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COMPACTNESS ESTIMATES FOR $\Box_b$ ON A CR MANIFOLD

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Abstract. This paper aims to state compactness estimates for the Kohn-Laplacian on an abstract CR manifold in full generality. The approach consists of a tangential basic estimate in the formulation given by the first author in his thesis, which refines former work by Nicoara. It has been proved by Raich that on a CR manifold of dimension $2n-1$ which is compact pseudoconvex of hypersurface type embedded in the complex Euclidean space and orientable, the property named "$(CR-P_q)$" for $1 \leq q \leq \frac{n-1}{2}$, a generalization of the one introduced by Catlin, implies compactness estimates for the Kohn-Laplacian $\Box_b$ in any degree k satisfying $q \leq k \leq n-1-q$. The same result is stated by Straube without the assumption of orientability. We regain these results by a simplified method and extend the conclusions to CR manifolds which are not necessarily embedded nor orientable. In this general setting, we also prove compactness estimates in degree $k=0$ and $k=n-1$ under the assumption of $(CR-P_1)$ and, when $n=2$, of closed range for $\bar\partial_b$. For $n \geq 3$, this refines former work by Raich and Straube and separately by Straube.

1. Introduction and statements

Let M be a compact pseudoconvex CR manifold of hypersurface type of real dimension $2n-1$ endowed with the Cauchy-Riemann structure $T^{1,0}M$. We choose a basis $L_1, ..., L_{n-1}$ of $T^{1,0}M$, the conjugated basis $\bar L_1, ..., \bar L_{n-1}$ of $T^{0,1}M$, and a transversal, purely imaginary, vector field T. We also take a hermitian metric on the complexified tangent bundle in which we get an orthogonal decomposition $\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T$. We denote by $\omega_1, ..., \omega_{n-1}, \bar\omega_1, ..., \bar\omega_{n-1}, \gamma$ the dual basis of 1-forms. We denote by $\mathcal{L}_M$ the Levi form defined by $\mathcal{L}_M(L, \bar L') := d\gamma(L, \bar L')$ for L, $L' \in T^{1,0}M$. The coefficients of the matrix (c_{ij}) of $\mathcal{L}_M$ in the above basis are described through the Cartan formula as

$$c_{ij} = \langle \gamma, [L_i, \bar L_j] \rangle.$$

We denote by $\mathcal{B}^k$ the space of $(0,k)$-forms u with C^∞ coefficients. They are expressed, in the local basis, as $u = \sum'_{|J|=k} u_J \bar\omega_J$ for $\bar\omega_J = \bar\omega_{j_1} \wedge ... \wedge \bar\omega_{j_k}$. Associated to the Riemaniann metric $\langle \cdot, \cdot \rangle_z$, $z \in M$, and to the element of volume dV, there is a L^2-inner product $(u, v) = \int_M \langle u, v \rangle_z dV$. We denote by $(L^2)^k$ the completion of $\mathcal{B}^k$ under this norm. We also use the notation $(H^s)^k$ for the completion under the Sobolev norm H^s. The de-Rham exterior derivative induces a complex $\bar\partial_b : \mathcal{B}^k \to \mathcal{B}^{k+1}$. We denote by $\bar\partial_b^* : \mathcal{B}^k \to \mathcal{B}^{k-1}$ the adjoint and set $\Box_b = \bar\partial_b \bar\partial_b^* + \bar\partial_b^* \bar\partial_b$. Let φ be a smooth function, denote by (φ_{ij}) the matrix of the

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Levi form $\mathcal{L}_\varphi = \frac{1}{2}(\partial_b\bar\partial_b - \bar\partial_b\partial_b)(\varphi)$ in the basis above, and by $\lambda_1^{\varphi^\epsilon} \le ... \le \lambda_{n-1}^{\varphi^\epsilon}$ the ordered eigenvalues of $\mathcal{L}_\varphi$. Let L^2_φ be the L^2 space weighted by $e^{-\varphi}$ and, for $\varphi_j := L_j(\varphi)$, denote by $L_j^\varphi = L_j - \varphi_j$ the L^2_φ-adjoint of $-\bar L_j$. The following is the tangential version of the celebrated Hörmander-Kohn-Morrey basic estimate. Here we present the refinement by Khanh [Kh10] of a former statement by Nicoara [N06]. Let $z_o \in M$. For a suitable neighborhood U of z_o and a constant $c > 0$, we have

