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Estimates for regularity of the tangential $\bar\partial$-system

Tran Vu Khanh* and **G. Zampieri****

Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy

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We study ellipticity in a weak sense, such as fractional or logarithmic, of the system $(\bar\partial_b, \bar\partial_b^*)$ tangential to a hypersurface or a generic higher codimensional submanifold $M \subset \mathbb{C}^n$. The geometric setting which assures the estimates is the q-pseudoconvexity/concavity of M in addition to the existence of a suitable family of weights in a strip or a tube around M. The basic estimates for the $\bar\partial$-Neumann problem on q-pseudoconvex/concave domains is related to the classical work by Shaw [17] and more recent by Zampieri [19]. The method of the weights is due to Catlin [3] and the relation between the tangential and the ambient $\bar\partial$ system on pseudoconvex domains is inspired to Kohn [14]. Both these techniques are adapted here to a general Levi signature.

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1 Regularity of the $\bar\partial$-system tangential to a hypersurface

Let M be a real smooth hypersurface of $\mathbb{C}^n$ defined by $r = 0$ with $\partial r \neq 0$. We denote by $D = D^+$ and $D = D^-$ the two sides of M in a neighborhood V of a point $z_o \in M$; we assume $r < 0$ in D^+ and $r > 0$ in D^-. Let $\omega_1, \dots, \omega_n$ be an orthonormal basis of $(1,0)$ forms in a neighborhood of z_o with $\omega_n = \partial r$, and let $\partial_{\omega_1}, \dots, \partial_{\omega_n}$ be the dual basis of $(1,0)$ vector fields. For $0 \leq k \leq n$, we write a general k-form u on V as

$$u = \sum_{|J|=k}{}' u_J \bar\omega_J,$$

where $\sum'$ denotes summation restricted to ordered multiindices $J = \{j_1, \dots, j_k\}$ and where $\bar\omega_J = \bar\omega_{j_1} \wedge \cdots \wedge \bar\omega_{j_k}$. When the multiindex is no more ordered, it is understood that the coefficient u_J is antisymmetric with respect to J; in particular, if J decomposes into jK, then $u_{jK} = \operatorname{sign}\binom{J}{jK} u_J$. We define a scalar product and a norm by $\langle u, u\rangle = |u|^2 = \sum'_{|J|=k} |u_J|^2$; this definition is independent of the choice of the orthonormal basis $\omega_1, \dots, \omega_n$. The coefficients of our forms are taken in various spaces such as $C^\infty(\bar D \cap V), C^\infty(D \cap V), C^\infty_c(\bar D \cap V)$, $L^2(D \cap V)$ and the corresponding spaces of k-forms are denoted by $C^\infty(\bar D \cap V)^k$ and so on. All our discussion is local; sometimes, we omit this specification. Though our a priori estimates are proved over smooth forms, they are stated in Hilbert norms. Thus, let $\|u\|_{H^0}$ or $\|u\|_0$ be the $H^0 = L^2$ norm and, for a real function φ, let the weighted L^2-norm be defined by

$$\|u\|^2_{H^0_\varphi} := \sum_{|J|=k}{}' \int_U e^{-\varphi} |u_J|^2 dv$$

where dv is the element of volume in $\mathbb{C}^n$.

Let $\bar\partial^*$ be the adjoint of $\bar\partial$. The operator $\bar\partial^*$ is still closed, densely defined but it is not necessarily the case the smooth forms belong to $D_{\bar\partial^*}$. For this, they must satisfy certain boundary conditions. Namely, integration by

* e-mail: khanh@math.unipd.it, Phone: +39 049 8271443, Fax: +39 049 8271499
** Corresponding author: e-mail: zampieri@math.unipd.it, Phone: +39 049 8271433, Fax: +39 049 8271499

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parts shows that a form u of degree k cannot belong to $D_{\bar\partial^*}$ unless

$$\sum_{j=1}^{n}\int_M e^{-\varphi}\partial_{\omega_j}(r)u_{jK}\bar\psi_K\,da=0\quad\text{for any } K \text{ and any } \psi_K \text{ of degree } k-1,$$

where da denotes the element of area in $M=\partial D$. This means that $\sum_{j=1}^n \partial_{\omega_j}(r)u_{jK}|_{\partial D}\equiv 0$ for any K. Since we have chosen our basis with the property $\partial_{\omega_j}(r)|_{\partial D}=\kappa_{jn}$ (the Kronecker's symbol), we then conclude

$$u \text{ belongs to } D_{\bar\partial^*}\quad\text{if and only if}\quad u_J|_{\partial D}=0\quad\text{whenever}\quad n\in J. \tag{1.1}$$

Let $\delta^{\varphi}_{\omega_j}$ be the formal H^0_φ-adjoint of $-\partial_{\bar\omega_j}$; over a form that belongs to $D_{\bar\partial^*}$, the action of the Hilbert adjoint of $\bar\partial$ coincides with that of its "formal adjoint" and is therefore expressed by a "divergence operator":

$$\bar\partial^*_\varphi u=-\sum_{|K|=k-1}{}^{'}\sum_j \delta^\varphi_{\omega_j}(u_{jK})\bar\omega_K+\cdots\quad\text{for any}\quad u\in D_{\bar\partial^*}, \tag{1.2}$$

where dots denote an error term in which u is not differentiated and φ does not occur. We now recall some inequalities which are needed for the proof of our estimates. The key technical tools of our discussion are the Hörmander-Kohn-Morrey estimates.

Proposition 1.1 *Fix arbitrarily an index q_o with $0\le q_o\le n-1$. Then for a suitable $C>0$ and any $u\in C^\infty_c(\bar D\cap V)^k\cap D_{\bar\partial^*}$, we have*

$$\begin{aligned}
&\|\bar\partial u\|^2_{H^0_\varphi}+\|\bar\partial^*_\varphi u\|^2_{H^0_\varphi}+C\|u\|^2_{H^0_\varphi}\\
&\quad\ge \underbrace{\sum_{|K|=k-1}{}^{'}\sum_{i,j=1}^{n}\int_D e^{-\varphi}\varphi_{ij}u_{iK}\bar u_{jK}\,dv-\sum_{|J|=k}{}^{'}\sum_{j=1}^{q_o}\int_D e^{-\varphi}\varphi_{jj}|u_J|^2\,dv}_{(I)_D}\\
&\qquad+\underbrace{\sum_{|K|=k-1}{}^{'}\sum_{i,j=1}^{n-1}\int_{\partial D} e^{-\varphi}r^\delta_{ij}u_{iK}\bar u_{jK}\,da-\sum_{|J|=q}{}^{'}\sum_{j=1}^{q_o}\int_{\partial D} e^{-\varphi}r^\delta_{jj}|u_J|^2\,da}_{(II)_D}\\
&\qquad+(1-\alpha)\underbrace{\left(\sum_{j=1}^{q_0}\|\delta^\varphi_{\omega_j}u\|^2_{H^0_\varphi(D)}+\sum_{j=q_o+1}^{n}\|\partial_{\bar\omega_j}u\|^2_{H^0_\varphi(D)}\right)}_{(III)_D}.
\end{aligned}\tag{1.3}$$

We refer for instance to [20] for the proof of Proposition 1.1; some ideas of the proof can also be found in [17].

We denote by $T^{1,0}M$ and $T^{0,1}M$ the holomorphic and antiholomorphic tangent bundles to M respectively; they are both isomorphic to the complex tangent bundle $T^{\mathbb{C}}M:=TM\cap iTM$. Let $\{\omega\}=\{\omega',\omega_n\}$ be a basis of forms such that $\omega_n=\partial r$. Thus for the dual system of vector fields $\{\partial_\omega\}$ we have that $\partial'_{\omega_j}|_M$ and $\partial'_{\bar\omega_j}|_M$ are a basis for the tangential bundles $T^{1,0}M$ and $T^{0,1}M$ respectively. Also, $T:=\frac{i}{\sqrt2}(\partial_{\omega_n}-\partial_{\bar\omega_n})$ and $N:=\frac{1}{\sqrt2}(\partial_{\omega_n}+\partial_{\bar\omega_n})$ are the vector fields totally real tangential and normal to M respectively.

