Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 228 (2011) 1938–1965

ADVANCES IN
Mathematics

www.elsevier.com/locate/aim

Subellipticity of the $\bar\partial$-Neumann problem on a weakly q-pseudoconvex/concave domain

Tran Vu Khanh [a], Giuseppe Zampieri [b,*]

[a] *Tan Tao University, Duc Hoa Ha Road, Tan Duc Industrial Park, Duc Hoa District, Long An Province, Viet Nam*
[b] *Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy*

Received 15 August 2009; accepted 30 May 2011

Communicated by Carlos E. Kenig

Abstract

For a domain D of $\mathbb{C}^n$ which is weakly q-pseudoconvex or q-pseudoconcave, we give a sufficient condition for subelliptic estimates for the $\bar\partial$-Neumann problem. This extends to domains which are not necessarily pseudoconvex, the results and the techniques of Catlin (1987) [3].

MSC: 32F20; 32F10; 32T25; 32N15

Keywords: $\bar\partial$-Neumann problem; Subelliptic estimates; q-Pseudoconvex/concave domains

1. Introduction

Let D be a bounded domain of $\mathbb{C}^n$ with smooth boundary. For a form f of degree k which satisfies $\bar\partial f=0$, to solve the $\bar\partial$-Neumann problem consists in finding a form of degree $k-1$ such that

$$\begin{cases}\bar\partial u=f,\\ u \text{ is orthogonal to } \operatorname{Ker}\bar\partial.\end{cases}\tag{1.1}$$

* Corresponding author.
E-mail addresses: khanh.tran@ttu.edu.vn (T.V. Khanh), zampieri@math.unipd.it (G. Zampieri).

0001-8708/$ – see front matter © 2011 Published by Elsevier Inc.
doi:10.1016/j.aim.2011.05.027

The main interest relies in the regularity at the boundary for this problem, that is, in stating under which condition u inherits from f the smoothness at the boundary ∂D (it certainly does in the interior). Let $\bar\partial^*$ be the formal adjoint of $\bar\partial$ under the choice of a smoothly varying hermitian metric on $\bar D$. Related to (1.1) is the problem

$$\begin{cases} (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)u = f, \\ u \in D_{\bar\partial} \cap D_{\bar\partial^*}, \\ \bar\partial u \in D_{\bar\partial^*}, \quad \bar\partial^* u \in D_{\bar\partial}, \end{cases} \tag{1.2}$$

where $D_{\bar\partial^*}$ and $D_{\bar\partial}$ are the domains of $\bar\partial^*$ and $\bar\partial$ respectively. This is a non-elliptic boundary value problem; in fact, the Kohn Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ itself is elliptic but the boundary conditions which are imposed by the membership to $D_\Box$ are not. If (1.1) has a solution for every f, then one defines the $\bar\partial$-Neumann operator $N := \Box^{-1}$; this commutes both to $\bar\partial$ and $\bar\partial^*$. If we then return back to (1.1) and define $u := \bar\partial^* N f$ we see that

$$\begin{aligned} \bar\partial u &= \bar\partial\bar\partial^* N f \\ &= \Box N f = f. \end{aligned}$$

Also, $\bar\partial^* u = \bar\partial^*\bar\partial^* N f = 0$ and therefore u is orthogonal to $\operatorname{Ker}\bar\partial$. One of the main methods used in investigating the regularity at the boundary of the solutions of (1.1) consists in certain a priori subelliptic estimates. We recall the tangential Sobolev norm $|||u|||_s^2$, $s \in \mathbb{R}$, defined in [6, p. 36]; recall that when s is integer, then $|||u|||_s^2 + \sum_{j=0}^s |||\partial_\nu^j u|||_{-j+s}^2$ is the usual norm $\|u\|_s^2$ (where ∂_ν is the normal derivative).

Definition 1.1. The $\bar\partial$-Neumann problem is said to satisfy a subelliptic estimate of order $\epsilon > 0$ at $z_o \in \bar D$ on k-forms if there exist a positive constant c and a neighborhood $V \ni z_o$ such that

$$|||u|||_\epsilon^2 \leqslant c\big(\|\bar\partial u\|^2 + \big\|\bar\partial^* u\big\|^2 + \|u\|_0^2\big) \quad \text{for any } u \in C_c^\infty(\bar D \cap V)^k \cap D_{\bar\partial^*}. \tag{1.3}$$

By Garding's inequality, subelliptic estimates of order 1, that is, elliptic estimates hold in the interior of D. So our interest is confined to the boundary ∂D. When the domain D is pseudoconvex, a great deal of work has been done about subelliptic estimates (cf. [2–4,8,11–15]). The most general results concerning this problem have been obtained by Kohn [14] and Catlin [3].

In [14], Kohn gave a sufficient condition for subellipticity over pseudoconvex domains with real analytic boundary by introducing a sequence of ideals of subelliptic multipliers.

In [3], Catlin proved, regardless whether ∂D is real analytic or not, that subelliptic estimates hold for k-forms at z_o if and only if a certain number $D_k(z_o)$ is finite. Note that the definition of $D_k(z_o)$ in [3] is closely related to that of $\Delta_k(z_o)$ introduced by D'Angelo [5]. In particular, when $k = 1$, these numbers do coincide.

However, not much is known in the case when the domain is not necessarily pseudoconvex except from the results related to the celebrated $Z(k)$ condition which characterizes the existence of subelliptic estimates for $\epsilon = \frac{1}{2}$ according to Hörmander [7] and Folland and Kohn [6]. Some further results, mainly related to the case of forms of top degree $n-1$ have been obtained by Ho [9].

We exploit here the full strength of Catlin's method to study subellipticity on domains which are not pseudoconvex. Let ∂D be defined by $r = 0$ with $r < 0$ on the side of D and let $T^{\mathbb{C}}\partial D$

be the complex tangent bundle to ∂D. We use the following notations: $L_{\partial D} = \partial\bar\partial r|_{T^{\mathbb{C}}\partial D}$ is the Levi form of the boundary, $s^+_{\partial D}$, $s^-_{\partial D}$, $s^0_{\partial D}$ are the numbers of eigenvalues of $L_{\partial D}$ which are > 0, < 0, $= 0$ respectively and finally $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n-1}$ are its ordered eigenvalues. We choose an orthonormal basis of $(1,0)$-forms $\omega_1, \ldots, \omega_n = \partial r$, the dual basis $L_1, \ldots, L_n$ of $(1,0)$ vector fields, and denote by $(r_{ij})_{i,j\leqslant n-1}$ the matrix of $L_{\partial D}$ in the above basis. We take a pair of indices $1 \leqslant q \leqslant n-1$ and $0 \leqslant q_o \leqslant n-1$ such that $q \neq q_o$. We assume that there is a bundle $\mathcal{V}^{q_o} \subset T^{1,0}\partial D$ of rank q_o with smooth coefficients in a neighborhood V of z_o, say the bundle spanned by $L_1, \ldots, L_{q_o}$, such that

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_o} r_{jj} \geqslant 0 \quad \text{on } \partial D \cap V. \tag{1.4}$$

Definition 1.2.

(i) If $q > q_o$ we say that D is q-pseudoconvex at z_o.
(ii) If $q < q_o$ we say that D is q-pseudoconcave at z_o.

What we first remark is that (1.4) for $q > q_o$ implies $\lambda_q \geqslant 0$; hence (1.4) is still true if we replace the first sum $\sum_{j=1}^{q} \cdot$ by $\sum_{j=1}^{k} \cdot$ for any k such that $q \leqslant k \leqslant n-1$. Similarly, if it holds for $q < q_o$, then $\lambda_{q+1} \leqslant 0$ and hence it also holds with q replaced by $k \leqslant q$ in the first sum.