$$\begin{aligned}
&\|\bar\partial_b u\|^2_\varphi + \|\bar\partial^*_{b,\varphi} u\|^2_\varphi + c\|u\|^2_\varphi \\
&\quad\ge \sum_{|K|=k-1}{}' \sum_{i,j}(\varphi_{ij}u_{iK}, u_{jK})_\varphi - \sum_{|J|=k}{}' \sum_{j=1}^{q_o}(\varphi_{jj}u_J, u_J)_\varphi \\
&\qquad + \sum_{|K|=k-1}{}' \sum_{i,j}(c_{ij}Tu_{iK}, u_{jK})_\varphi - \sum_{|J|=k}{}' \sum_{j=1}^{q_o}(c_{jj}Tu_J, u_J)_\varphi \\
&\qquad + \frac{1}{2}\Big(\sum_{j=1}^{q_o}\|L_j^\varphi u\|^2_\varphi + \sum_{j=q_o+1}^{n-1}\|\bar L_j u\|^2_\varphi\Big),
\end{aligned} \tag{1.1}$$

for any $u \in \mathcal{B}^k_c(U)$ where q_o is any integer with $0 \le q_o \le n-1$. We now introduce a potential-theoretical condition which is a variant of the "P-property" by Catlin [C84]. In the present version it has been introduced by Raich [R10].

Definition 1.1. Let z_o be a point of M and q an index in the range $1 \le q \le n-1$. We say that M satisfies property $(CR - P_q)$ at z_o if there is a family of weights $\{\varphi^\epsilon\}$ in a neighborhood U of z_o such that

$$\begin{cases} |\varphi^\epsilon(z)| \le 1, & z \in U, \\ \sum\limits_{j=1}^{q} \lambda_j^{\varphi^\epsilon}(z) \ge \epsilon^{-1}, & z \in U \text{ and } \ker \mathcal{L}_M(z) \ne \{0\}. \end{cases} \tag{1.2}$$

It is obvious that $(CR - P_q)$ implies $(CR - P_k)$ for any $k \ge q$.

Remark 1.2. Outside a neighborhood V_ϵ of $\ker d\gamma$, the sum $\sum\limits_{j \le q} \lambda_j^{\varphi^\epsilon}$ can get negative; let $-b_\epsilon$ be a bound from below. Now, if c_ϵ is a bound from below for $d\gamma$ outside V_ϵ, by setting $a_\epsilon := \frac{\epsilon^{-1}+b_\epsilon}{qc_\epsilon}$, we have

$$\sum_{j \le q} \lambda_j^{\varphi^\epsilon} + a_\epsilon d\gamma = \sum_{j\le q} \lambda_j^{\varphi^\epsilon} + q a_\epsilon c_\epsilon \ge \epsilon^{-1} \quad \text{on the whole } U. \tag{1.3}$$

Conversely, (1.3) readily yields the second line of (1.2). This equivalence was already noticed in [S10] and justifies our abuse of notation. In fact, (1.3) is named $(CR - P_q)$ by [S10] in accordance with [R10], whereas (1.2) is named "property (P_q) in the nullspace of the Levi form".

Again, (1.3) for q implies (1.3) for any $k \ge q$.

We now state one of the two main results of the paper.

Theorem 1.3. *Let M be a compact pseudoconconvex CR manifold of hypersurface type of dimension $2n-1$. Assume that $(CR-P_q)$ holds for a fixed q with $1 \leq q \leq \frac{n-1}{2}$ over a covering $\{U\}$ of M. Then we have compactness estimates: given ϵ there is C_ϵ such that*

(1.4)
$$\|u\|^2 \leq \epsilon(\|\bar\partial_b u\|^2+\|\bar\partial_b^* u\|^2)+C_\epsilon\|u\|^2_{-1} \quad \text{for any } u\in D^k_{\bar\partial_b^*}\cap D^k_{\bar\partial_b} \text{ and } k\in[q,n-1-q],$$

where $D^k_{\bar\partial_b^*}$ and $D^k_{\bar\partial_b}$ are the domains of $\bar\partial_b^*$ and $\bar\partial_b$ respectively.

