\| \le 1\},$$

and the self-adjointness of N and the Cauchy–Schwarz inequality yield

$$
\begin{aligned}
|(\chi_2^t N\alpha, \beta)| &= |(\alpha, N\chi_2^t\beta)| \\
&= |(\alpha, \tilde{\chi}_0^t N\chi_2^t\beta)| \\
&\lesssim \|\alpha\|_{-s_2} \|\tilde{\chi}_0^t N\chi_2^t\beta\|_{s_2}, \qquad (4.14)
\end{aligned}
$$

where $\tilde{\chi}_0^t$ is a cut-off function such that $\tilde{\chi}_0^t \equiv 1$ on supp α. Let $\tilde{\chi}_0^t \prec \tilde{\chi}_1^t \prec \tilde{\chi}_2^t$ with supp $\tilde{\chi}_1^t \Subset \mathbb{B}_{\frac{1}{4t}}(z_o)$; in particular, supp $\tilde{\chi}_1^t \cap$ supp $\chi_2^t = \emptyset$. Using again Theorem 2.1 for the triplet of cut-off functions $\tilde{\chi}_0^t$, $\tilde{\chi}_1^t$ and χ_2^t, we obtain

$$
\begin{aligned}
\|\tilde{\chi}_0^t N\chi_2^t\beta\|_{s_2}^2 &\lesssim t^{2s_2}\|\tilde{\chi}_1^t\chi_2^t\beta\|_{s_2}^2 + g^*(t)^{2(s_2+1)}\|\tilde{\chi}_2^t N\chi_2^t\beta\|^2 \\
&\lesssim g^*(t)^{2(s_2+1)}\|\tilde{\chi}_2^t N\chi_2^t\beta\|^2 \\
&\lesssim g^*(t)^{2(s_2+1)}\|\beta\|^2. \qquad (4.15)
\end{aligned}
$$

Taking supremum over $\|\beta\| \le 1$, we get (4.11). □

Proof of Theorem 1.4(ii) We follow the guidelines of [16] and also [14]. Let (ζ, z) be local complex coordinates in a neighborhood of (z_o, z_o) in which $X(z_o) = \partial_{\zeta_1}$ with the normalization $\partial_{\zeta_n} r|_{z_o} = 1$. If $z \in U$ and $z \notin b\Omega$, we define

$$
h_z(\zeta) = \frac{K_\Omega(\zeta, z)}{\sqrt{K_\Omega(z, z)}}
$$

so that $\|h_z\| = 1$ and $\frac{|h_z(z)|}{\sqrt{K_\Omega(z,z)}} = 1$. We also define

$$
\gamma_z(\zeta) = R(z)(\zeta_1 - z_1)h_z(\zeta) \quad \text{for } R(z) = g(\delta^{-1+\eta}(z)).
$$

It is obvious that $\gamma_z \in \mathrm{hol}(\Omega)$ and $\gamma_z(z) = 0$. We claim that $\|\gamma_z\| \le 1$; once this is proved, then (4.4) assures that

$$
B_\Omega(z, X) \ge \frac{|X\gamma_z(z)|}{\sqrt{K_\Omega(z, z)}} = \frac{|R(z)h_z(z)|}{\sqrt{K_\Omega(z, z)}} = |R(z)| = g(\delta^{-1+\eta}(z)), \qquad (4.16)
$$

and the proof of Theorem 1.4(ii) is complete. We prove the claim. In all what follows, z is fixed in U; we set $t = g(\delta^{-1+\eta}(z))$ and, for ψ^t as in Proposition 4.2, put $\psi_z^t(\zeta) = \psi^t(\zeta - z)$. We decompose

$$
\gamma_z(\zeta) = \psi_z^t(\zeta)\gamma_z(\zeta) + (1 - \psi_z^t(\zeta))\gamma_z(\zeta). \qquad (4.17)
$$

The second term satisfies

$$
\|(1 - \psi_z^t)\gamma_z\| \lesssim |R(z)|t^{-1} = 1. \qquad (4.18)
$$

As for the first term, multiplying and dividing by $\bar{G}^m := \frac{\partial^m}{\partial \bar{z}_n^m}\bar{G}$ where G is the function introduced in the beginning of Sect. 3, we get

$$\psi_z^t(\zeta)\gamma_z(\zeta) = R(z)(\zeta_1 - z_1)\frac{1}{\bar{G}^m(z)}\frac{1}{\sqrt{K_\Omega(z,z)}}(\psi_z^t(\zeta)K_\Omega(\zeta,z)\bar{G}^m(z)). \tag{4.19}$$