We want to exploit (1.3) when we have a geometric setting which gives a good control from below of the terms $(I)_D$ and $(II)_D$. Let $L_M:=\partial\bar\partial r|_{T^{\mathbb{C}}M}$ be the Levi form of M (from the side of $D=D^+$ defined by $r<0$) and let $\lambda_1\le\lambda_2<\cdots\le\lambda_{n-1}$ be its ordered eigenvalues and $s^\pm$ the numbers of the λ_j's which are $\gtrless 0$. We assume that there is a bundle $\mathcal{V}^{q_o}\subset T^{1,0}M$ of rank q_o with smooth coefficients on $V\cap M$ for a neighborhood V of z_o, say the bundle of the first q_o coordinate tangential vector fields $\partial_{\omega_1},\dots,\partial_{\omega_{q_o}}$, such that for $q>q_o$ or $q<q_o$

$$\sum_{j=1}^{q}\lambda_j(z)-\sum_{j=1}^{q_o}r_{jj}(z)\ge 0\quad\text{for}\quad z\in M\cap V. \tag{1.4}$$

Definition 1.2 We say that the hypersurface M is *q-pseudoconvex* (resp. *q-pseudoconcave*) from the side D in a neighborhood of z_o, when (1.4) holds for $q>q_o$ (resp. $q<q_o$).

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Note that the definition differs from [19] and [20] where (1.4) defines $(q-1)$-pseudoconvexity ($(q-1)$-pseudoconcavity). There is a basic relation between pseudoconvexity/concavity of the two sides $D^\pm$ of M.

Proposition 1.3 *The domain D^+ is q-pseudoconvex if and only if D^- is $(n-1-q)$-pseudoconcave in a neighborhood of z_o.*

Proof. We have

$$\sum_{j=1}^{q}\lambda_j-\sum_{j=1}^{q_o}r_{jj}=\left(\sum_{j=1}^{n-1}\lambda_j-\sum_{j=q+1}^{n-1}\lambda_j\right)-\left(\sum_{j=1}^{n-1}r_{jj}-\sum_{j=q_o+1}^{n-1}r_{jj}\right)$$
$$=-\sum_{j=q+1}^{n-1}\lambda_j+\sum_{j=q_o+1}^{n-1}r_{jj},$$

where the second equality follows from the identity $\sum_{j=1}^{n-1}\lambda_j=\sum_{j=1}^{n-1}r_{jj}$. On the other hand, $-r$ is the defining function for D^- and

$$-\lambda_{n-1}\le-\lambda_{n-2}\le\cdots\le-\lambda_1,$$

the ordered eigenvalues of D^-. $\square$

Remark 1.4 The notion of q-pseudoconvexity and q-pseudoconcavity was used in [19] to prove the existence of solutions to the equation $\bar\partial u=f$ smooth up to the boundary in $\bar D$. Here the problem is different: we search for estimates which assure local hypoellipticity or compactness, that is, local or global regularity of the canonical solution.

Remark 1.5 Assume that (1.4) holds for $q>q_o$. Then, $\lambda_q\ge 0$. Thus (1.4) still holds with q replaced by k for any $k\ge q$. Similarly, if $q<q_o$, then $\lambda_q^-\le 0$. In this case, (1.4) holds with q replaced by k for any $k\le q$.

Remark 1.6 When we have strict inequality "$>$" in (1.4) for $q>q_o$ (resp. $q<q_o$), it means that we have in fact $\lambda_q^+>0$ (resp. $\lambda_{q+1}^-<0$). It follows

$$q\ge n-s^+\quad\text{(resp. } q\le s^--1).\tag{1.5}$$

We refer to these two situations as "strong" q-pseudoconvexity (resp. strong q-pseudoconcavity). Note that this amounts as to say, in the terminology of Folland-Kohn, that M satisfies $Y(k)$ for any $k\ge q$ (resp. $k\le q$).

Example 1.7 (1) Let M be pseudoconvex in the usual sense; then D^+ is 1-pseudoconvex and D^- is $(n-2)$-pseudoconcave.

(2) More generally, let the number of negative eigenvalues s^- be constant in a neighborhood of z_o; then D^+ is (s^-+1)-pseudoconvex and D^- is $(n-1-s^-)$-pseudoconcave.

We use the notation $Q_D(u,u):=\|\bar\partial u\|_D^2+\|\bar\partial^*u\|_D^2$ (with $\bar\partial^*$ unweighted); we also write Q instead of Q_D. We denote by k the degree of a form u. We make a first crucial remark: if M is q-pseudoconvex, then in the estimate (1.3) for u we have $(II)_{D^+}\ge 0$. By the trivial choice $\varphi\equiv 0$ (which yields $(I)_D=0$) we then get for $u\in D_{\bar\partial^*}$

$$\sum_{j=1}^{q_o}\left\|\partial_{\omega_j}u\right\|^2+\sum_{j=q_o+1}^{n}\left\|\partial_{\bar\omega_j}u\right\|^2\lesssim Q(u,u)+\|u\|^2\quad\text{if}\quad k\ge q\quad\text{(resp. } k\le q).$$

We also notice that

$$\left\|\partial_{\omega_j}u\right\|^2=\left\|\partial_{\bar\omega_j}u\right\|^2+\text{error},\quad j=1,\dots,n\quad\text{if}\quad u|_{\partial D}=0,$$

where "error" stands for the integral of the product of a first derivative of u by u. Since ∂_{ω_j} and $\partial_{\bar\omega_j}$ for $j=1,\dots,n$ are a full basis of derivatives in $\mathbb{R}^{2n}\simeq\mathbb{C}^n$, then we get in particular, if M is q-pseudoconvex (resp. -pseudoconcave) and $k\ge q$ (resp. $k\le q$)

$$\|u\|_1^2\lesssim Q(u,u)+\|u\|^2\quad\text{if}\quad u|_{\partial D}=0.\tag{1.6}$$

We deal with forms in M that we denote by u_b; these are related to forms u on D belonging to $D_{\bar\partial^*}$ by

$$u|_{\partial D} = u_b.$$

Of course, in the above relation, there is no uniqueness between "extension" u and "trace" u_b; not even if we reduce a form u of $D_{\bar\partial^*}$ to its "tangential component" in which we discard the coefficients u_J with $n \in J$. We will show a way to take a "distinguished" tangential extension. First, we recall the microlocal decomposition by Kohn [14]. We introduce special coordinates $(a, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ so that a serves as a local coordinate for M and, moreover, $\partial_{a_{2n-1}} = T_{z_o}$ for the totally real tangential vector field T. We denote by $\xi = (\xi_a, \xi_r)$ the dual coordinates and by $\mathcal{F}_\tau$ the partial Fourier transform with respect to a. We denote by $\Lambda_{\xi_a} = \left(1 + |\xi_a|^2\right)^{\frac{1}{2}}$ the standard elliptic symbol of order 1 and by $\mathcal{F}_\tau$ the tangential Fourier trnsform. We first introduce the "s-tangential" derivative Λ^s_τ, $s \in R$. This is the pseudodifferential operator of symbol $\Lambda^s_{\xi_a}$ whose action on C^∞_c forms is defined, coefficientwise, by $\Lambda^s_\tau u(a, r) := \mathcal{F}^{-1}_\tau\left(\Lambda^s_{\xi_a} \mathcal{F}_\tau u(\xi_a, r)\right)$. This way of relating operators to symbols will be used throughout the paper. We also define the tangential Sobolev norm by $|||u|||_s := \|\Lambda^s_\tau u\|^2_0$. We now introduce standard microlocalization operators. We consider a conic partition of the unity in $\mathbb{R}^{2n-1}_{\xi_a}$