By Hodge duality between forms of complementary degree, we need the double constraint $k \geq q$ (for the positive microlocalization) and $k \leq n-1-q$ (for the negative one); this forces $q \leq \frac{n-1}{2}$. For M embedded and orientable, Theorem 1.3 is contained in [R10]. The same is proved in [S10] without the assumption of orientability. The proof of this, as well as of the theorem which follows, is given in Section 2. Let $\mathcal{H}^k = \ker\bar\partial_b \cap \ker\bar\partial_b^*$ be the space of harmonic forms of degree k. As a consequence of (1.4), we have that for $q \leq k \leq n-1-q$, the space $\mathcal{H}^k$ is finite-dimensional, $\Box_b$ is invertible over $\mathcal{H}^{k\,\perp}$ (cf. [N06] Lemma 5.3) and its inverse G_k is a compact operator. When $k=0$ and $k=n-1$ it is no longer true that it is finite-dimensional. However, if $q=1$, we have a result analogous to (1.4) also in the critical degrees $k=0$ and $k=n-1$.

Theorem 1.4. *Let M be a compact, pseudoconvex CR manifold of hypersurface type of dimension $2n-1$. Assume that property $(CR-P_q)$ holds for $q=1$ over a covering $\{U\}$ of M and, in case $n=2$, make the additional hypothesis that $\bar\partial_b$ has closed range. Then for any ϵ there is C_ϵ such that*

(1.5)
$$\|u\|^2 \leq \epsilon(\|\bar\partial_b u\|^2+\|\bar\partial_b^* u\|^2)+C_\epsilon\|u\|^2_{-1} \quad \text{for any } u\in \mathcal{H}^{k\,\perp},\ k=0 \text{ and } k=n-1.$$

In particular, G_k is compact for $k=0$ and $k=n-1$.

For $n\geq 3$ and M a boundary of a domain in $\mathbb{C}^n$, resp. embedded and orientable, Theorem 1.4 is contained in [RS08] (resp. [S10]).

2. Proofs

Proof of Theorem 1.3. We choose a local patch U where a local frame of vector fields is found for which (1.1) is fulfilled. The key point is to specify the convenient choices of q_o and φ in (1.1). Let $1=\psi^{+\,2}+\psi^{-\,2}+\psi^{0\,2}$ be a conic, smooth partition of the unity in the space $\mathbb{R}^{2n-1}$ dual to the space in which U is identified in local coordinates. Let $\Psi^{\overset{\pm}{0}}$ be the pseudodifferential operators with symbols $\psi^{\overset{\pm}{0}}$ and let $\text{id}=\Psi^+\Psi^{+\,*}+\Psi^-\Psi^{-\,*}+\Psi^0\Psi^{0\,*}$ be the corresponding microlocal decomposition of the identity. For a cut off function $\zeta^1\in C^\infty_c(U)$ we decompose a form u as

$$u^{\overset{\pm}{0}}=\zeta^1\Psi^{\overset{\pm}{0}}u \quad u\in\mathcal{B}^k_c(U),\ \zeta^1|_{\text{supp}\,u}\equiv 1. \tag{2.1}$$

For u^+ we choose $q_o=0$ and $\varphi=\varphi^\epsilon$. We also need to go back to Remark 1.2. Now, if a_ϵ has been chosen so that (1.3) is fulfilled, we remove T from our scalar products observing that, for large ξ, we have $\xi_{2n+1}>a_\epsilon$ over $\text{supp}\,\psi^+$. In the same way as in Lemma 4.12 of [N06], we conclude that for $k \geq q$

$$\begin{aligned}
\sideset{}{'}\sum_{|K|=k-1} \sum_{i,j=1,\dots,n-1} &((c_{ij}T + \varphi^\epsilon_{ij})u^+_{iK}, u^+_{jK})_{\varphi^\epsilon} \\
&\geq \sideset{}{'}\sum_{|K|=k-1} \sum_{i,j=1,\dots,n-1} ((a_\epsilon c_{ij} + \varphi^\epsilon_{ij})u^+_{iK}, u^+_{jK})_{\varphi^\epsilon} \\
&\quad - C\|u^+\|^2_{\varphi^\epsilon} - C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^+\|^2_{\varphi^\epsilon} \\
&\geq \epsilon^{-1}\|u^+\|^2_{\varphi^\epsilon} - C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^+\|^2_{\varphi^\epsilon},
\end{aligned}$$