Remark 1.3. An "adapted frame" is a basis of $(1,0)$-forms, not necessarily orthonormal, in which $\omega_n = \partial r$. We remark that q-pseudoconvexity/concavity is invariant under a change of an orthonormal basis but not of an adapted frame. In fact, not only the number, but also the size of the eigenvalues comes into play. Likewise, $\bar\partial^*$ and hence also the subelliptic estimates, depend on the choice of the frame but $D_{\bar\partial^*}$ is invariant as far as the frame is "adapted to the boundary". Thus, when we say that ∂D is q-pseudoconvex/concave, we mean that there is an adapted frame in which (1.4) is fulfilled; the same is meant when we deal with (1.2). Sometimes, it is more convenient to put our calculations in an orthonormal frame. In this case, it is meant that the metric has been changed so that the adapted frame has become orthonormal.

Example 1.4. It is readily seen that for $q_o = s^- + s^0$ and for any $q > q_o$ (resp. $q_o = s^-$ and any $q < q_o$), (1.4) is satisfied in a suitable local boundary frame. Thus any index $q \notin [s^-, s^- + s^0]$ satisfies (1.4) for either choice of q_o. Note that $s^- + s^0 = n - 1 - s^+$ and thus $q \notin [s^-, n-1-s^+]$ coincides, in the terminology of Folland and Kohn, with condition $Z(q)$. Instead, we use the terminology of strong q-pseudoconvexity (concavity) when $q > n - 1 - s^+$ (resp. $q < s^-$) because this is the same as to ask that (1.4) holds with strict inequality.

Example 1.5. The interesting new point about (1.4) is when weak inequality occurs, that is, when $q \in [s^-, s^- + s^0]$. Thus, for instance, let $s^-(z)$ be constant for $z \in \partial D$ close to z_o; then (1.4) holds for $q_o = s^-$ and $q = s^- + 1$. In fact, we have $\lambda_{s^-} < 0 \leqslant \lambda_{s^-+1}$, and therefore the negative eigenvectors span a bundle $\mathcal{V}^{q_o}$ for $q_o = s^-$ that, identified to the span of $L_1, \ldots, L_{q_o}$, yields $\sum_{j=1}^{q_o+1} \lambda_j(z) \geqslant \sum_{j=1}^{q_o} r_{jj}(z)$. Note that a pseudoconvex domain is characterized by $s^-(z) \equiv 0$, thus, it is 1-pseudoconvex in our terminology.

In the same way, if $s^+(z)$ is constant at z_o, then $\lambda_{s^-+s^0} \leqslant 0 < \lambda_{s^-+s^0+1}$. Then, the eigenspace of the eigenvectors $\leqslant 0$ is a bundle which, identified to the span of $L_1, \ldots, L_{q_o}$, yields (1.4) for

$q = q_o - 1$. In particular a pseudoconcave domain, that is a domain which satisfies $s^+ \equiv 0$, is $(n-2)$-pseudoconcave in our terminology.

Remark 1.6. The notion of q-pseudoconvexity was introduced in [16] and further refined in [1] in order to discuss existence of $C^\infty(\bar{D})$ solutions to the equation $\bar\partial u = f$. Though the notion of q-pseudoconcavity is formally symmetric to q-pseudoconvexity, it is useless in the existence problem. The reason is intrinsic. Existence is a "global" problem but bounded domains are never globally q-pseudoconcave. Owing to the local nature of subelliptic estimates and the related hypoellipticity of $\bar\partial$, here is the first occurrence where q-pseudoconcavity comes successfully into play.

We define the δ-strip of D along the boundary by $S_\delta = \{z \in D\colon\ r(z) > -\delta\}$. The main result in this paper is the following.

Theorem 1.7. *Let* (1.4) *be satisfied in a neighborhood of z_o for $q > q_o$ (resp. $q < q_o$) and suppose that for any small $\delta > 0$ there exists a weight $\phi = \phi^\delta$ in $C^2(\bar{S}_\delta \cap V)$ with $|\phi| \leqslant 1$, such that, if $\lambda_1(z) \leqslant \lambda_2(z) \leqslant \cdots$ are the ordered eigenvalues of the Levi form $\partial\bar\partial\phi = (\phi_{ij})$, we have*

$$\sum_{j=1}^{q} \lambda_j^\phi(z) - \sum_{j=1}^{q_o} \phi_{jj}(z) \geqslant c\left(\delta^{-2\epsilon} + \sum_{j=1}^{q_o} \left|\phi_j(z)\right|^2\right),$$

$$\textit{for any } z \in \bar{S}_\delta \cap V \textit{ and for } c > 0 \textit{ independent of } \delta. \tag{1.5}$$

Then, ϵ-subelliptic estimates at z_o hold for forms of degree $k \geqslant q$ (resp. $k \leqslant q$).

The proof is the content of Section 3.

Remark 1.8. Similarly as observed before Remark 1.3, we notice that if condition (1.5) holds for forms in some degree $q > q_o$ (resp. $q < q_o$), then it also holds in any degree $k \geqslant q$ (resp. $k \leqslant q$). In fact, (1.5) forces $\lambda_q^\phi \geqslant 0$ (resp. $\lambda_q^\phi \leqslant 0$) which implies $\lambda_k^\phi \geqslant 0$ for any $k \geqslant q$ (resp. $\lambda_k^\phi \leqslant 0$ for any $k \leqslant q$).

It is not restrictive to assume, as we will do all throughout the paper, that $L_j|_{z_o} = \partial_{z_j}$ for any j. We have a large class of q-pseudoconvex/concave domains to which Theorem 1.7 applies and produces subelliptic estimates.

Theorem 1.9. *Let D be the "rigid" domain defined, in a neighborhood of $z_o = 0$, by $r < 0$ with $r = 2\operatorname{Re} z_n + h(z_1, \ldots, z_{q_o})$ for $(h_{ij}) \leqslant 0$ (resp. $r = 2\operatorname{Re} z_n + h(z_{q_o+1}, \ldots, z_{n-1})$ for $(h_{ij}) \geqslant 0$); thus D is q-pseudoconvex in a neighborhood of $z_o = 0$ for any $q \geqslant q_o + 1$ (resp. q-pseudoconcave for any $q \leqslant q_o - 1$). (Here, as always, (h_{ij}) is the matrix of $\partial\bar\partial h$.) Let $h^j(z_j)$ for $j = q, \ldots, n-1$ (resp. $j = 1, \ldots, q+1$) be real positive subharmonic, non-harmonic, functions of vanishing order $2m_j$ with respect to $|z_j|$ that, by reordering, we may assume to be decreasing $\ldots m_j \geqslant m_{j+1} \ldots$ (resp. increasing $\ldots m_j \leqslant m_{j+1} \ldots$). Set $g := \sum_{j=q}^{n-1} h^j$ (resp. $g := \sum_{j=1}^{q+1} h^j$), put $\tilde r := r + g$ (resp. $\tilde r := r - g$) and define $\tilde D$ by $\tilde r < 0$.*

Then subelliptic estimates hold for $\tilde{D}$ *at* $z_o = 0$ *in degree* $k \geqslant q$ *(resp.* $k \leqslant q$*) of any order* $< \epsilon_k$ *for* $\epsilon_k := \frac{1}{2m_k}$ *(resp.* $\epsilon_k := \frac{1}{2m_{k+1}}$*). In both cases, when* $\epsilon_k = \frac{1}{2}$*, we have in fact estimates including for order* $\frac{1}{2}$.

The proof is given in Section 4. When $\epsilon_k = \frac{1}{2}$ it means that $Z(k)$ is satisfied; thus we regain, for these particular domains, a classical result by Hörmander and Folland and Kohn. We will use the notation $h^j \cong |z_j|^{2m_j}$ for a function h^j which has exactly vanishing order $2m_j$ in $|z_j|$ at 0.

Example 1.10. Let a and b be integers between 1 and $n-1$ with $a < b$ and let D be defined by

$$2\,\mathrm{Re}\,z_n - \sum_{j=1}^{a} |z_j|^{2m_j} + \sum_{j=b}^{n-1} |z_j|^{2m_j} < 0,$$

where the two groups of indices $\{m_1, \dots, m_a\}$ and $\{m_b, \dots, m_{n-1}\}$ have increasing and decreasing order respectively. Then subelliptic estimates hold in degree $k < a$ of order $\epsilon < \frac{1}{2m_{k+1}}$ and in degree $k \geqslant b$ of order $\epsilon < \frac{1}{2m_k}$.