$$1 = \psi^+ + \psi^- + \psi^0,$$

where $\text{supp}(\psi^\pm) \subset \left\{\xi \in \mathbb{R}^{2n-1} : \pm\xi_{a_{2n-1}} \geq \left(\sum_{j=1}^{2n-1} |\xi_j|^2\right)^{\frac{1}{2}}\right\}$ and $\text{supp}(\psi^0) \subset \left\{\xi \in \mathbb{R}^{2n-1} : |\xi_{a_{2n-1}}| \leq 2\left(\sum_{j=1}^{2n-1} |\xi_j|^2\right)^{\frac{1}{2}}\right\}$. We further consider the associated pseudodifferential decomposition of the identity

$$\text{id} = \Psi^+ + \Psi^- + \Psi^0; \tag{1.7}$$

here $\Psi^{\overset{\pm}{0}}$ are the operators with symbols $\sigma\left(\Psi^{\overset{\pm}{0}}\right) = \psi^{\overset{\pm}{0}}$. Let $U \subset\subset U' \subset\subset M$; we assume that $\text{supp}\, u \subset\subset U$ and take a cut off function $\zeta \in C^\infty_c(U')$ with $\zeta|_U \equiv 1$. Formula (1.7) yields two decompositions

$$u_b = \zeta\Psi^+ u_b + \zeta\Psi^- u_b + \zeta\Psi^0 u_b = u^+_b + u^-_b + u^0_b,$$

and

$$u = \zeta\Psi^+ u + \zeta\Psi^- + \zeta\Psi^0 u = u^+ + u^- + u^0.$$

We define

$$\tilde u^+(a, r) = \zeta(a)\, \frac{1}{(2\pi)^{2n-1}} \int e^{ia\cdot\xi_a + r\sigma(T)} \psi^+(\xi_a)\mathcal{F}_\tau u(\xi_a, 0)\, d\xi_a. \tag{1.8}$$

Note that $\sigma(T) \sim |\xi_a|$ over $\text{supp}(\psi^+)$ and recall that $r < 0$ on D; thus the absolute value of the exponential is $\leq e^{-|r||\xi_a|}$. We point out that we can think of $\tilde u^+$ in two different ways: either as an extension of $u^+_b = u^+(a, 0)$ or as a modification of u^+ satisfying $\tilde u^+|_{\partial D} = u^+|_{\partial D}$; in any case we have on U

$$\tilde u^+|_{\partial D} = u^+|_{\partial D} = u^+_b.$$

There are three basic properties for $\tilde u^+$. First, notice that (cf. [14] p. 241 line 8 from the bottom)

$$\partial_{\bar\omega_n} = \frac{1}{\sqrt{2}}(\partial_r + iT),$$

owing to

$$\begin{cases} T = \dfrac{i}{\sqrt{2}}(\partial_{\omega_n} - \partial_{\bar\omega_n}), \\ \partial_r = \dfrac{1}{\sqrt{2}}(\partial_{\omega_n} + \partial_{\bar\omega_n}). \end{cases}$$

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It follows

$$\partial_{\bar{\omega}_n} \tilde{u}^+ \equiv 0. \tag{1.9}$$

The second, is a relation between $\tilde{u}^+$ and u_b^+; this is not related to the specific choice of the extension $\tilde{u}^+$ but just to the property $\tilde{u}^+|_{\partial D} = u_b^+$. This is

$$\|u_b^+\|^2 \lesssim \frac{1}{\epsilon}|||\tilde{u}^+|||^2_{\frac{1}{2}} + \epsilon\, |||\partial_r \tilde{u}^+|||^2_{-\frac{1}{2}}. \tag{1.10}$$

To prove it, we take a cut-off function $\chi = \chi(r)$ with $\chi(0) = 1$ and remark that

$$|u_b^+|^2 \leq \int_{-\infty}^0 \partial_r |\chi \tilde{u}^+|^2 dr \lesssim \frac{1}{\epsilon}\int_{-\infty}^0 |\chi\tilde{u}^+|^2 dr + \epsilon \int_{-\infty}^0 |\partial_r \chi \tilde{u}^+|^2 dr, \tag{1.11}$$

where the second inequality is a consequence of the small/large constant argument. We can also take tangential Fourier transform of the two sides of (1.11) and replace ϵ by $\epsilon\big(1 + |\xi_a|^2\big)^{\frac{1}{2}}$ (and similarly for the inverse) since this is constant with respect to the integration variable r. Taking inverse Fourier transform we get (1.10). Variants of (1.10) are obtained by replacing the fractional $\frac{1}{2}$-derivative by a real s-derivative or a more general pseudodifferential operator.

The third relation between $\tilde{u}^+$ and u_b^+, which is specific of $\tilde{u}^+$, is

$$\|\tilde{u}^+\|^2 \lesssim \|u_b^+\|^2_{-\frac{1}{2}}. \tag{1.12}$$

To prove (1.12), we notice that by the change $r|\xi_a| = r'$, we get

$$\begin{aligned}
\|\tilde{u}^+\|^2 &\lesssim \iint e^{2r|\xi_a|} \big|\mathcal{F}_\tau u_b^+(\xi_a, r)\big|\, d\xi_a\, dr \\
&\lesssim \iint e^{2r'}\big(1 + |\xi_a|^2\big)^{-\frac{1}{2}} \left|\mathcal{F}_\tau u_b^+\left(\xi_a, \frac{r'}{|\xi_a|}\right)\right| d\xi_a\, dr' \\
&\lesssim \int \big(1 + |\xi_a|^2\big)^{-\frac{1}{2}} \big|\mathcal{F}_\tau u_b^+(\xi_a, 0)\big|\, d\xi_a = \|u_b^+\|^2_{-\frac{1}{2}} \quad \text{(by Plancherel).}
\end{aligned}$$

A similar calculation shows that $|||\partial_r \tilde{u}^+|||_{-\frac{1}{2}} \lesssim \|\tilde{u}_b^+\|^2_0$. Thus, for the specific extension $\tilde{u}^+$, (1.10) converts into the better estimate (1.12). Again, we have several variants of (1.12) with the $\frac{1}{2}$-fractional derivative replaced by general pseudodifferential operators. We introduce these operators. We consider a real smooth monotonic increasing function $f(t),\ t \geq 1$; since we wish these operators to be dominated by the $\frac{1}{2}$-fractional derivative, we take $f \lesssim t^{\frac{1}{2}}$. We define $f(\Lambda_\tau)$ as the pseudodifferential operator with symbol $f(\Lambda_{\xi_a})$; in particular, $\Lambda_\tau^{\frac{1}{2}}$ and Λ_τ^s are the $\frac{1}{2}$-fractional and s-real derivative respectively. With the operator $f(\Lambda_\tau)$ in our hands, we pass to consider the estimate

$$\|f(\Lambda_\tau)u\|^2 \lesssim Q(u, u) + \|u\|^2 \quad \text{for any} \quad u \in D_{\bar{\partial}^*} \cap C_c^\infty(\bar{D} \cap V)^k \quad \text{with} \quad k \geq q \,(\text{or}\, k \leq q). \tag{1.13}$$

We are also interested in the tangential version of (1.13), that is,

$$\|f(\Lambda_\tau)u_b\|^2 \lesssim Q_b(u_b, u_b) + \|u_b\|^2 \quad \text{for} \quad u_b \in C_c^\infty(M \cap V)^k.$$

We wish to compare this estimate with its tangential version for u_b. Classically, the main interest in (1.13) is when $f\big(\delta^{-1}\big)$ has a sufficiently high rate to infinity when $\delta \to 0$. Thus, if $f\big(\delta^{-1}\big) = \delta^{-\epsilon}$ then (1.13) are the celebrated "subelliptic estimates". As it is classical, they yield the local hypoellipticity of the system $\big(\bar{\partial}_b, \bar{\partial}_b^*\big)$. The solution u to $\big(\bar{\partial}u = f, \bar{\partial}^* u = 0\big)$ or the solution u_b to $\big(\bar{\partial}_b u_b = f_b, \bar{\partial}_b^* u_b = 0\big)$ are smooth exactly where f or f_b are smooth (cf. [7]). Another classical case is $\frac{f(\delta^{-1})}{\log(\delta^{-1})} \to \infty$; in this situation (1.13) readily implies the so-called "superlogarithmic estimates" and they still suffice for hypoellipticity. The last case is when we simply have $f\big(\delta^{-1}\big) \to \infty$. In this case (1.13) implies the so-called "compactness estimate"; this does not always suffice for local hypoellipticity (cf. e.g. the discussion in [10] about Christ example [5]).