where $\tilde{\Psi}^0 \succ \Psi^0$ and $\zeta^2 \succ \zeta^1$ in the sense that $\tilde{\psi}^0|_{\operatorname{supp}\psi^0} \equiv 1$ and $\zeta^2|_{\operatorname{supp}\zeta^1} \equiv 1$ respectively. (Here $\|u^+\|_{-1,\varphi^\epsilon} = \|\Lambda^{-1}u^+\|_{\varphi^\epsilon}$, where Λ^{-1} is the standard tangential pseudodifferential operator of order -1 in the local patch U.) Note that there is an additional term $-C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon}$ with respect to [N06]. The reason is that $(c_{ij}\xi_{2n-1} + \varphi^\epsilon_{ij})$ can get negative values, even on $\operatorname{supp}\psi^+$, when $\xi_{2n-1} < a_\epsilon$. Integration in this compact region produces the above error term. It follows that

(2.2)
$$\|u^+\|^2_{\varphi^\epsilon} \leq \epsilon(\|\bar\partial_b u^+\|^2_{\varphi^\epsilon} + \|\bar\partial^*_{b,\varphi^\epsilon} u^+\|^2_{\varphi^\epsilon}) + C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon} + C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^+\|^2_{\varphi^\epsilon},\ k=1,\dots,n-1.$$

By taking the composition $\chi(\varphi^\epsilon)$ where $\chi = \chi(t)$ is a smooth function on $\mathbb{R}^+$ satisfying $\dot\chi > 0$ and $\ddot\chi > 0$, we get

$$(\chi(\varphi^\epsilon))_{ij} = \dot\chi\varphi^\epsilon_{ij} + \ddot\chi|\varphi^\epsilon_j|^2\kappa_{ij},$$

where κ_{ij} is the Kronecker symbol. We also notice that

$$|\bar\partial^*_{b,\chi(\varphi^\epsilon)}u|^2 \leq 2|\bar\partial^*_b u|^2 + 2\dot\chi^2 \sideset{}{'}\sum_{|K|=k-1} |\sum_{j=1}^{n-1}\varphi^\epsilon_j u_{jK}|^2.$$

Remember that $\{\varphi^\epsilon\}$ are uniformly bounded by 1. Thus, if we choose $\chi = \frac{1}{4}e^{(t-1)}$, then we have that $\ddot\chi \geq 2\dot\chi^2$ for $t = \varphi^\epsilon$. For this reason, with this modified weight, we can replace the weighted adjoint $\bar\partial^*_{b,\varphi^\epsilon}$ by the unweighted $\bar\partial^*_b$ in (2.2). By the uniform boundedness of the weights, we can also remove them from the norms and end up with the estimate

$$\|u^+\|^2 \leq \epsilon(\|\bar\partial_b u^+\|^2 + \|\bar\partial^*_b u^+\|^2) + C_\epsilon\|u^+\|^2_{-1} + C_\epsilon\|\zeta^2\tilde{\Psi}^0 u\|^2, \quad k = q,\dots,n-1. \tag{2.3}$$

For u^-, we choose $q_o = n-1$ and $\varphi = -\varphi^\epsilon$. Observe that for $|\xi|$ large we have $-\xi_{2n-1} \geq a_\epsilon$ over $\operatorname{supp}\psi^-$ (cf. [N06], Lemma 4.13). Thus, we have in the current case, for $k \leq n-1-q$,

$$\begin{aligned}
&\sideset{}{'}\sum_{|K|=k-1} \sum_{i,j=1,\dots,n-1} ((c_{ij}T - \varphi^\epsilon_{ij})u^-_{iK}, u^-_{jK})_{-\varphi^\epsilon} - \sideset{}{'}\sum_{|J|=k} \sum_{j=1}^{n-1}((c_{jj}T - \varphi^\epsilon_{jj})u^-_J, u^-_J)_{-\varphi^\epsilon} \\
&\quad\geq - \sideset{}{'}\sum_{|K|=k-1} \sum_{i,j=1,\dots,n-1} ((a_\epsilon c_{ij} + \varphi^\epsilon_{ij})u^-_{iK}, u^-_{jK})_{-\varphi^\epsilon} \\
&\qquad + \sideset{}{'}\sum_{|J|=k} \sum_{j=1}^{n-1}((a_\epsilon c_{jj} + \varphi^\epsilon_{jj})u^-_J, u^-_J)_{-\varphi^\epsilon} \\
&\qquad - C\|u^-\|^2_{\varphi^\epsilon} - C_\epsilon\|u^-\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^-\|^2_{\varphi^\epsilon} \\
&\quad\geq \epsilon^{-1}\|u^-\|^2_{\varphi^\epsilon} - C\|u^-\|^2_{\varphi^\epsilon} - C_\epsilon\|u^-\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^-\|^2_{\varphi^\epsilon}.
\end{aligned}$$