Corollary 1.11. *Let* D *be a domain in* $\mathbb{C}^n$ *defined by*

$$2\,\mathrm{Re}\,z_n + g + |z_{n-1}|^{2m} < 0$$

where $g = g(z_1, \dots, z_{n-1})$ *is a real* C^∞*-function such that* $g_{n-1\,n-1} = o(|z_{n-1}|^{2(m-1)})$*. Then subelliptic estimates of order* $\epsilon < \frac{1}{2m}$ *hold at* $z_o = 0$ *for any* $(n-1)$*-form.*

Proof. Put $r := 2\,\mathrm{Re}\,z_n + g + \frac{1}{2}|z_{n-1}|^{2m}$; we claim that r satisfies (1.4) for $q_o = n-2$ and $q = n-1$. In fact

$$\sum_{j=1}^{n-1} \lambda_j - \sum_{j=1}^{n-2} r_{jj} = \left(\frac{1}{2}|z_{n-1}|^{2(m-1)} + o\big(|z_{n-1}|^{2(m-1)}\big)\right)$$

which is $\geqslant 0$. We are thus in position to apply Theorem 1.9. □

Example 1.12. Let D be defined by

$$2\,\mathrm{Re}\,z_3 - \left|z_1^2 + z_2^3\right|^2 \pm |z_1|^{2m} + |z_2|^4 < 0 \quad \text{or} \quad 2\,\mathrm{Re}\,z_3 - \left|z_1^2 z_2^3\right|^2 \pm |z_1|^{2m} + |z_2|^4 < 0;$$

then subelliptic estimates hold at $z_o = 0$ on 2-forms for any order $\epsilon < \frac{1}{4}$.

Remark. Corollary 1.11 is more general than Corollary 3.4 in [10] where g cannot depend on z_{n-1}.

We decompose the coordinates as $z = (z', z'', z_n) \in \mathbb{C}^{q_o} \times \mathbb{C}^{n-q_o-1} \times \mathbb{C}$. The conclusion contained in Theorem 1.9 is sharp.

Theorem 1.13.

(i) *Let* $r = 2\,\mathrm{Re}\, z_n - Q(z')$ *for* $Q \geqslant 0$ *and set* $\tilde{r} = r + \sum_{j=q_o+1}^{n-1} |z_j|^{2m_j}$ *with* $m_j \geqslant m_{j+1} \geqslant \cdots$ *(decreasing) and with* $Q = O(|z'|^{2m_{q_o+1}})$*. If* ϵ*-subelliptic estimates at* $z_o = 0$ *hold in degree* $k \geqslant q_o + 1$ *for the standard metric, then we must have* $\epsilon \leqslant \frac{1}{2m_k}$.

(ii) *Let* $r = 2\,\mathrm{Re}\, z_n + Q(z'')$ *for* $Q \geqslant 0$ *and set* $\tilde{r} = r - \sum_{j=1}^{q_o} |z_j|^{2m_j}$ *with* $m_j \leqslant m_{j+1} \leqslant \cdots$ *(increasing). We also assume* $m_1 \geqslant \frac{m_{q_o-1}}{2} + \frac{1}{4}$ *and* $Q = O(|z''|^{2m_{q_o-1}})$*. If* ϵ*-subelliptic estimates hold at* $z_o = 0$ *in degree* $k \leqslant q_o - 1$*, then* $\epsilon \leqslant \frac{1}{2m_k}$.

The proof is given in Section 5. We point out that the metric has not been changed: the same conclusion in other metrics would be easier. Necessary conditions for subellipticity in degree $k = n - 1$ are also stated, for a modified metric, in [9].

The paper is structured as follows. In Section 2 we derive some basic inequalities which are useful for the proof of Theorem 1.7. Sections 3, 4 and 5 are devoted to the proof of Theorem 1.7, Theorem 1.9 and Theorem 1.13 respectively.

2. The basic estimates on q-pseudoconvexity/concavity

In this section we prepare some inequalities which are needed for the subelliptic estimates of our Theorem 1.7. The key technical tool of our discussion are the so-called Hörmander–Kohn–Morrey estimates contained in the following proposition. Let D be a domain with smooth boundary defined by $r = 0$ in a neighborhood of z_o. Let $\omega_1, \dots, \omega_n = \partial r$ be an orthonormal basis of $(1,0)$-forms and $L_1, \dots, L_n$ the dual basis of $(1,0)$ vector fields.

For $0 \leqslant k \leqslant n$, we write a general k-form u as

$$u = \sum_{|J|=k}{}' u_J \bar{\omega}_J,$$

where $\sum'$ denotes summation restricted to ordered multiindices $J = \{j_1, \dots, j_k\}$ and where $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_k}$. When the multiindex is no more ordered, it is understood that the coefficient u_J is an antisymmetric function of J; in particular, if J decomposes into jK, then $u_{jK} = \mathrm{sign}\binom{J}{jK} u_J$. We define $\langle u, u\rangle$ by $\langle u, u\rangle = |u|^2 = \sum'_{|J|=k} |u_J|$; this definition is independent of the choice of orthonormal basis $\omega_1, \dots, \omega_n$. The coefficients of our forms are taken in various spaces Λ such as $C^\infty(\bar{D})$, $C^\infty(D)$, $C_c^\infty(\bar{D})$, $L^2(D)$ and the corresponding spaces of k-forms are denoted by Λ^k. Though our a priori estimates are proved over smooth forms, they are stated in Hilbert norms. Thus, let $\|u\|$ be the $H^0 = L^2$ norm and, for a real function ϕ, let the weighted L^2-norm be defined by

$$\|u\|^2_{H^0_\phi} := \sum_{|J|=k}{}' \int_D e^{-\phi} |u_J|^2 \, dv$$

where dv is the element of volume in $\mathbb{C}^n$. We begin by noticing that $\bar{\partial}$ is closed, densely defined. Also, its domain $D_{\bar{\partial}}$ certainly contains smooth forms and its action is expressed by

$$\bar\partial u=\sum_{|K|=k-1}'\sum_{\substack{ij=1,\dots,n\\ i<j}}(\bar L_i u_{jK}-\bar L_j u_{iK})\bar\omega_i\wedge\bar\omega_j\wedge\bar\omega_K+\cdots,\tag{2.1}$$

where dots denote terms in which no differentiation of u occurs.

Let $\bar\partial^*$ be the adjoint of $\bar\partial$. The operator $\bar\partial^*$ is still closed, densely defined but it is no more true that smooth forms belong to $D_{\bar\partial^*}$. For this, they must satisfy certain boundary conditions. Namely, integration by parts shows that a form u of degree k cannot belong to $D_{\bar\partial^*}$ unless

$$\sum_{j=1}^n\int_{\partial D}e^{-\phi}L_j(r)u_{jK}\psi_K\,ds=0\quad\text{for any } K \text{ and any } \psi_K \text{ of degree } k-1.$$

This means that $\sum_{j=1}^n L_j(r)u_{jK}|_{\partial D}\equiv 0$ for any K. (Here ds is the element of hypersurface in ∂D.) Since we have chosen our basis with the property $L_j(r)|_{\partial D}=\kappa_{jn}$ (the Kronecker's symbol), we then conclude

$$u\text{ belongs to }D_{\bar\partial^*}\quad\text{iff}\quad u_J|_{\partial D}=0\quad\text{whenever } n\in J.\tag{2.2}$$

We decompose any form as $u=u^\tau+u^\nu$ where u^τ (resp. u^ν) is the "tangential" (resp. "normal") component which collects the coefficients u_J such that $n\notin J$ (resp. $n\in J$). Thus we have $u\in D_{\bar\partial^*}$ precisely when $u^\nu|_{\partial D}\equiv 0$; in particular, by Garding's inequality, u^ν enjoys elliptic estimates and is therefore negligible in our discussion. Let $\mathcal{L}_j^\phi$ be the formal H^0_ϕ-adjoint of $-L_j$; on a tangential form the action of the Hilbert adjoint of $\bar\partial$, coincides with that of its "formal adjoint" and is therefore expressed by a "divergence operator":

$$\bar\partial^*_\phi u=-\sum_{|K|=k-1}'\sum_j\mathcal{L}_j^\phi(u_{jK})\bar\omega_K+\cdots\quad\text{for any } u\in D_{\bar\partial^*},\tag{2.3}$$

where dots denote an error term in which u is not differentiated and ϕ does not occur. By developing the equalities (2.1) and (2.3) by means of integration by parts, we get the proof of the following crucial result.