The following theorem is contained in [14] under the choice $f(\Lambda_\tau) = \Lambda_\tau^\epsilon$ and for pseudoconvex domains.

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Theorem 1.8 *Let D be q-pseudoconvex (resp. -pseudoconcave). Then, for any pseudodifferential operator $f(\Lambda_\tau)$ associated to a general smooth monotonic increasing function f as described above, we have*

$$\begin{aligned}\big\|f(\Lambda_\tau)u^+\big\|^2 \lesssim Q\big(u^+,u^+\big)+\big\|u^+\big\|^2 \quad &\textit{for any}\quad u\in D_{\bar\partial^*}\cap C^\infty_c(\bar D\cap V)^k\\ &\textit{with}\quad k\ge q\quad (resp. k\le q),\end{aligned}\tag{1.14}$$

if and only if

$$\begin{aligned}\big\|f(\Lambda_\tau)u_b^+\big\|_b^2 \lesssim Q_b\big(u_b^+,u_b^+\big)+\big\|u_b^+\big\|_b^2 \quad &\textit{for any}\quad u_b\in C^\infty_c(M\cap V)^k\\ &\textit{with}\quad k\ge q\quad (resp. k\le q).\end{aligned}\tag{1.15}$$

Proof. We first prove that (1.14) implies (1.15). We recall that $\partial_r = \frac{2}{\sqrt2}\partial_{\bar\omega} - iT$ and $\big\|\partial_{\bar\omega_n}\tilde u^+\big\|^2 \le Q\big(\tilde u^+,\tilde u^+\big)$. It follows

$$\begin{aligned}\big\|f(\Lambda_\tau)u_b^+\big\|_b^2 &\lesssim \big|\big|\big|f(\Lambda_\tau)\chi\tilde u^+\big|\big|\big|^2_{\frac12}+\big|\big|\big|f(\Lambda_\tau)\chi\partial_r\tilde u^+\big|\big|\big|^2_{-\frac12}\\ &\lesssim Q\big(\chi\tilde u^+,\chi\tilde u^+\big)+2\big|\big|\big|f(\Lambda_\tau)\chi\tilde u^+\big|\big|\big|^2_{\frac12}\\ &\lesssim Q\big(\chi\Lambda_\tau^{\frac12}\tilde u^+,\chi\Lambda_\tau^{\frac12}\tilde u^+\big)+\big|\big|\big|\chi\tilde u^+\big|\big|\big|^2_{\frac12}\\ &\lesssim Q_b\big(u_b^+,u_b^+\big)+\big\|u_b^+\big\|^2,\end{aligned}$$

where the first inequality follows from (1.10), the second and the third from (1.14), and the fourth from (1.9) combined with (1.12) respectively.

We prove that (1.15) implies (1.14). We denote by $\{M_j\}$ the system $\{\delta_{\omega_j}\}_{1\le j\le q_o}\cup\{\partial_{\bar\omega_j}\}_{q_o+1\le j\le n}$ and denote by ζ a tangential cut-off with the property $\zeta\equiv1$ over supp(u). We first point our attention to $\tilde u^+$; we have

$$\begin{aligned}\big\|f(\Lambda_\tau)\tilde u^+\big\|_D^2 &\lesssim \big\|f(\Lambda_\tau)u_b^+\big\|^2_{-\frac12\, b}\\ &\lesssim Q_b\big(\zeta\Lambda_\tau^{-\frac12}u_b^+,\zeta\Lambda_\tau^{-\frac12}u_b^+\big)+\big\|\Lambda_\tau^{-\frac12}u_b^+\big\|_b^2\\ &\lesssim \underbrace{\sum_{j=1}^{n-1}\big\|M_j\zeta\Lambda_\tau^{-\frac12}u_b^+\big\|_b^2}_{(I)} + \big\|\Lambda_\tau^{-\frac12}u^+b\big\|_b^2\\ &\quad+\underbrace{\sum_{ij}\sideset{}{'}\sum_{|K|=k-1}\big(c_{ij}T\zeta\Lambda_\tau^{-\frac12}u_{iK},\zeta\Lambda_\tau^{-\frac12}u_{jK}\big)_b-\sum_{j=1}^{q}\big(c_{jj}T\zeta\Lambda_\tau^{-\frac12}u_J,\zeta\Lambda_\tau^{-\frac12}u_J\big)_b}_{(II)},\end{aligned}\tag{1.16}$$

where the first inequality follows from (1.12), the second from (1.15) and the third from the basic estimate (1.3) respectively. Now, to estimate (I), we use

$$\begin{aligned}\big\|M_j\Lambda_\tau^{-\frac12}u_b^+\big\|^2 &\lesssim \big(\partial_rM_j\Lambda_\tau^{-\frac12}u^+,M_j\Lambda_\tau^{-\frac12}u^+\big)_D\\ &\lesssim \big(\partial_r\Lambda_\tau^{-1}M_ju^+,M_ju^+\big)_D+\big\|u^+\big\|_D^2\\ &\lesssim \big(\Lambda_\tau^{-1}M_jM_nu^+,M_ju^+\big)_D+\big(\Lambda_\tau^{-1}TM_ju^+,M_ju^+\big)_D+\big\|u^+\big\|_D^2\\ &\lesssim \big\|M_ju^+\big\|_D^2+\big\|M_nu^+\big\|_D^2+\big\|u^+\big\|_D^2\\ &\le Q\big(u^+,u^+\big)+\big\|u^+\big\|_D^2.\end{aligned}\tag{1.17}$$

On the other hand it is evident that

$$(II)\lesssim Q\big(u^+,u^+\big)+\big\|u^+\big\|^2.\tag{1.18}$$

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Finally we have

$$\begin{aligned}\big\|\Lambda_\tau^{-\frac{1}{2}}u_b^+\big\|^2 &\le \big\||\partial_r u^+|\big\|_{-1}+\big\|u^+\big\|^2\\ &\le \big\|M_n u^+\big\|^2+\big\|u^+\big\|^2 \le Q\big(u^+,u^+\big)+\big\|u^+\big\|^2.\end{aligned}\tag{1.19}$$

This proves that $\big\|f(\Lambda_\tau)\tilde u^+\big\|^2 \lesssim Q\big(u^+,u^+\big)+\big\|u^+\big\|^2$. To conclude the proof, we have to estimate $u^{(0)}:=u^+-\tilde u^+$. We have

$$\begin{aligned}\big\|u^{(0)}\big\|_1^2 &\lesssim Q\big(u^{(0)},u^{(0)}\big)+\big\|u^{(0)}\big\|^2\\ &\le Q\big(u^+,u^+\big)+\big\|u^+\big\|^2+\underbrace{Q\big(\tilde u^+,\tilde u^+\big)+\big\|\tilde u^+\big\|^2}_{(III)}.\end{aligned}$$

It remains to estimate (III). For this, we use

$$(III)\lesssim Q_b\big(\zeta\Lambda_\tau^{-\frac{1}{2}}u_b^+,\zeta\Lambda_\tau^{-\frac{1}{2}}u^+b\big)+\big\|\Lambda_\tau^{-\frac{1}{2}}u_b^+\big\|_b^2,$$

and then estimate the right-hand side by $Q\big(u^+,u^+\big)+\big\|u^+\big\|^2$ in the same way as in (1.16)–(1.19).