Thus, we get the analogue of (2.2) for u^+ replaced by u^- and, again removing the weight from the adjoint $\bar\partial^*_{b,\varphi^\epsilon}$ and from the norms, we conclude that
(2.4)
$$\|u^-\|^2 \le \epsilon(\|\bar\partial_b u^-\|^2+\|\bar\partial^*_b u^-\|^2)+C_\epsilon\|u^-\|^2_{-1,\varphi^\epsilon}+C_\epsilon\|\zeta^2\tilde\Psi^0 u\|^2, \quad k=0,...,n-1-q.$$
In addition to (2.3) and (2.4), we have elliptic estimates for u^0:
$$\|u^0\|^2_1 \underset{\sim}{<} \|\bar\partial u^0\|^2+\|\bar\partial^*_b u^0\|^2+\|u\|^2_{-1}. \tag{2.5}$$
The same estimate also holds for u^0 replaced by $\zeta^2\tilde\Psi^0 u$. We put together (2.3), (2.4) and (2.5) and notice that
$$\begin{aligned}\|\bar\partial_b(\zeta^1\Psi^{\overset{\pm}{0}}u)\|^2 &\le \|\zeta^1\Psi^{\overset{\pm}{0}}\bar\partial_b u\|^2+\|[\bar\partial_b,\zeta^1\Psi^{\overset{\pm}{0}}]u\|^2\\ &\le \|\zeta^1\Psi^{\overset{\pm}{0}}\bar\partial_b u\|^2+\|\zeta^2\tilde\zeta\Psi^{\overset{\pm}{0}}u\|^2+\|\zeta^2\tilde\Psi^0 u\|^2,\end{aligned} \tag{2.6}$$
for $\zeta^2\succ\zeta^1$ and $\tilde\Psi^0\succ\Psi^0$. A similar estimate holds for $\bar\partial_b$ replaced by $\bar\partial^*_b$. Since $\zeta^1|_{\operatorname{supp}u}\equiv 1$, then
$$\begin{aligned}\|u\|^2 &\le \sum_{+,-,0}\|\zeta^1\Psi^{\overset{\pm}{0}}u\|^2+Op^{-\infty}(u)\\ &\le \epsilon\sum_{+,-,0}(\|(\bar\partial_b u)^{\overset{\pm}{0}}\|^2+\|(\bar\partial^*_b u)^{\overset{\pm}{0}}\|^2)+C_\epsilon\|u\|^2_{-1},\end{aligned}$$
and therefore
$$\|u\|^2\le\epsilon(\|\bar\partial_b u\|^2+\|\bar\partial^*_b u\|^2)+C_\epsilon\|u\|^2_{-1},\quad q\le k\le n-1-q. \tag{2.7}$$
We now consider u globally defined on the whole M instead of a local patch U. To pass from local to global compactness estimates is immediate (cf. e.g. [S10]). We cover M by $\{U_\nu\}$ so that in each patch there is a basis of forms in which the basic estimate holds. In the identification of U_ν to $\mathbb{R}^{2n-1}$, we suppose that the microlocal decomposition by the operators $\Psi^{\overset{\pm}{0}}$ which occur in (2.6) is well defined. We then get (2.7) and apply it to a decomposition $u=\sum_\nu\zeta_\nu u$ for a partition of the unity $\sum_\nu\zeta_\nu=1$ on M. We point out that we first take summation over $+,-,0$ on each patch U_ν and then summation over ν; this is why orientability of M is needless.