Proposition 2.1. *Let D be a smooth domain in a neighborhood of z_o and fix an arbitrary index q_o with $0\leqslant q_o\leqslant n-1$. Then for a suitable $C>0$ and any $u\in C_c^\infty(\bar D\cap V)^k\cap D_{\bar\partial^*}$, we have*

$$\begin{aligned}
&\|\bar\partial u\|^2_{H^0_\phi}+\|\bar\partial^*_\phi u\|^2_{H^0_\phi}+C\|u\|^2_{H^0_\phi}\\
&\quad\geqslant\sum_{|K|=k-1}'\sum_{i,j=1}^n\int_D e^{-\phi}\phi_{ij}u_{iK}\bar u_{jK}\,dv-\sum_{j=1}^{q_o}\int_D e^{-\phi}\phi_{jj}|u|^2\,dv\\
&\qquad+\sum_{|K|=k-1}'\sum_{i,j=1}^{n-1}\int_{\partial D}e^{-\phi}r_{ij}u_{iK}\bar u_{jK}\,ds-\sum_{j=1}^{q_o}\int_{\partial D}e^{-\phi}r_{jj}|u|^2\,ds\\
&\qquad+\frac12\Bigg(\sum_{j=1}^{q_0}\|\mathcal{L}_j^\phi u\|^2_{H^0_\phi}+\sum_{j=q_o+1}^n\|\bar L_j u\|^2_{H^0_\phi}\Bigg).
\end{aligned}\tag{2.4}$$

We refer for instance to [17, Proposition 1.9.1] for the proof of Proposition 2.1. We note that there is no relation between k and q_o in the above inequality and that C and α are independent of ϕ (and u). By choosing ϕ so that $e^{-\phi}$ is bounded, we may remove the weight functions in (2.4) to get some inequalities that are useful for the proof of Theorem 1.7. We use the notation $Q(u,u) = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$. We also use the symbols $\lesssim$ and $\gtrsim$ to denote an estimate up to a constant which is independent of relevant parameters and $\cong$ for the combination of $\lesssim$ and $\gtrsim$.

We have already introduced the notation S_δ for the strip $\{z \in D\colon\ r(z) > -\delta\}$.

Theorem 2.2. *Assume that the hypotheses of Theorem* 1.7 *be fulfilled. Then, for a suitable neighborhood V of z_o and for δ small, we have*

$$\|u\|^2 + \delta^{-2\epsilon} \int\limits_{S_{\delta/2}} |u|^2\, dv + \sum_{j=1}^{q_o} \|L_j u\|^2 + \sum_{j=q_o+1}^{n} \|\bar L_j u\|^2 \lesssim Q(u,u) \tag{2.5}$$

for any $u \in C_c^\infty(\bar D \cap V)^k \cap D_{\bar\partial^*}$ with $k \geqslant q$ (resp. $k \leqslant q$) when $q > q_o$ (resp. $q < q_o$).

Proof. To begin the proof, we have to rephrase (1.4) and (1.5) in terms of the action of (r_{ij}) and (ϕ_{ij}) over a form u. Precisely, we have that (1.4) for $q > q_o$ (resp. $q < q_o$) is equivalent to

$$\begin{gathered}\sum_{|K|=k-1}{}' \sum_{i,j=1}^{n-1} r_{ij} u^\tau_{iK} \bar u^\tau_{jK} - \sum_{|K|=k-1}{}' \sum_{i=1}^{q_o} r_{jj} \left|u^\tau_J\right|^2 \geqslant 0 \\ \text{on } \partial D \cap V \text{ for any } u^\tau \text{ of degree } k \geqslant q > q_o \text{ (resp. } k \leqslant q < q_o\text{)}.\end{gathered} \tag{2.6}$$

This claim is readily proved once one remarks that u having degree k, the first sum in the left side of (2.6) is $\geqslant \sum_{j=1}^k \lambda_j |u^\tau|^2$ (cf. also [17, formula (1.9.23)]). In the same way we can check that (1.5) for $q > q_o$ (resp. $q < q_o$) is equivalent to

$$\begin{gathered}\sum_{|K|=k-1}{}' \sum_{i,j=1}^{n} \phi_{ij}(z) u^\tau_{iK} \bar u^\tau_{jK} - \sum_{j=1}^{q_o} \phi_{jj}(z) \left|u^\tau\right|^2 \gtrsim \delta^{-2\epsilon} \left|u^\tau\right|^2 + \sum_{j=1}^{q_o} \left|\phi_j(z)\right|^2 \left|u^\tau\right|^2, \\ \text{for } z \in \bar S_\delta \cap V \text{ and } u^\tau \text{ of degree } k \geqslant q > q_o \text{ (resp. } k \leqslant q < q_o\text{)}.\end{gathered} \tag{2.7}$$

Next, we have to extend $\phi = \phi^\delta$ from $\bar S_\delta \cap V$ to $\bar D \cap V$, keeping (1.5) in a half strip $\bar S_{\delta/2} \cap V$ and satisfying, for a suitable $c > 0$ and for $k \geqslant q > q_o$ (resp. $k \leqslant q < q_o$),

$$\sum_{|K|=k-1}{}' \sum_{ij=1}^{n-1} \phi_{ij} u^\tau_{iK} \bar u^\tau_{jK} - \sum_{j=1}^{q_o} \phi_{jj} \left|u^\tau\right|^2 \geqslant \begin{cases} c \sum'_{|K|=k-1} |\partial\phi \cdot u^\tau_{\cdot K}|^2 & \text{in } D \cap V, \\ c\delta^{-2\epsilon} |u^\tau|^2 & \text{in } \bar S_{\delta/2} \cap V. \end{cases} \tag{2.8}$$

For this, we first remark that by an additive constant we can make ϕ positive and by a multiplicative one we can renormalize so that $0 \leqslant \phi \leqslant 1$. Next, we take a smooth decreasing cut-off function θ satisfying $\theta \equiv 1$ on $[0, \frac12]$ and $\theta \equiv 0$ on $[\frac23, 1]$, and define $\tilde\phi := \theta(-\frac{r}{\delta})\phi$. Recall that $L_j r = 0$ for $j \leqslant n-1$ and $u^\tau_{nK} \equiv 0$. Then, over such forms we have

$$\left(\sum_{|K|=k-1}' \sum_{i,j=1}^{n-1} \tilde{\phi}_{ij} u^\tau_{iK} \bar{u}^\tau_{jK} - \sum_{j=1}^{q_o} \tilde{\phi}_{jj} |u^\tau|^2\right)$$
$$\geqslant \theta \cdot \left(\sum_{|K|=k-1}' \sum_{i,j=1}^{n} \phi_{ij} u^\tau_{iK} \bar{u}^\tau_{jK} - \sum_{j=1}^{q_o} \phi_{jj} |u^\tau|^2\right). \tag{2.9}$$

In fact, if i and j denote derivation in L_i and $\bar{L}_j$ respectively, we have

$$\begin{aligned}\left(\theta\left(-\frac{r}{\delta}\right)\phi\right)_{ij} &= \ddot{\theta}\frac{r_i r_j}{\delta^2}\phi - \dot{\theta}\frac{\phi}{\delta} r_{ij} - \dot{\theta}\frac{r_j \phi_i}{\delta} - \dot{\theta}\frac{r_i \phi_j}{\delta} + \theta\phi_{ij} \\ &= -\dot{\theta}\frac{\phi}{\delta} r_{ij} + \theta\phi_{ij} \quad \text{on a tangential form } u^\tau.\end{aligned} \tag{2.10}$$

Since $-\dot{\theta} \geqslant 0$, then (2.10) implies (2.9). Note that $\partial\tilde{\phi} = -\dot{\theta}\frac{\partial r}{\delta}\phi + \theta\partial\phi$ and recall that $\partial r \cdot u^\tau \equiv 0$. It follows from (1.5) that $\tilde{\phi}$ satisfies (2.8) for a suitable $c > 0$; we keep denoting by ϕ this modified weight $\tilde{\phi}$.