This concludes the proof. □

Remark 1.9 According to the theory by Kohn, we have

$$|||u^-|||_1^2 \lesssim \sum_{j=1}^n \|M_j u\|^2+\|u\|^2 \le Q(u,u)+\|u\|^2.\tag{1.20}$$

The same result for u^0 is easy; its argument is contained in the proof of Theorem 1.11 below. Since $f(\Lambda_\tau)$ is dominated by Λ_τ^1 and since $Q\big(u^+,u^+\big)\lesssim Q(u,u)+\|u\|^2$ $\big($because $\big[Q,\Psi^+\big]$ is an error term$\big)$, then (1.14) is equivalent to

$$\|f(\Lambda_\tau)u\|^2 \lesssim Q(u,u)+\|u\|^2.\tag{1.21}$$

In particular, (1.15) implies (1.21).

We introduce now the “Hodge-star” correspondence of forms

$$\begin{array}{ccc}\wedge^k T^{*\,(0,1)}M & \longrightarrow & \wedge^{n-1-k}T^{*\,(0,1)}M\\ \sum u_J\bar\omega_J & \longmapsto & \sum \epsilon^{JJ'}_{(1,\dots,n)}\bar u_J\bar\omega_{J'},\end{array}$$

where J' is the index complementary to J. We have

Proposition 1.10 *Estimate* (1.15) *is equivalent to*

$$\begin{aligned}\big\|f(\Lambda_\tau)\bar u_b^-\big\|^2 \lesssim Q_b\big(u_b^-,u_b^-\big)+\big\|u_b^-\big\|^2 \quad &\textit{for any}\quad u_b\in C_c^\infty(M\cap V)^k\\ &\textit{with}\quad k\le n-q-1\quad (\textit{resp. } k\ge n-q-1).\end{aligned}\tag{1.22}$$

P r o o f. The proof is a direct consequence of the identity

$$\overline{u_J^+}=(\bar u_J)^-.$$

□

Assume, without loss of generality, that $q\le n-1-q$. We have

Theorem 1.11 *Let D be q-pseudoconvex or $(n-q-1)$-pseudoconcave. Then*

$$\big\|f(\Lambda_\tau)u^+\big\|^2 \lesssim Q\big(u^+,u^+\big)+\big\|u^+\big\|^2,\quad u\in C_c^\infty(\bar D\cap V)^k\quad \textit{with}\quad q\le k\le n-1-q\tag{1.23}$$

if and only if

$$\|f(\Lambda_\tau)u_b\|^2 \lesssim Q(u_b,u_b)+\|u_b\|^2,\quad u_b\in C_c^\infty(M\cap V)^k\quad \textit{with}\quad q\le k\le n-1-q.\tag{1.24}$$

Proof. Estimate (1.23) implies (1.24) for u_b^+ and, according to (1.22), this implies (1.24) for u_b^- (since the interval $[q, n-1-q]$ is stable under complementation over $n-1$). Thus, it only remains to estimate u_b^0. We give the detail of the proof of this result which was already announced in Remark 1.9 for u^0 instead of u_b^0. For this, we remark that

$$\|Tu_b^0\|^2 \leq \sum_{j=1}^{n-1} \|\partial_{\bar{\omega}_j} u_b^0\|^2 + \sum_{j=1}^{n-1} \|\partial_{\omega_j} u_b^0\|^2,$$

which is a consequence of the fact that for the symbols of the operators we have the estimate $|\sigma(T)| \lesssim \sum_{j=1}^{n-1}(|\sigma(\partial_{\omega_j})| + |\sigma(\partial_{\bar{\omega}_j})|)$ over supp(ψ^0). On the other hand, integration by parts yields

$$\|\partial_{\omega_j} u_b^0\|^2 \leq \|\partial_{\bar{\omega}_j} u_b^0\|^2 + \epsilon\|Tu_b^0\|^2 + C_\epsilon\|u_b^0\|^2.$$

It follows that

$$\begin{aligned} \|u_b^0\|_1^2 &\lesssim \sum_{j=1}^{n-1} \|\partial_{\bar{\omega}_j} u_b^0\|^2 + \|u_b^0\|^2 \\ &\leq Q(u_b^0, u_b^0) + \|u_b^0\|^2 \leq Q(u_b, u_b) + \|u_b\|^2, \end{aligned} \tag{1.25}$$

which follows from the fact that the commutator $[Q, \Psi^0]$ only introduces an error term. □

As already pointed out in Remark 1.9, (1.24) implies in fact (1.21) for $q \leq k \leq n-1-q$.

Definition 1.12 Let D be q-pseudoconvex (-pseudoconcave) in a neighborhood of z_o; it is said to satisfy $(f\text{-}P\text{-}q)$ when there is a family of weights $\varphi = \varphi^\delta$, absolutely bounded on $S_\delta \cap V$ where S_δ is the δ strip $\{z \in D : \text{dist}(z, \partial D) < \delta\}$ with the eigenvalues λ_j^φ of $\partial\bar{\partial}\varphi$ satisfying

$$\sum_{j=1}^{q} \lambda_j^\varphi - \sum_{j=1}^{q_o} \varphi_{jj} \geq f^2(\delta^{-1}) + \sum_{j=1}^{q_o} |\partial_{\omega_j}\varphi|^2 \quad \text{for} \quad q > q_o \quad (\text{resp. } q < q_o). \tag{1.26}$$

Similarly as for (1.4) we have that if (1.26) holds for q, then it also holds for any $k \geq q$. This is obvious once one notices that (1.26) forces $\lambda_q^\varphi \geq 0$. For this reason, any estimate which comes from the combination of (1.4) with (1.26) and which is true for q-forms is also true for k-forms for any $k \geq q$. We recall now our result of [10]: if D is q-pseudoconvex (-pseudoconcave) and satisfies $(f\text{-}P\text{-}q)$ then (1.21) holds for any $k \leq q$ (resp. $k \geq q$). In particular, (1.23) holds for $q \leq k \leq n-1-q$. In combination with Theorem 1.11 this yields

Theorem 1.13 *Let D be q-pseudoconvex (-pseudoconcave) and satisfy $(f\text{-}P\text{-}q)$; then*

$$\|f(\Lambda_\tau)u_b\|^2 \lesssim Q(u_b, u_b) + \|u_b\|^2 \quad \textit{for any} \quad u_b \in C_c^\infty(M \cap V)^k \quad \textit{with} \quad q \leq k \leq n-1-q. \tag{1.27}$$

(Again, we are supposing $q \leq n-1-q$; otherwise, we have to take in (1.27) k which satisfies $n-1-q \leq k \leq q$.) In the particular case in which (1.4) and (1.26) hold for $q_o = 0$, Theorem 1.13 is contained in [16] Theorem 1.4. A general condition of regularity without compactness, but still assuming $q_o = 0$ in (1.4), is discussed in [18].

Remark 1.14 Recall that according to Proposition 1.3, q-pseudoconvexity of D^+ implies $(n-1-q)$-pseudoconcavity of D^-. However, $(f\text{-}P\text{-}q)$ of D^+ does not imply $(f\text{-}P\text{-}(n-1-q))$ for D^-. This is why (1.27) makes a full use of Theorem 1.11 and could not be obtained as a combination of Theorem 1.8 from the two sides D^+ and D^- $\big($in addition to the estimate (1.25) for $\|u_b^0\|_1^2\big)$. Instead, $f(\Lambda_\tau)$-estimates are in correspondence from D^+ to D^- (in complementary degrees). This follows from Theorem 1.8 and the equivalence of (1.15) with (1.22).