We observe that $[\bar\partial_b,\zeta_\nu]$ and $[\bar\partial^*_b,\zeta_\nu]$ are 0-order operators and, since they come with a factor of ϵ, they are absorbed in the left side of (2.7); thus (2.7) holds for any $u\in\mathcal{B}^k$. Finally, we use the density of smooth forms $\mathcal{B}^k$ into Sobolev forms $(H^1)^k$ of $D^k_{\bar\partial^*_b}\cap D^k_{\bar\partial^*_b}$ for the graph norm and get (2.7). The proof is complete. □

Proof of Theorem 1.4. We prove estimates in degree 0 (those in degree $n-1$ being similar). We first discuss the case $n>2$. We make repeated use of (2.7) in degree 1. This first implies that $\bar\partial^*_b$ has closed range on 1-forms, that is,
$$\begin{aligned}\mathcal{H}^{0\,\perp}&=(\ker\bar\partial_b)^\perp\\ &=\operatorname{range}\bar\partial^*_b.\end{aligned}$$
(Thus, if $u\in\mathcal{H}^{0\,\perp}$, then there exists a solution $v\in(L^2)^1$ to the equation $\bar\partial^* v=u$. Moreover, we can choose v belonging to $(Ker(\bar\partial^*_b))^\perp$.) This is a consequence of the following estimate:
$$\|v\|^2_0\underset{\sim}{<}\|\bar\partial^*_b v\|^2_0\qquad\text{for any } v\in(\ker\,\bar\partial^*_b)^\perp. \tag{2.8}$$

This can be proved by contradiction. If (2.8) is violated, there exists a sequence $v_\nu \in (\ker \bar\partial_b^*)^\perp$ such that $\|v_\nu\|_0^2 \equiv 1$ and $\|\bar\partial_b^* v_\nu\|_0 \to 0$. Take a subsequential L^2-weak limit v_0 of v_ν; it satisfies $v_0 \in Ker(\bar\partial_b^*) \cap (Ker(\bar\partial_b^*))^\perp$ and in particular $\|v_\nu\|_{-1} \to 0$. This violates (2.7) and proves (2.8). We also have

$$\|v\|_{-1}^2 \le \epsilon \|\bar\partial_b^* v\|_0^2 + c_\epsilon \|\bar\partial_b^* v\|_{-1}^2, \qquad \text{for any } v \in (\ker \bar\partial_b^*)^\perp. \tag{2.9}$$

The argument is similar. If (2.9) is violated, then there is a sequence $v_\nu \in (\ker \bar\partial_b^*)^\perp$ such that $\|v_\nu\|_{-1} \equiv 1$, $\|\bar\partial_b^* v_\nu\|_{-1} \to 0$ and $\|\bar\partial_b^* v_\nu\|_0 \le c$. By (2.7), $\|v_\nu\|_0 \le C'$; hence there is a subsequential L^2-weak limit $v_{\nu_k} \to v_0 \in (Ker(\bar\partial_b^*))^\perp \cap Ker(\bar\partial_b^*)$; thus $v_0 = 0$ and $\|v_{\nu_k}\|_{-1} \to 0$, a contradiction.

We now point out that $(Ker(\bar\partial_b^*))^\perp = \overline{range(\bar\partial_b)} \subset Ker(\bar\partial)$; in particular, our solution v satisfies $\bar\partial_b v = 0$. We are ready to conclude the proof for $n > 2$. We use the notation lc and sc for a large and small constant respectively. We have for any function $u \in \mathcal{H}^\perp$

$$\begin{aligned}
\|u\|^2 &= (u, \bar\partial_b^* v)\\
&= (\bar\partial_b u, v)\\
&\le \|\bar\partial_b u\| \|v\|\\
&\underset{(2.7)\text{ for } v}{\le} \|\bar\partial_b u\|(\epsilon \|\bar\partial_b^* v\| + c_\epsilon \|v\|_{-1})\\
&\underset{(2.9)}{\lesssim} \|\bar\partial_b u\|(\epsilon\|u\| + c_\epsilon \|u\|_{-1})\\
&\le lc_1\, \epsilon^2 \|\bar\partial_b u\|^2 + sc_1 \|u\|^2 + lc_2\, c_\epsilon^2 \|u\|_{-1}^2 + sc_2 \|\bar\partial_b u\|^2\\
&\le \epsilon' \|\bar\partial_b u\|^2 + c_{\epsilon'} \|u\|_{-1}^2 + sc_1 \|u\|^2,
\end{aligned} \tag{2.10}$$

for $\epsilon' = lc_1\, \epsilon^2 + sc_2$ and $c_{\epsilon'} = lc_2\, c_\epsilon^2$. By choosing sc_1 so that $sc_1 \|u\|^2$ is absorbed in the left, (2.10) yields (2.7) for u in degree 0. This concludes the proof of the case $n > 2$ for functions.