We use now twice Proposition 2.1 and in both cases, owing to the assumption of q-pseudoconvexity (resp. q-pseudoconcavity) we have the crucial fact that the boundary integrals are $\geqslant 0$ for any $k \geqslant q > q_o$ (resp. $k \leqslant q < q_o$). We first use Proposition 2.1 under the choice $\phi \equiv 0$ and get

$$Q(u,u) + C\|u\|^2 \gtrsim \sum_{j=1}^{q_o} \|L_j u\|^2 + \sum_{j=q_o+1}^{n} \|\bar{L}_j u\|^2, \quad u \in C^\infty_c(\bar{D} \cap V)^k \cap D_{\bar{\partial}^*}. \tag{2.11}$$

This is immediate for u^τ and then also for $u \in D_{\bar{\partial}^*}$ because $u^\nu = u - u^\tau$ is 0 at ∂D and hence satisfies elliptic estimates. For this reason, we will not distinguish between $u \in D_{\bar{\partial}^*}$ and u^τ in what follows. We use again Proposition 2.1, this time for the weight $\chi(\phi^\delta)$ obtained by composing ϕ^δ satisfying (2.8) with a convex increasing function χ which will be chosen later. In this case, the second line of (2.4) splits into two terms

$$\int_D e^{-\chi(\phi^\delta)} \dot{\chi}\left(\sum_{|K|=k-1}' \sum_{ij=1}^{n} \phi^\delta_{ij} u_{iK} \bar{u}_{jK} - \sum_{j=1}^{q_o} \phi^\delta_{jj} |u|^2\right) dv$$
$$+ \int_D e^{-\chi(\phi^\delta)} \ddot{\chi}\left(\sum_{|K|=k-1}' \left|\sum_{j=1}^{n} \phi^\delta_j u_{jK}\right|^2 - \sum_{j=1}^{q_o} |\phi^\delta_j|^2 |u|^2\right) dv. \tag{2.12}$$

We also have

$$\|\bar{\partial}^*_{\chi(\phi^\delta)} u\|^2_{H^0_{\chi(\phi^\delta)}} \leqslant 2\|\bar{\partial}^* u\|^2_{H^0_{\chi(\phi^\delta)}} + 2\sum_{|K|=k-1}' \left\|\dot{\chi}\sum_j \phi^\delta_j u_{jK}\right\|^2_{H^0_{\chi(\phi^\delta)}}. \tag{2.13}$$

Remark that $|\sum_{j=1}^{q_o} (\chi(\phi^\delta))_j u|^2 = |\dot{\chi}|^2 |\sum_{j=1}^{q_o} \phi^\delta_j|^2 |u|^2$. Thus we get from (2.4), under the choice of the weight $\chi(\phi^\delta)$, and taking into account (2.12) and (2.13):

$$
\begin{aligned}
&\|\bar\partial u\|^2_{H^0_{\chi(\phi^\delta)}}+2\|\bar\partial^* u\|^2_{H^0_{\chi(\phi^\delta)}}+2C\|u\|^2_{H^0_{\chi(\delta)}}\\
&\geqslant \int\limits_D \dot\chi e^{-\chi(\phi^\delta)}\left(\sum_{|K|=k-1}{}' \sum_{i,j=1}^n \phi^\delta_{ij}u_{iK}\bar u_{jK}\,dv-\sum_{j=1}^{q_o}\phi^\delta_{jj}|u|^2\right)dv\\
&\quad+\int\limits_D \left(\ddot\chi-2\dot\chi^2\right)e^{-\chi(\phi^\delta)}\sum_{|K|=k-1}{}'\left|\sum_{j=1}^n \phi^\delta_j u_{jK}\right|^2 dv\\
&\quad-\int\limits_D \ddot\chi e^{-\chi(\phi^\delta)}\sum_{j=1}^{q_o}\left|\phi^\delta_j\right|^2|u|^2\,dv.
\end{aligned}
\tag{2.14}
$$

We now specify our choice of χ. First, we want $\ddot\chi \geqslant 2\dot\chi^2$ so that the first sum in the third line can be disregarded. Keeping this condition, we need an opposite estimate which assures that the absolute value of the last negative term in the fourth line of (2.14) is controlled by one half of the second line. If c is the constant of (2.8), the above condition is fulfilled as soon as $\frac{2\ddot\chi}{\dot\chi}\leqslant c$. If we then set $\chi := \frac{1}{2}e^{\frac{c}{2}(t-1)}$ then both requests are satisfied (we also notice that $\dot\chi^2 \ll \dot\chi$ because $c \ll 1$). Thus our inequality continues as

$$
\begin{aligned}
&\geqslant \frac{1}{2}\int\limits_D \dot\chi e^{-\chi(\phi^\delta)}\left(\sum_{|K|=k-1}{}' \sum_{i,j=1}^n \phi^\delta_{ij}u_{iK}\bar u_{jK}\,dv-\sum_{j=1}^{q_o}\phi^\delta_{jj}|u|^2\right)dv\\
&\geqslant \frac{1}{2}\int\limits_{S_{\delta/2}} \dot\chi e^{-\chi(\phi^\delta)}\left(\sum_{|K|=k-1}{}' \sum_{i,j=1}^n \phi^\delta_{ij}u_{iK}\bar u_{jK}-\sum_{j=1}^{q_o}\phi^\delta_{jj}|u|^2\right)dv\\
&\geqslant \delta^{-2\epsilon}\int\limits_{S_{\delta/2}}\frac{c}{2}\dot\chi e^{-\chi(\phi^\delta)}|u|^2\,dv.
\end{aligned}
\tag{2.15}
$$

Here we are using the two main assumptions for our weights ϕ^δ, that is, the first inequality $\geqslant c\sum'_{|K|=k-1}|\partial\phi\cdot u^\tau_{\cdot K}|^2$ in the right of (2.8) to get the second line and the second inequality $\geqslant c\delta^{-2\epsilon}|u^\tau|^2$ to get the third. Thus the first line of (2.12) is bigger or equal to the last of (2.15). We want to remove the weight from the resulting inequality. The first term can be handled owing to $e^{-\chi(\phi^\delta)}\leqslant 1$ on $\bar D\cap V$ and the second owing to $\dot\chi e^{-\chi(\phi^\delta)}\geqslant c\geqslant 0$ on $S_{\delta/2}\cap V$ which follows in turn from $|\phi^\delta|<1$. We end up with the unweighted estimate

$$
\|\bar\partial u\|^2+\left\|\bar\partial^* u\right\|^2+C\|u\|^2 \gtrsim \delta^{-2\epsilon}\int\limits_{S_{\delta/2}}|u|^2\,dv. \tag{2.16}
$$

Now, for fixed δ_o and for V contained in the δ_o-ball centered at $z_o=0$, the term $C\|u\|^2$ in the left of (2.16) can be absorbed in the right. Thus we end up with the estimate

$$\|\bar\partial u\|^2+\left\|\bar\partial^* u\right\|^2 \gtrsim \delta^{-2\epsilon}\int\limits_{S_{\delta/2}}|u|^2\,dv+\|u\|^2 \tag{2.17}$$

for any $u\in C_c^\infty(\bar D\cap V)^k\cap D_{\bar\partial^*}$ and $\delta\leqslant\delta_o$.

Combining (2.11) and (2.17), we get (2.5) which concludes the proof of the theorem. □

Theorem 2.2 is the essential tool for the proof of Theorem 1.7. This will be given in the section below; what we remark here is that the conclusion of Theorem 1.7 is better than it looks. Let ∂_ν be the normal derivative.