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Example 1.15 We regain many results of the literature about subelliptic, superlogarithmic and compactness estimates in CR manifolds. Here are a few new examples.

(i) Let D^+ be "q-decoupled-pseudoconvex of finite type $2m$". The model is the domain defined by $2\mathrm{Re}\, z_n > h(z_1,\dots,z_{q_o}) + \sum_{j=q_o+1}^{n-1}|z_j|^{2m_j}$ for $\partial\bar\partial h \le 0$; in this case the type is $2m = 2\max_j m_j$. It satisfies $(f\text{-}P\text{-}q)$ for $f = t^\epsilon$ with $\epsilon = \frac{1}{m}$. The complement D^- is $(n-q-1)$-pseudoconcave and satisfies $(f\text{-}P\text{-}(n-1-q))$. Thus (1.27) holds with $f(\Lambda_\tau) = \Lambda^\epsilon$ in degree $q \le k \le (n-q-1)$.

(ii) Let D^+ be defined by $\mathrm{Re}\, z_n = \sum_{j=1}^{n-1} e^{-\frac{1}{|z_j|^\alpha}}$ or $\mathrm{Re}\, z_n = \sum_{j=1}^{n-1} e^{-\frac{1}{|x_j|^\alpha}}$. The domain D^+ is pseudoconvex and satisfies $(f\text{-}P\text{-}1)$ for $f = \log t^{\frac{1}{\alpha}}$. The domain D^- is $(n-2)$-pseudoconcave and satisfies $(f\text{-}P\text{-}(n-1-q))$. Thus (1.24) holds with $f = \log t^{\frac{1}{\alpha}}$ for $1 \le k \le n-1-q$. In particular, we have superlogarithmic estimates when $\alpha < 1$ and compactness when $\alpha \ge 1$. Note that in [14] the boundary is less flat: it is defined by $\mathrm{Re}\, z_n = e^{\frac{1}{\sum_j |z_j|^\alpha}}$ with $\sum_j$ at exponent.

All discussion we did has local character. In particular, Theorem 1.11 relates local estimates from $\bar D$ to ∂D and vice-versa. On the other hand, local estimates over a covering $\{V_j\}$ of $\partial D \subset\subset \mathbb{C}^n$ yield global estimates. For the tangential estimates this is obvious: one decomposes $u = \sum_j \zeta_j u$ where $\sum_j \zeta_j \equiv 1$ is a partition of the unity associated to the covering, and applies the local estimates to each $\zeta_j u$. As for the estimates over $\bar D$, one supplements the ζ_j's by an additional $\zeta_o \in C_c^\infty(D)$ such that $\zeta_o + \sum_j \zeta_j \equiv 1$ on a neighborhood of $\bar D$. Each $\zeta_j u$ is estimated by local estimates. On the other hand, $\zeta_o u|_{\partial D} \equiv 0$ and thus it enjoys elliptic estimates which are stronger than f-estimates.

2 Regularity of the tangential $\bar\partial$ system to a higher codimensional generic submanifold

We consider a generic smooth submanifold $M \subset \mathbb{C}^n$ of $\mathrm{codim}(M) = l > 1$; our discussion is here totally different from Section 1. We denote by $r = 0$ for $r = (r_1,\dots,r_l)$ a system of independent defining functions in a neighborhood V of a point $z_o \in M$ and by $D = D_\delta$ the "tube" $\{z \in \mathbb{C}^n : r^2 - \delta^2 < 0\}$ and also set $r^\delta := r^2 - \delta^2$. Our spaces of forms have now coefficients in $\bar D_\delta \cap V$. We choose a local basis $\{\omega_j\}$ of $(1,0)$-forms such that $\omega_{n-l+k} = \partial r_k,\ k = 1,\dots,l$; we also use the notation ω'_j when $j < n-l+1$ and ω''_j when $j \ge n-l+1$. We use the similar notation $\partial_{\omega'_j},\ j < n-l+1$, and $\partial_{\omega''_j},\ j \ge n-l+1$, for the dual basis of $(1,0)$ vector fields. This decomposition induces an obvious decomposition of the operators $\bar\partial = (\bar\partial', \bar\partial'')$ and $\bar\partial^* = (\bar\partial'^*, \bar\partial''^*)$. We denote by $T_j := \frac{i}{\sqrt 2}\big(\partial''_{\omega_j} - \partial''_{\bar\omega_j}\big)$ and $\partial_{r_j} = \frac{1}{\sqrt 2}\big(\partial''_{\omega_j} + \partial''_{\bar\omega_j}\big)$ the real tangential and normal vector fields respectively. Similarly as in Section 1, we have that $u \in D_{\bar\partial^*}$ if and only if $u_J|_{\partial D_\delta} \equiv 0$ whenever $j \in J$ for $j \ge n-l+1$; also, the basic estimate (1.3) applies without modification to ∂D_δ. We wish to relate now forms on M to restrictions to M of forms on $\mathbb{C}^n$ in a neighborhood V of z_o. Starting from a form u_b in M which belongs to $\overset{k}{\wedge} T^{1,0}M^*$, we complete the set of its coefficients by putting $(u_b)_{jK} = 0$ for any $j = n-l+1,\dots n$ and take an extension u that we call "tangential", that is, with the property

$$u_{jK} = 0 \quad \text{on} \quad V \quad \text{for any} \quad j = n-l+1,\dots,n.$$

In particular, $\sum_j \partial_{\omega_j}(r^\delta) u_{jK}|_{\partial D_\delta} \equiv 0$; thus, if $\bar\partial^*$ is the adjoint of $\bar\partial$ over D_δ, we have $u \in D_{\bar\partial^*}$.We also suppose that M is without boundary or that u_b has compact support; by the above choice of vector fields we get

$$\bar\partial^* u = \bar\partial'^* u_b + O(|r|). \tag{2.1}$$

We note that we have

$$\begin{cases} u = u_b + O(|r|), \\ \partial'_{\omega_j} u = \partial'_{\omega_j} u_b + O(|r|), \\ T_j u = T_j u_b + O(|r|). \end{cases}$$

Among these extensions u, we have a choice of some distinguished ones with the property

$$\partial''_{\bar\omega_i} u = O(|r|) \ \ i = n-l+1,\dots,n. \tag{2.2}$$

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This is a result of Whitney type whose proof can be found in [1]. Under this choice we have

$$\bar\partial u = \bar\partial' u_b + O(|r|). \tag{2.3}$$

Recall again that, since $u_{jK} = 0$ for any $j \geq n - l + 1$, we have (2.1). We now denote by $\bar\partial_b$ the system induced by $\bar\partial$ over forms tangential to M and by $\bar\partial_b^*$ its adjoint. We wish to translate an estimate for the system $(\bar\partial, \bar\partial^*)$ over forms tangential to ∂D_δ into an estimate for the induced system on M. Here there are the key relations which are a consequence of (2.1) and (2.3):

$$\begin{cases} \|u\|^2_{D_\delta} = \delta^l \|u_b\|^2_M + O\big(\delta^{l+1}\big), \\ \big\|\bar\partial u\big\|^2_{D_\delta} = \delta^l \big\|\bar\partial_b u_b\big\|^2_M + O\big(\delta^{l+1}\big), \\ \big\|\bar\partial^* u\big\|^2_{D_\delta} = \delta^l \big\|\bar\partial_b^* u_b\big\|^2_M + O\big(\delta^{l+1}\big). \end{cases} \tag{2.4}$$

In particular, with the notation $Q_{D_\delta}(u,u) = \big\|\bar\partial u\big\|^2_{D_\delta} + \big\|\bar\partial^* u\big\|^2_{D_\delta}$ and $Q_b(u_b,u_b) = \big\|\bar\partial_b u_b\big\|^2_b + \big\|\bar\partial_b^* u_b\big\|^2_b$, the estimate

$$\|f(\Lambda_\tau)u\|^2_{D_\delta} \lesssim Q_{D_\delta}(u,u) + \|u\|^2_{D_\delta},$$

implies

$$\|f(\Lambda_\tau)u_b\|^2_b \lesssim Q_b(u_b,u_b) + \|u_b\|^2_b.$$

We now make the geometric assumtion on M which makes the boundary integrals positive.