Let $n = 2$. We have only estimates for positively microlocalized 1-forms and for negatively microlocalized functions. We have to show how to get estimates for positively microlocalized functions (the argument for negative 1-forms being similar). We use our extra assumption of closed range for $\bar\partial_b$; thus for any $u \in (\ker \bar\partial_b)^\perp$ there is $v \in (\ker \bar\partial_b^*)^\perp$ such that $\bar\partial_b^* v = u$. On each U_ν we consider the positive microlocalization Ψ^+, take a pair of cut-off functions ζ_ν, $\zeta_\nu^1 \in C_c^\infty(U_\nu)$ with $\zeta_\nu^1|_{\text{supp }\zeta_\nu} \equiv 1$, and define $\Psi_\nu^+ := \zeta_\nu^1 \Psi^+ \zeta_\nu$. Note that the commutators $[\bar\partial_b^*, \Psi_\nu^+]$ and $[\bar\partial_b, \Psi_\nu^+]$ are operators with symbols of types $\dot\zeta_\nu^1 \psi^+ \zeta_\nu$, $\zeta_\nu^1 \dot\psi^+ \zeta_\nu$ and $\zeta_\nu^1 \psi^+ \dot\zeta_\nu$. All these symbols have support contained in the positive half-space $\xi_{2n-1} > 0$, and hence we have compactness estimates for 1-forms if their coefficients are subjected to the action of the corresponding pseudodifferential operators. We denote by a common symbol Φ_ν^+ all these operators coming from commutators. We have

$$\begin{aligned}
\|\Psi_\nu^+ v\| &\le \epsilon \|\bar\partial_b^* \Psi_\nu^+ v\| + c_\epsilon \|\Psi_\nu^+ v\|_{-1} + c_\epsilon \|\zeta_\nu^2 \tilde\Psi^0 \zeta_\nu v\|\\
&\le \epsilon \|\Psi_\nu^+ \bar\partial_b^* v\| + \epsilon \|\Phi_\nu^+ v\| + c_\epsilon \|\Psi_\nu^+ v\|_{-1} + c_\epsilon \|\zeta_\nu^2 \tilde\Psi^0 \zeta_\nu v\|\\
&\underset{(2.8)\text{ and }(2.9)\text{ for }+}{\le} \epsilon \|u\| + c_\epsilon \|u\|_{-1}.
\end{aligned} \tag{2.11}$$

The same estimate also holds for $\|\Phi^+_\nu v\|$. It follows that

$$
\begin{aligned}
\|\Psi^+_\nu u\|^2 &= (\Psi^+_\nu u, \Psi^+_\nu \bar\partial_b^* v)\\
&= (\Psi^+_\nu \bar\partial_b u, \Psi^+_\nu v) + (\Phi^+_\nu u, \Psi^+_\nu v) + (\Psi^+_\nu u, \Phi^+_\nu v)\\
&\le (\|\Psi^+_\nu \bar\partial_b u\| + \|\Phi^+_\nu u\| + \|\Psi^+_\nu u\|)(\|\Phi^+_\nu v\| + \|\Psi^+_\nu v\|)\\
&\underset{(2.11)}{\le} (\|\Psi^+_\nu \bar\partial_b u\| + \|u\|)(\epsilon\|u\| + c_\epsilon\|u\|_{-1})\\
&\lesssim \epsilon\|\Psi^+_\nu \bar\partial_b u\|\|u\| + c_\epsilon\|\Psi^+_\nu \bar\partial_b u\|\|u\|_{-1} + \epsilon\|u\|^2 + c_\epsilon\|u\|_{-1}\|u\|\\
&\le lc_1\,\epsilon^2\|\Psi^+_\nu \bar\partial_b u\|^2 + sc_1\|u\|^2 + sc_2\|\Psi^+_\nu \bar\partial_b u\|^2 + lc_2\,c_\epsilon^2\|u\|^2_{-1}\\
&\quad + \epsilon\|u\|^2 + sc_3\|u\|^2 + lc_3\,c_\epsilon^2\|u\|^2_{-1}\\
&\le \epsilon'\|\Psi^+_\nu \bar\partial_b u\|^2 + sc_4\|u\|^2 + c_{\epsilon'}\|u\|^2_{-1},
\end{aligned}
\tag{2.12}
$$