Proposition 2.3. *Assume that we have the subelliptic estimate* (1.4)*; then we have in fact*

$$|||u|||_\epsilon^2+|||\partial_\nu u|||_{-1+\epsilon}^2\leqslant c\left(\|\bar\partial u\|^2+\left\|\bar\partial^* u\right\|^2\right)+c_\epsilon\|u\|_0^2. \tag{2.18}$$

Proof. We remark that ∂_ν can be expressed as a linear combination of $\bar L_n$ and a suitable "totally real tangential" vector field that we denote by T. We then have

$$\begin{cases} Q(u,u)\gtrsim\|\bar L_n u\|^2,\\ |||u|||_\epsilon^2\gtrsim\|Tu\|_{\epsilon-1}^2.\end{cases}$$

It follows

$$\begin{aligned}\|u\|_\epsilon^2 &:= \left\|D_r(u)\right\|_{\epsilon-1}^2+|||u|||_\epsilon^2\\ &\lesssim Q(u,u)+|||u|||_\epsilon^2,\end{aligned}$$

which proves the claim. □

3. Proof of Theorem 1.7

Let V be a neighborhood of a given point $z_o\in\partial D$, let (t,r) be smooth coordinates in V with $t=(t_1,\dots,t_{2n-1})$ and let τ be dual coordinates to t. For a function u supported in V, one defines the tangential Fourier transform by

$$\hat u(\tau,r)=\int\limits_{\mathbb{R}^{2n-1}}e^{-it\tau}u(t,r)\,dt,$$

and the tangential H^s-Sobolev norm by

$$|||u|||_s^2=\left\|\Lambda^s u\right\|^2=\int\limits_{-\infty}^{0}\int\limits_{\mathbb{R}^{2n-1}}\left(1+|\tau|^2\right)^s\left|\hat u(\tau,r)\right|^2\,d\tau\,dr,$$

where Λ^s is the tangential Bessel potential of order s. We note that when $s=0$ then $|||u|||_0=\|u\|$ is the usual L^2-norm. We refer to [6] for further details.

We remark that if D_i is $\frac{\partial}{\partial t_j}$ or $\frac{\partial}{\partial r}$ and s is integer, then

$$\begin{aligned}\|u\|_{s+1}^2 &= \sum_i \|D_i u\|_s^2 \\ &\cong |||u|||_{s+1}^2 + \|D_r u\|_s^2.\end{aligned}$$

The next result contains the key estimate in the proof of Theorem 1.7.

Lemma 3.1. *Let U be a special boundary chart for D. Then for all $z_o \in \partial D \cap U$ there exists a neighborhood $V \subseteq U$ of z_o such that*

$$|||u|||_\epsilon^2 \lesssim \sum_{j\leqslant q_0} |||L_j u|||_{\epsilon-1}^2 + \sum_{j\geqslant q_0+1} |||\bar{L}_j u|||_{\epsilon-1}^2 + \|u_b\|_{\epsilon-\frac{1}{2}}^2 \quad \textit{for any function } u \in C_c^\infty(V \cap \bar{D}),$$

where $u_b := u|_{\partial D}$ and $\epsilon \leqslant \frac{1}{2}$.

The above lemma is a variant of Theorem 2.4.5 of [6] to which we refer for the proof. Notice that on one hand our statement is more general because we choose any $\epsilon \leqslant \frac{1}{2}$ instead of $\epsilon = \frac{1}{2}$. On the other, we specialize the choice of a general elliptic system to the case of $\{L_j\}_{j\leqslant q_o} \cup \{\bar{L}_j\}_{q_o+1\leqslant j\leqslant n}$.

For the proof of Theorem 1.7, we use a method derived from [3]. Let $p_k(t)$, $k = 0, 1, \ldots,$ be a sequence of functions with $\sum_{k=0}^\infty p_k^2(t) = 1$, $p_k(t) \equiv 0$ if $t \notin (2^{k-1}, 2^{k+1})$ with $k \geqslant 1$ and $p_0(t) \equiv 0, t \geqslant 2$. We can also choose p_k so that

$$\left|p_k'(t)\right| \leqslant C2^{-k}.$$

Let P_k denote the operator defined by

$$(\widehat{P_k u})(\tau, r) = p_k\big(|\tau|\big)\hat{u}(\tau, r)$$

where $\hat{u}$ is the tangential Fourier transform. Note that, induced by the partition $\sum_{k=0}^\infty p_k^2(t) = 1$, there is a decomposition of the Sobolev norms $|||u|||_\epsilon^2 \simeq \sum_{k=0}^{+\infty} 2^{k\epsilon}\|P_k u\|_0^2$. Let $\mathbb{R}_-^{2n} := \{z\colon r(z) < 0\}$ and denote by $S(\mathbb{R}_-^{2n})$ the Schwartz space of $C^\infty(\mathbb{R}_-^{2n})$-functions which are rapidly decreasing at ∞.

Lemma 3.2. *For $f, u \in S(\mathbb{R}_-^{2n})$ and $\sigma \in \mathbb{R}$ then*

$$\sum_{k=0}^\infty 2^{2k\sigma}\left\|[P_k, f]u\right\|^2 \lesssim |||u|||_{\sigma-1}^2.$$

Lemma 3.3. *Let T be a tangential vector field with coefficients in $C_0^\infty(\mathbb{R}_-^{2n})$. Then*

$$\sum_{k=0}^\infty \left\|[P_k, T]u\right\|^2 \leqslant C\|u\|^2.$$

The proof of Lemmas 3.2 and 3.3 can be found in [3, Lemmas 2.4 and 2.5] respectively. We remark that if we replace $u \in S(\mathbb{R}^{2n}_-)$ by $u \in C^\infty_c(\bar{D} \cap V)^k \cap D_{\bar\partial^*}$, then the two lemmas above still hold.

Proof of Theorem 1.7. By Lemma 3.1 and Theorem 2.2, we get for any $u \in C^\infty_c(\bar{D} \cap V)^k \cap D_{\bar\partial^*}$ with $k \geqslant q > q_o$ (resp. $k \leqslant q < q_o$)

$$\begin{aligned}
|||u|||^2_\epsilon &\lesssim \sum_{j=0}^{q} |||L_j u|||^2_{\epsilon-1} + \sum_{j=q_o+1}^{n} |||\bar{L}_j u|||^2_{\epsilon-1} + \|u_b\|^2_{\epsilon-1/2} \\
&\lesssim Q(u,u) + \|u_b\|^2_{\epsilon-1/2}.
\end{aligned}$$

Now, we estimate $\|u_b\|^2_{\epsilon-1/2}$. We have the elementary inequality

$$\left|g(0)\right|^2 \leqslant \frac{2^k}{\eta} \int_{-2^{-k}}^{0} \left|g(r)\right|^2 dr + 2^{-k}\eta \int_{-2^{-k}}^{0} \left|g'(r)\right|^2 dr,$$

which holds for any g such that $g(-2^{-k}) = 0$. If we apply it for $g(r) = \chi_k(r) P_k u(\cdot, r)$, where $\chi_k \in C^\infty_c(-2^{-k}, 0]$ with $0 \leqslant \chi_k \leqslant 1$ and $\chi_k(0) = 1$, we get

$$\begin{aligned}
\|u_b\|^2_{\epsilon-1/2} &\cong \sum_{k=0}^{\infty} 2^{2k(\epsilon-1/2)} \left\|\chi_k(0) P_k u_b\right\|^2 \\
&\leqslant \eta^{-1} \sum_{k=0}^{\infty} 2^{2k\epsilon} \int_{-2^{-k}}^{0} \left\|\chi_k P_k u(.,r)\right\|^2 dr + \eta \sum_{k=0}^{\infty} 2^{2k(\epsilon-1)} \int_{-2^{-k}}^{0} \left\|D_r\big(\chi_k P_k u(.,r)\big)\right\|^2 dr \\
&= \underbrace{\eta^{-1} \sum_{k=0}^{\infty} 2^{2k\epsilon} \int_{-2^{-k}}^{0} \left\|\chi_k P_k u(.,r)\right\|^2 dr}_{I} + \underbrace{\eta \sum_{k=0}^{\infty} 2^{2k(\epsilon-1)} \int_{-2^{-k}}^{0} \left\|D_r(\chi_k) P_k u(.,r)\right\|^2 dr}_{II} \\
&\quad \underbrace{+ \eta \sum_{k=0}^{\infty} 2^{2k(\epsilon-1)} \int_{-2^{-k}}^{0} \left\|\chi_k D_r\big(P_k u(.,r)\big)\right\|^2 dr}_{III}.
\end{aligned}$$