For $z_o \in M$ we identify $\dot{\mathbb{R}}^l := \mathbb{R}^n \setminus \{0\}$ with the conormal space $\big(\dot T^*_M \mathbb{C}^n\big)_{z_o}$ by

$$\eta \longmapsto \partial r^\eta|_{z_o} := \sum_{j=1}^{l} \eta_j \partial r_j|_{z_o}.$$

We denote by L^η_M the Levi form $\partial\bar\partial r^\eta(z)|_{T^{\mathbb{C}}_z M}$, by $\lambda^\eta_1 \leq \lambda^\eta_2 \leq \cdots$ its ordered eigenvalues and by s^η_+, s^η_- and s^η_0 the numbers of those which are > 0, < 0 and $= 0$ respectively.

Definition 2.1 We say that the higher codimensional submanifold M is *q-pseudoconvex* (resp. *-pseudoconcave*) at z_o when there is a bundle $\mathcal{V}^{q_o}_{(z,\eta)} \subset T^{1,0}_z M$ of rank $q_o < q$ (resp. $q_o > q$) homogeneous in η, say the bundle of the first q_o coordinate tangential vector fields $\partial_{\omega_1}, \ldots \partial_{\omega_{q_o}}$, such that

$$\sum_{j=1}^{q} \lambda^\eta_j(z) - \sum_{j=1}^{q_o} r^\eta_{jj}(z) \geq 0 \quad \text{for} \quad (z,\eta) \in (M \cap V) \times \dot{\mathbb{R}}^l, \tag{2.5}$$

where V is a neighborhood of z_o.

Remark 2.2 The coefficients of the ω_j, $j \leq n-l$, may be singular at $\eta = 0$ but they are assumed to be "tangentially regular" in the sense that their derivatives $\partial_{\omega_j}, \partial_{\bar\omega_j}$, $j \leq n-l$, and $\partial_{\omega_j} - \partial_{\bar\omega_j}$, $j = n-l+1, \ldots, n$ are bounded and, instead, only their normal derivatives $\partial_{\omega_j} + \partial_{\bar\omega_j}$, $j = n-l+1, \ldots, n$, may be unbounded. As for the coefficients of the ω_j, $j \geq n-l-1$, they are assumed to be regular.

The boundary of the tube ∂D_δ keeps track of the assumption of q-pseudoconvexity (resp. q-pseudoconcavity) of M. Similarly as in the case of codimension $l = 1$, it is easy to check that we have now $(II)_{D_\delta} \geq -O(\delta)|u|^2$. As before, the crucial point has become to make the right choice of the φ in order to take advantage of $(I)_{D_\delta}$.

Definition 2.3 Let the higher codimensional submanifold M be q-pseudoconvex (resp. -pseudoconcave) and suppose that there is a family of weights $\varphi = \varphi^\delta$ for $\delta \to 0$ which fulfill (1.26) for any $z \in D_\delta \cap V$. We then say that M satisfies $(f\text{-}P\text{-}q)$ (resp. $(f\text{-}P\text{-}q)$) in a neighborhood of z_o.

We discuss an example. We divide coordinates in $\mathbb{C}^n$ as $z = (z', z'', z''', z^{iv}, z_{n-1}, z_n)$, suppose that each group z', z'', z''' and z^{iv} has a number a of components and consider the 2-codimensional manifold M defined by

$$\begin{cases} 2\mathsf{Re}\, z_{n-1} = |z'|^{2m} - |z''|^{2m} - |z'''|^{2m} + |z^{iv}|^{2m}, \\ 2\mathsf{Re}\, z_n = |z'|^{2m} - |z''|^{2m} + |z'''|^{2m} - |z^{iv}|^{2m}. \end{cases}$$

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One readily checks that M is q-pseudoconvex for $q = n-a$ and q-pseudoconcave for $q = a-1$ in a neighborhood of $z_o = 0$. We write a point $z \in D_\delta$ as $z = s + t\partial r^\eta(s)$ for $(s, \eta, t) \in M \times S^{l-1} \times (0, \delta)$ and use this triplet as new system of coordinates. We denote $z_j^\pm = z_j^{\eta\,\pm}$ the directions in which $\partial\bar\partial r^\eta$ is $\gtrless 0$. We choose our family of weights as

$$\varphi^\delta = -\log\big(-|r|^2 + 2\delta^2\big) + \sum_j \log\big(|z_j^\pm|^2 + \delta^{\frac{1}{m}}\big).$$

We omit the normalization which makes the weights bounded. We have the estimate

$$\begin{aligned}\partial_{z_j^+}\partial_{\bar z_j^+}\varphi^\delta &\geq \frac{|z_j^+|^{2m-2}}{(-|r|^2 + 2\delta^2)} + \frac{\delta^{\frac{1}{m}}}{\big(|z_j^+|^2 + \delta^{\frac{1}{m}}\big)^2}\\ &=: A + B,\end{aligned}$$

where the second line serves as a definition of A and B.

Observe that in D_δ we have $2\delta^2 \leq |r|^2 + 2\delta^2 < 3\delta^2$. Now, if $\big|z_j^+\big|^2 < \delta^{\frac{1}{m}}$, then

$$B \gtrsim \delta^{-\frac{1}{m}}.$$

If, instead, $\big|z_j^+\big|^2 \geq \delta^{\frac{1}{m}}$, then

$$A \gtrsim \delta^{-1-\frac{1}{m}}.$$

Thus $(f\text{-}P\text{-}q)$ is satisfied for $f\big(\delta^{-1}\big) = \delta^{-\epsilon}$ with $\epsilon = \frac{1}{2m}$ (more precisely with any $\epsilon < \frac{1}{2m}$ because of the normalization which is omitted). Then ϵ-subelliptic estimates hold by Theorem 2.6 below.

We first see what the basic estimates produce.

Theorem 2.4 *Let M be q-pseudoconvex (-pseudoconcave) and satisfy $(f\text{-}P\text{-}q)$ at z_o. Let $q \leq k$ $(k \leq q)$; then for the extensions u of the forms u_b in degree k with support in a neighborhood V of z_o in $\mathbb{C}^n$, we have*

$$f^2\big(\delta^{-1}\big)\|u\|^2_{D_\delta} \lesssim Q_{D_\delta}(u, u) + \|u\|^2_{D_\delta}. \tag{2.6}$$

Proof. By looking at the terms in the right of (1.3) and using (1.26) for $\tilde\varphi^\delta_{ij}$ in the first term and the q-pseudoconvexity or -pseudoconcavity for r_{ij} in the second, we get

$$f^2\big(\delta^{-1}\big)\|u\|^2_{H^0_\varphi(D_\delta)} \lesssim (I)_{D_\delta} + (II)_{D_\delta}. \tag{2.7}$$

At this point, we need to modify our weights by taking the composition $\theta \circ \tilde\varphi^\delta$ for $\theta = \frac{1}{2}e^{c(t-1)}$ for a suitable c so that we can replace $\bar\partial^*_\varphi$ by $\bar\partial^*$ (and H^0_φ- by H^0-norms). By following word by word the argument of Theorem 3.2 of [9] we can see that (2.7) implies (2.6). $\square$

We choose now a partition of the unity $\sum_k p_k^2 = 1$ where the p_k's are a sequence of functions with $p_k \equiv 0$ in $\mathbb{R}^+ \setminus \big(2^{k-1}, 2^{k+1}\big)$ for $k \geq 1$ and $p_0 \equiv 0$ in $[2, +\infty)$. We can also choose p_k so that

$$|p_k'| \leq C2^{-k}.$$

Associated to these functions there are the pseudodifferential operators

$$P_k u = (\mathcal{F}_\tau)^{-1}(p_k(|\xi_a|)\mathcal{F}_\tau u);$$

this definition does not differ from that of $f(\Lambda_\tau)$. We remark that the action of $f(\Lambda_\tau)$ and P_k is obviously defined not only over u but also over u_b. We have the following result which is a slight generalization of the corresponding statement of Catlin [3].