We denote by c_z the term, constant in ζ, before parentheses; since $K_\Omega(z,z) \geq |G(z)|^2 \gtrsim \delta^{-1}(z)$, then $|c_z| \lesssim g(\delta^{-1+\eta}(z))\delta^{m+1}(z)$. On the other hand, if φ_z^t is a cut-off with support in $\mathbb{B}_{\frac{1}{10t}}(0)$ with unit mass, then

$$\begin{aligned} K_\Omega(\zeta,z)\bar{G}^m(z) &= \int K(\zeta,w)\bar{G}^m(w)\varphi_z^t(w)\,dV_w \\ &= P(\bar{G}^m(\zeta)\varphi_z^t(\zeta)) \\ &= \bar{G}^m(\zeta)\varphi_z^t(\zeta) - \bar{\partial}^* N\bar{\partial}(\bar{G}^m(\zeta)\varphi_z^t(\zeta)), \end{aligned} \tag{4.20}$$

where the first equality follows from the mean value theorem for antiholomorphic functions, the second from the definition of P and the third from the relation of P with N. Notice that the supports of ψ_z^t and φ_z^t are disjoint, and that supp $\bar{\partial}(\bar{G}^m\varphi_z^t)$ is contained in $\mathbb{B}_{\frac{1}{8t}}$ for all $z \in U$. We call the attention of the reader to the fact that in Theorem 1.4(ii) and (iii), it is assumed that an f-estimate holds in degree $q = 1$. We may therefore apply Proposition 4.2 to the 1-form $\bar{\partial}(\bar{G}^m\varphi_z^t)$ for z_o replaced by z and for $s_1 = 1$, and obtain

$$\begin{aligned} \|\psi_z^t K_\Omega(\cdot,z)\bar{G}^m(z)\|^2 &= \|\psi_z^t\bar{\partial}^* N\bar{\partial}(\bar{G}^m\varphi_z^t)\|^2 \\ &\lesssim \|\psi_z^t N\bar{\partial}(\bar{G}^m\varphi_z^t)\|_1^2 + \|[\psi_z^t,\bar{\partial}^*]N\bar{\partial}(\bar{G}^m\varphi_z^t)\|^2 \\ &\lesssim g^*(t)^{2(s_2+5)}\|\bar{\partial}(\bar{G}^m\varphi_z^t)\|_{-s_2}^2 \\ &\lesssim g^*(t)^{2(s_2+5)}t^2\|\bar{G}^m\varphi_z^t\|_{-s_2+1}^2 \\ &\lesssim g^*(t)^{2(s_2+6)}\|\bar{G}^m\|_{-m}\|\varphi_z^t\|_{-s_2+m+1}, \end{aligned} \tag{4.21}$$

where the last inequality follows from the Cauchy–Schwartz inequality and from $g(t) \lesssim t$. We notice that $\|\bar{G}^m\|_{-m} \lesssim \|\bar{G}\| \leq 1$ (because $\bar{G}^m = \frac{\partial^m}{\partial^m_{z_n}}\bar{G}$); besides, for $s_2 - m - 1 > n$ we have by Sobolev's Lemma

$$\begin{aligned} \|\varphi_z^t\|_{-s_2+m+1}^2 &= \sup\{(|(\varphi_z^t,h)| : h \in C_c^\infty, \|h\|_{s_2-m-1} \leq 1\} \\ &\lesssim \|\varphi_z^t\| = 1. \end{aligned} \tag{4.22}$$

Therefore, remembering that $t = g(\delta^{-1+\eta}(z))$,

$$\|\psi_z^t K_\Omega(\cdot,z)\bar{G}^m(z)\|^2 \lesssim \delta(z)^{(-1+\eta)2(m+n+8)}. \tag{4.23}$$