Observe that $\chi_k \leqslant 1$ and recall Theorem 2.2 that we apply for $P_k u$ and $\delta = 2^{-k}$. Thus the first sums above can be estimated by

$$(I) \leqslant \eta^{-1} \sum_{k=0}^{\infty} 2^{2k\epsilon} \int_{-2^{-k}}^{0} \left\|P_k u(.,r)\right\|^2 dr \lesssim \eta^{-1} \sum_{k=0}^{\infty} Q(P_k u, P_k u).$$

We note that Q can be written as a finite sum of terms of the type

$$M_i = a_i T_i + b_i D_r + c_i,$$

where T_i are tangential vector fields. Hence

$$\begin{aligned}\sum_{k=0}^{\infty} Q(P_k u, P_k u) &\leqslant \sum_{k=0}^{\infty}\big(\|P_k\bar\partial u\|^2 + \|P_k\bar\partial^* u\|^2\big) + \sum_i\sum_{k=0}^{\infty}\big\|[M_i, P_k]u\big\|^2\\ &\lesssim Q(u,u) + \sum_i\sum_{k=0}^{\infty}\big\|[a_iT_i, P_k]u\big\|^2 + \sum_i\sum_{k=0}^{\infty}\big\|[b_i, P_k]D_r(u)\big\|^2 + |||u|||^2_{-1}\\ &\lesssim Q(u,u) + \|u\|^2 + \big|\big|\big|D_r(u)\big|\big|\big|^2_{-1},\end{aligned}$$

where the estimates on the commutator terms follow by Lemmas 3.2 and 3.3. As it has already been remarked, $D_r(u)$ can be expressed as a linear combination of $\bar L_n u$ and Tu for some tangential vector field T. Then

$$\begin{aligned}\big|\big|\big|D_r(u)\big|\big|\big|^2_{-1} &\lesssim |||\bar L_n u|||^2_{-1} + |||Tu|||^2_{-1}\\ &\lesssim \|\bar L_n u\|^2 + \|u\|^2\\ &\lesssim Q(u,u)\end{aligned}$$

where the last line follows from Theorem 2.2.

We now estimate (*II*). Since $D_r(\chi_k) \leqslant 2^k$, we get

$$(II) \leqslant \eta\sum_{k=0}^{\infty}2^{2k\epsilon}\int_{-2^{-k}}^{0}\big\|P_k u(.,r)\big\|^2\,dr \leqslant \eta\sum_{k=0}^{\infty}2^{2k\epsilon}\|P_k u\|^2 \cong \eta|||u|||^2_{\epsilon}.$$

As for the term (*III*), we have $D_r P_k = P_k D_r$ and $\chi_k \leqslant 1$. Also $D_r = a\bar L_n + bT$ as before. Thus

$$\begin{aligned}(III) &\leqslant \eta\sum_{k=0}^{\infty}2^{2k(\epsilon-1)}\big\|P_k D_r(u)\big\| \cong \eta\big|\big|\big|D_r(u)\big|\big|\big|_{\epsilon-1}\\ &\lesssim \eta\big(|||\bar L_n u|||^2_{\epsilon-1} + |||Tu|||^2_{\epsilon-1}\big)\\ &\lesssim \eta Q(u,u) + \eta|||u|||^2_{\epsilon}.\end{aligned}$$

Combining all our estimates of $\|u_b\|_{\epsilon-1/2}$, we obtain

$$\|u_b\|_{\epsilon-1/2} \lesssim \eta^{-1}Q(u,u) + \eta|||u|||_{\epsilon}.$$

Summarizing up, we have shown that

$$|||u|||_{\epsilon} \lesssim \eta^{-1}Q(u,u) + \eta|||u|||^2_{\epsilon}.$$

Choosing $\eta > 0$ sufficiently small, we can move the term $\eta |||u|||_\epsilon^2$ into the left-hand side and get

$$|||u|||_\epsilon^2 \lesssim Q(u,u).$$

The proof is complete. □

4. Proof of Theorem 1.9

We start by pointing out that, in the assumptions of Theorem 1.9 the domain $\tilde{D}$ inherits from D the property of q-pseudoconvexity or q-pseudoconcavity. In fact, consider the basis of vector fields

$$L_j = \partial_{z_j} - r_{z_j}\partial_{z_n}, \quad j = 1,\dots,n-1, \qquad L_n = \sum_{j=1}^n r_{\bar{z}_j}\partial_{z_j}.$$

This is a "boundary frame", that is, it satisfies $\langle L_j, \partial r\rangle = 0$ for any $j = 1,\dots,n-1$ and $\langle L_n, \partial r\rangle = |\partial r|^2 \simeq 1$, but not an orthonormal basis. However, as we have already noticed, for the solution of the $\bar\partial$-Neumann problem, we can allow non-unitary changes of basis, provided that they preserve the $\bar\partial$-Neumann conditions as the changes of boundary frames do. In the frame that we have chosen, it is obvious that both D and $\tilde{D}$ are q-pseudoconvex with respect to the same bundle $\mathcal{V}$ and this suffices for the conclusion. (The same is true for the case q-pseudoconcave.) Note that we could use as well the basis in which the L_j's are unchanged for $j \leqslant n-1$ and, instead, $L_n = \partial_{z_n}$. In the sequel, it is understood that the metric has been changed so that the boundary frame has become orthonormal.

We now construct the weight ϕ which satisfies the assumptions of Theorem 1.7; we distinguish $q > q_o$ from $q < q_o$.

The case q-pseudoconvex. We set

$$\psi = -\log(-\tilde{r}+\delta) + \sum_{j=q}^{n-1} \log\big(|z_j|^2 + \delta^{\frac{1}{m_j}}\big), \tag{4.1}$$

and define $\phi := c|\log\delta|^{-1}\psi$ where c is an irrelevant constant needed to get the bound 1 in (1.5) or (2.8). We set $\psi^I = -\log(-\tilde{r}+\delta)$ and denote by ψ^{II} the remaining term in the right of (4.1); thus $\psi = \psi^I + \psi^{II}$. We have

$$\begin{aligned}\psi^I_{ij} &= (-\tilde{r}+\delta)^{-1}\tilde{r}_{ij}\\ &= (-\tilde{r}+\delta)^{-1}r_{ij} + (-\tilde{r}+\delta)^{-1}\big(\partial_{z_j}\partial_{\bar{z}_j}h^j\big)\kappa_{ij} + \mathcal{E} \quad \text{for } i,j \leqslant n-1,\end{aligned} \tag{4.2}$$

where $\mathcal{E}$ is an error of type $\mathcal{E} = O(|z|)(-\tilde{r}+\delta)^{-1}\sum_j(\partial_{z_j}\partial_{\bar{z}_j}h^j)$. We also have

$$\psi^I_{nn} = (-\tilde{r}+\delta)^{-2} \tag{4.3}$$

(where κ_{ij} continues to denote the Kronecker's symbol) and

$$\psi^{II}_{ij} = \left(\frac{\delta^{\frac{1}{m_j}}}{(|z_j|^2 + \delta^{\frac{1}{m_j}})^2}\right)\kappa_{ij}. \tag{4.4}$$