Proposition 2.5 *We have*

$$\|f(\Lambda_\tau)u_b\|^2_b \lesssim \sum_k f\big(2^k\big)^2\|P_k u_b\|^2_b. \tag{2.8}$$

We refer to [3] for the proof. In particular, by the choice $f = 1$, we have $\|u_b\|_b^2 = \sum_k \|P_k u_b\|_b^2$; in the same way, $\|u\|_{D_\delta}^2 = \sum_k \|P_k u\|_{D_\delta}^2$.

We have also to recall the estimates for commutators which are the same as in [3]. Let T be a vector field tangential to M and g a function in $C_c^\infty(M)$; then, for any $u_b \in C_c^\infty(M)$:

$$\sum_k \|[T, P_k]u_b\|_{H^0(M)}^2 \lesssim \|u_b\|_{H^0(M)}^2, \tag{2.9}$$

$$\sum_k \|[g, P_k]u_b\|_{H^0(M)}^2 \leq \|u_b\|_{H^{-1}(M)}^2. \tag{2.10}$$

We are ready for the main theorem of this section.

Theorem 2.6 *Let $l > 1$ and assume that M is q-pseudoconvex (resp. -pseudoconcave) and satisfies $(f\text{-}P\text{-}q)$ in a neighborhood of z_o. Then for any k such that $q \leq k$ (resp. $k \geq q$), we have*

$$\|f(\Lambda_\tau)u_b\|_b^2 \lesssim Q_b(u_b, u_b) + \|u_b\|_b^2, \quad \textit{for} \quad u_b \in C_c^\infty(M \cap V)^k. \tag{2.11}$$

As already noticed in Section 1, the main interest in (2.11) is when $f(\delta^{-1})$ has a sufficiently high rate to infinity when $\delta \to 0$ such as $f = t^\epsilon$ or $f > k \log t$ for $t > c_k$ which yield subelliptic and superlogarithmic estimates respectively.

P r o o f. We decompose norms according to Proposition 2.5

$$\begin{aligned}\|f(\Lambda_\tau)u_b\|_b^2 &\lesssim \sum_k f(2^k)^2 \|P_k u_b\|_b^2 \\ &\lesssim \sum_{2^{-k} \geq \delta^{1-\epsilon}} \left(f(2^k)^2 \delta^{-l} \|P_k u\|_{D_\delta}^2 + O(\delta) f(2^k)^2\right) \\ &\quad + \sum_{2^{-k} < \delta^{1-\epsilon}} \left(f(2^k)^2 3^{kl} \|P_k u\|_{D_{3^{-k}}}^2 + O(3^{-k}) f(2^k)^2\right).\end{aligned} \tag{2.12}$$

Since $f(t) \lesssim t^{\frac{1}{2}}$, then

$$\sum_{2^{-k} \geq \delta^{1-\epsilon}} O(\delta^{1-\epsilon}) f(2^k)^2 + \sum_{2^{-k} < \delta^{1-\epsilon}} O(3^{-k}) f(2^k)^2 = \mu(\delta),$$

where the notation μ is used to denote terms which are infinitesimal. In the first sum of the second line, owing to $2^{-k} \geq \delta$, we use the estimate $f(2^k)^2 \leq f(\delta^{-1})^2$. Next, using the $(f\text{-}P\text{-}q)$ property and Theorem 2.4, we get the estimate

$$\begin{cases} f(\delta^{-1})^2 \|P_k u\|_{D_\delta}^2 \leq Q_{D_\delta}(P_k u, P_k u) + \|P_k u\|_{D_\delta}^2, \\ f(2^k)^2 \|P_k u\|_{D_{2^{-k}}}^2 \leq Q_{D_{2^{-k}}}(P_k u, P_k u) + \|P_k u\|_{D_{2^{-k}}}^2. \end{cases} \tag{2.13}$$

We wish to replace $Q_{D_\delta}(P_k u, P_k u)$ with $\|P_k \bar\partial u\|_{D_\delta}^2 + \|P_k \bar\partial^* u\|_{D_\delta}^2$ taking into account the errors which come from the commutators that we denote by dots. Now, for the two sums in the second line of (2.12), we have

$$\begin{aligned}\sum_{2^{-k} \geq \delta} \cdot + \sum_{2^{-k} < \delta} \cdot &\lesssim \sum_{2^{-k} \geq \delta} \delta^{-l} \left(\|P_k \bar\partial u\|_{D_\delta}^2 + \|P_k \bar\partial^* u\|_{D_\delta}^2\right) \\ &\quad + \sum_{2^{-k} < \delta} 2^{kl} \left(\|P_k \bar\partial u\|_{D_{2^{-k}}}^2 + \|P_k \bar\partial^* u\|_{D_{2^{-k}}}^2\right) + \mu(\delta) + \cdots\end{aligned} \tag{2.14}$$

Also, the analogous of (2.12) for $f \equiv 1$ with a quick sight to (1.6), shows that the two sums in the right of (2.14) equal $\|\bar\partial' u_b\|_{H^0(M)}^2 + \|\bar\partial'^* u\|_{H^0(M)}^2 + \mu(\delta)$. This yields

$$\sum_{2^{-k} \geq \delta} \cdot + \sum_{2^{-k} < \delta} \cdot \lesssim Q_b(u_b, u_b) + \|u_b\|_b^2 + \mu(\delta) + \cdots \tag{2.15}$$

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We come now to estimate the errors from commutators which are denoted by dots. Now, $[\bar\partial, P_k]$ and $[\bar\partial^*, P_k]$ can be represented as expressions of the type

$$[aN + bT + c, P_k] \quad \text{for suitable functions } a, b \text{ and } c.$$

The commutation by the c's are 0-order operators. As for bT, we have

$$[bT, P_k] = [b, P_k]T + b[T, P_k]$$

and use the commutation relations (2.9) and (2.10). When taking summation over k, the contribution of these terms is estimated by

$$\|Tu_b\|^2_{H^{-1}(M)} + \|u_b\|^2_{H^0(M)} + O(\delta).$$

As for aN, we notice that $N = \partial''_{\bar\omega_o} + T$, where $\partial''_{\bar\omega_o}$ is a suitable combination of the $\partial''_{\bar\omega_j}$'s and that T and the P_k's commute. Also, $\partial''_{\bar\omega_o} u = O(\delta)$ by our choice of u. Thus, $[N, P_k] = O(\delta)$ and therefore

$$[aN, P_k] = [a, P_k]N + O(\delta) = [a, P_k](\partial'_{\bar\omega_o} + T) + O(\delta).$$

Hence the error from the first commutator is estimated up to a constant by

$$\left\|\partial'_{\bar\omega_o} u_b\right\|^2_{H^{-1}(M)} + \|Tu_b\|^2_{H^{-1}(M)} + O(\delta),$$

which is in turn estimated by $\|u_b\|^2_{H^0(M)} + O(\delta)$. By combining (2.12) with (2.15) and by using the above estimates of the commutator error terms, we get (2.11). This concludes the proof of the theorem. □

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