We go back to (4.19); combining (4.23) with the estimate for c_z and with $R = g(\delta^{-1+\eta}(z)) \le \delta^{-1}(z)$, we obtain

$$\begin{aligned}\|\psi_z^t \gamma_z\| &\lesssim \delta(z)^{-1+(m+1)+(-1+\eta)(m+n+8)} \\ &\lesssim 1, \end{aligned} \tag{4.24}$$

for $m \to \infty$. We thus conclude that $\|\gamma_z\| \lesssim 1$, and then from (4.16) we get $B_\Omega(z, X) \gtrsim |R(z)| = g(\delta^{-1+\eta})$ which concludes the proof of Theorem 1.4(ii). □

5 From estimate to P-property—proof of Theorem 1.4(iii)

Proof of Theorem 1.4(iii) The notations $K_\Omega(z, z)$, $\delta(z)$, η and U_η are the same as in the section above. Again, the hypothesis is that an f-estimate holds in degree $q = 1$. Recall from the introduction that u^τ denotes a "tangential" form. Define

$$\varphi(z) = \frac{\log K_\Omega(z, z)}{(\log(\delta^{-1}(z)))^{1+2\eta}} - \frac{1}{(\log(\delta^{-1}(z)))^{\eta}} \tag{5.1}$$

for $z \in U$. Recall that $K_\Omega(z, z) \gtrsim \delta^{-1}(z)$ whereas $K_\Omega(z, z) \lesssim \delta^{-(n+1)}(z)$ is obvious because Ω contains an osculating ball at any boundary point. Thus $\varphi(z) \to 0$ as $\delta(z) \to 0$ (and in particular, φ is bounded). We wish to prove Property $(\tilde{f} - P)$. We begin by noticing that it suffices to check it over tangential forms, that is, $\partial\bar\partial\varphi(z)(u^\tau) \gtrsim \tilde{f}(\delta^{-1}(z))|u^\tau|^2$ for any u^τ in degree 1. (In fact, if this holds for u^τ, then the same holds for the full u once one adds the additional term $-\log(\frac{-r}{\delta} + 1)$ to the weight φ.)

Now,

$$\begin{aligned}\partial\bar\partial\varphi(z)(u^\tau) &= \frac{\partial\bar\partial \log K_\Omega(z, z)(u^\tau)}{(\log(\delta^{-1}(z)))^{1+2\eta}} + (1+2\eta)\frac{\log K_\Omega(z, z)\cdot\partial\bar\partial\delta(z)(u^\tau)}{\delta(z)(\log(\delta^{-1}(z)))^{2+2\eta}} \\ &\quad - \eta\frac{\partial\bar\partial\delta(z)(u^\tau)}{\delta(z)(\log(\delta^{-1}(z)))^{1+\eta}} \\ &= \frac{\partial\bar\partial \log K_\Omega(z, z)(u^\tau)}{(\log(\delta^{-1}(z)))^{1+2\eta}} + \frac{\partial\bar\partial\delta(z)(u^\tau)}{\delta(z)(\log(\delta^{-1}(z)))^{1+2\eta}} \\ &\quad \times \left((1+2\eta)\frac{\log K_\Omega(z, z)}{\log \delta^{-1}(z)} - \eta\big(\log \delta^{-1}(z)\big)^{\eta}\right). \end{aligned} \tag{5.2}$$

Here, the last line between brackets is negative when z approaches $b\Omega$ because its first term stays bounded whereas the second diverges to $-\infty$.

Since Ω is pseudoconvex at z_o, then $\partial\bar{\partial}\delta(z)(u^\tau) \leq 0$. Combining with Theorem 1.4(ii), we obtain

$$\begin{aligned}\partial\bar{\partial}\varphi(z)(u^\tau) &\geq \frac{B_\Omega(z,u^\tau)^2}{\log(\delta^{-1}(z))^{1+2\eta}}\\ &\gtrsim \frac{(f(\delta^{-1+\eta}(z)))^2}{(\log\delta^{-1+\eta}(z))^2(\log(\delta^{-1}(z)))^{1+2\eta}}|u^\tau|^2\\ &\sim \left(\frac{f}{\log^{\frac{3}{2}+\eta}}(\delta^{-1+\eta}(z))\right)^2|u^\tau|^2, \quad z \text{ near } b\Omega. \qquad (5.3)\end{aligned}$$

The inequality (5.3) implies the proof of the theorem. □

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