where $\epsilon' = lc_1\,\epsilon^2 + sc_2,\ c_{\epsilon'} = lc_2\,c_\epsilon^2 + lc_3\,c_\epsilon^2$ and $sc_4 = sc_1 + \epsilon + sc_3$. We have to recall now that the same estimate as (2.12) also holds for $\|\Psi^-_\nu u\|^2$ (the one for $\|\Psi^0_\nu u\|^2$ being trivial by ellipticity). Taking summation over $+$, $-$ and 0 on each U_ν, we get

$$\|\zeta_\nu u\|^2 \le \epsilon\|\zeta^1_\nu \bar\partial_b u\|^2 + c_\epsilon\|u\|^2_{-1} + sc\|u\|^2.$$

We now take summation over ν and choose sc so that the related term is absorbed by $\sum_\nu \|\zeta_\nu u\|^2 \sim \|u\|^2$ and end up with

$$\|u\|^2 \le \epsilon\|\bar\partial_b u\|^2 + c_\epsilon\|u\|^2_{-1} \quad \text{for any function } u. \qquad \square$$

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References

[C84] D. Catlin, Global regularity of the $\bar\partial$-Neumann problem. *Complex analysis of several variables* (Madison, Wis., 1982), *Proc. Sympos. Pure Math.* **41**, Amer. Math. Soc. (1984), 39–49. MR740870 (85j:32033)

[C87] D. Catlin, Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains, *Ann. of Math. (2)* **126** (1987), 131–191. MR898054 (88i:32025)

[FK72] G. B. Folland and J. J. Kohn, The Neumann problem for the Cauchy-Riemann complex, Ann. Math. Studies, **75**, Princeton Univ. Press, Princeton N.J. (1972). MR0461588 (57:1573)

[Kh10-] T.V. Khanh, A general method of weights in the $\bar\partial$-Neumann problem, Ph.D. thesis, arXiv:1001.5093v1

[Kh10] T.V. Khanh, Global hypoellipticity of the Kohn-Laplacian $\square_b$ on pseudoconvex CR manifolds, (2010) preprint.

[KZ09] T. V. Khanh and G. Zampieri, Estimates for regularity of the tangential $\bar\partial$ system, to appear in *Math. Nach.*

[K79] J. J. Kohn, Subellipticity of the $\overline{\partial}$-Neumann problem on pseudo-convex domains: Sufficient conditions, *Acta Math.* **142** (1979), 79–122. MR512213 (80d:32020)

[K02] J. J. Kohn, Superlogarithmic estimates on pseudoconvex domains and CR manifolds, *Annals of Math. (2)* **156** (2002), 213–248. MR1935846 (2003i:32059)

[KN65] J. J. Kohn and L. Nirenberg, Non-coercive boundary value problems, *Comm. Pure Appl. Math.* **18** (1965), 443–492. MR0181815 (31:6041)

[KN06] J.J. Kohn and A. Nicoara, The $\bar\partial_b$ equation on weakly pseudoconvex CR manifolds of dimension 3, *J. Funct. Analysis* **230** (2006), 251–272. MR2186214 (2006j:32046)

[N06] A. Nicoara, Global regularity for $\overline{\partial}_b$ on weakly pseudoconvex CR manifolds, *Adv. Math.* **199**, no. 2 (2006), 356–447. MR2189215 (2006h:32034)

[RS08] A. S. Raich and E. J. Straube, Compactness of the complex Green operator, *Math. Res. Lett.* **15**, no. 4 (2008), 761–778. MR2424911 (2009i:32044)

[R10] A.S. Raich, Compactness of the complex Green operator on CR-manifolds of hypersurface type, *Math. Ann.* **348** (2010), 81–117. MR2657435

[S10] E.J. Straube, The complex Green operator on CR submanifolds of $\mathbb{C}^n$ of hypersurface type: compactness, Trans. Amer. Math. Society, to appear.

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