When taking $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot$ of $(-\tilde{r}+\delta)^{-1} r_{ij}$ from (4.2) and of $(-r+\delta)^{-2}$ from (4.3) the result is $\geqslant 0$. This is true for the forms $(r_{ij}(z))_{ij}|_{T^{\mathbb{C}}\partial D}$ and $(\tilde{r}_{ij}(z))_{ij}|_{T^{\mathbb{C}}\partial \tilde{D}}$ according to the first part of the proof. But it remains true also outside the boundaries for $(r_{ij}(z))_{ij}|_{\partial r^{\perp}(z)}$ and $(\tilde{r}_{ij}(z))_{ij}|_{\partial \tilde{r}^{\perp}(z)}$ where $\partial r^{\perp}$ and $\partial \tilde{r}^{\perp}$ denote the bundles orthogonal to $\partial r = \omega_n$ and $\partial \tilde{r}^{\perp}$ respectively. We prove it for r. We note that $r = 2\,\mathsf{Re}\, z_n + h$ is a graphing function and denote by $z \mapsto z^*$ the projection $\mathbb{C}^n \to \partial D$ in a neighborhood of z_o along the x_n-axis. We have the evident equalities

$$\begin{cases} \big(r_{ij}(z)\big)_{ij=1}^{n-1} = \big(r_{ij}\big(z^*\big)\big)_{ij=1}^{n-1}, \\ \partial r^{\perp}(z) = \partial r^{\perp}\big(z^*\big). \end{cases} \tag{4.5}$$

Thus (4.5) relates $L_r|_{T^{\mathbb{C}}\partial D}$ on $\partial D \cap V$ to $L_r|_{\partial r^{\perp}}$ on the whole of $\bar{D} \cap V$; in particular, (1.4) passes from $\partial D \cap V$ to the whole of $\bar{D} \cap V$. The same is true for $\tilde{r}$ and so the afore-mentioned sums for r and $\tilde{r}$ are positive; so we can discard them in (4.2) and (4.3). But what is left is just

$$(-\tilde{r}+\delta)^{-2}|\partial r|^2 + (-\tilde{r}+\delta)^{-1} 2\,\mathsf{Re} \sum_{j=1}^{n} r_{nj}\, \partial r \otimes \bar{\omega}_j,$$

where the first term is positive and the second is 0. We also discard all terms of type $(\partial_{z_j}\partial_{\bar{z}_j} h^j)\kappa_{ij}$ and $\delta^{\frac{1}{m_j}}\kappa_{ij}$ for i or $j \leqslant k-1$ in addition to $\mathcal{E}$ because they can be made positive by adding a small amount of terms for which $i, j \geqslant k$ on account of the estimates (4.7) and (4.8) which follow. For the remaining terms $(\partial_{z_j}\partial_{\bar{z}_j} h^j)$, we note that we have $(\partial_{z_j}\partial_{\bar{z}_j} h^j) \gtrsim |z_j|^{2m_j-2}$. We end up with the estimate

$$\begin{aligned} &\sum_{|K|=k-1}' \sum_{ij=1}^{n} \psi_{ij} u_{iK} \bar{u}_{jK} - \sum_{j=1}^{q_o} \psi_{jj} |u|^2 \\ &\quad \geqslant \sum_{j=k}^{n-1} \left((-\tilde{r}+\delta)^{-1} |z_j|^{2m_j-2} + \frac{\delta^{\frac{1}{m_j}}}{(|z_j|^2 + \delta^{\frac{1}{m_j}})^2} \right) \\ &\qquad \cdot \sum_{|K|=k-1}' |u_{jK}|^2 + (-r+\delta)^{-2} \sum_{|K|=k-1}' |u_{nK}|^2. \end{aligned} \tag{4.6}$$

We now inspect the coefficients in the right of (4.6). First, let $z \in S_\delta$, that is, $-r > \delta$. Given a coefficient u_J of u, the index J contains for sure at least one j such that $k \leqslant j \leqslant n-1$ and thus $u_J = \operatorname{sign}\binom{J}{jK} u_{jK}$ for a suitable K. If, for this j, $|z_j|^2 \geqslant \delta^{\frac{1}{m_j}}$, then

$$(-\tilde{r}+\delta)^{-1} |z_j|^{2m_j-2} \gtrsim \delta^{-\frac{1}{m_j}}. \tag{4.7}$$

On the contrary, if $|z_j|^2 \leqslant \delta^{\frac{1}{m_j}}$, then

$$\frac{\delta^{\frac{1}{m_j}}}{(|z_j|^2+\delta^{\frac{1}{m_j}})^2} \gtrsim \delta^{-\frac{1}{m_j}}. \tag{4.8}$$

In both cases, the terms in the left are $\geqslant \delta^{-2\epsilon_k}$ since $-\frac{1}{m_j} \leqslant -\frac{1}{m_k} = -2\epsilon_k$. We are ready to prove (2.7) (which is equivalent to (1.5)). We split the inequality $\# \gtrsim \delta^{-2\epsilon}|u^\tau|^2 + \sum_{j=1}^{q_o} |\phi_j(z)|^2|u^\tau|^2$ into two inequalities $\# \gtrsim \sum_{j=1}^{q_o} |\phi_j(z)|^2|u^\tau|^2$ and $\# \gtrsim \delta^{-2\epsilon}|u^\tau|^2$ that we denote by (2.7)(i) and (2.7)(ii) respectively. Now, by combining (4.7) with (4.8), we get (2.7)(ii) for $\epsilon = \epsilon_k$. On the other hand, for any $j \leqslant q_o$, we have $\frac{r_j}{-r+\delta} = 0$ and we also have $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot \geqslant 0$ (all over $\bar{D} \cap V$). This proves (2.7)(i).

Finally, a normalization by a factor $c|\log\delta|^{-1}$ makes the weight bounded at the expenses of passing from $\delta^{-2\epsilon_k}$ to $\frac{\delta^{-2\epsilon_k}}{|\log\delta|}$ in the second of (2.8). Thus the weight ψ satisfies the requirements of Theorem 1.7 for any $\epsilon < \epsilon_k$ which implies subelliptic estimates of the corresponding order. Incidentally, we notice that when $\epsilon_k = \frac{1}{2}$, the term ψ^{II} is needless and we can take a different normalization by defining $\phi = -\log(\frac{-r+\delta}{2\delta})$; thus we get an even δ^{-1} on the right of (2.8). For $\epsilon_k = \frac{1}{2}$, a similar argument applies also to the case q-pseudoconcave which follows and we will not insist on it.

The case q-pseudoconcave. We now define

$$\psi = -\log(-\tilde{r}+\delta) - \sum_{j=1}^{k+1} \log\big(-\log\big(|z_j|^2+\delta^{\frac{1}{m_j}}\big)\big)$$

where we point out the attention to the double log. Comparing with the case q-pseudoconvex, there is now an extra difficulty for the weight to satisfy the first of (2.8) (whereas the second remains substantially unchanged) because we do not have any longer $\phi_j = 0$ for $j \leqslant q_o$. We write $\psi = \psi^I + \psi^{II}$ in the same way as in the previous case and will eventually define ϕ by a normalization $\phi = c|\log\delta|^{-1}\psi$. We have the analogues of (4.2) and (4.4) with the suitable sign. We apply $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot$ to $\psi^I + \psi^{II}$. When taking $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot$ we discard the contribution of $(-\tilde{r}+\delta)^{-1} r_{ij}$ in addition to the normal term $(-\tilde{r}+\delta)^{-2}$ because this contribution is positive as before. We discard the error term $\mathcal{E}$ because it can be made positive by the aid of a small amount of the remainder. This argument is the same as for the case q-pseudoconvex. What we are left with is

$$\begin{aligned}\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot \geqslant \sum_{j=1}^{k+1} \Bigg(&(-r+\delta)^{-1}|z_j|^{2m_j-2} + \frac{\delta^{\frac{1}{m_j}}}{(|z_j|^2+\delta^{\frac{1}{m_j}})^2} \frac{1}{|\log(|z_j|^2+\delta^{\frac{1}{m_j}})|} \\ &+ \frac{|z_j|^2}{(|z_j|^2+\delta^{\frac{1}{m_j}})^2} \frac{1}{|\log(|z_j|^2+\delta^{\frac{1}{m_j}})|^2} \Bigg) \Bigg(|u|^2 - \sum_{|K|=k-1}{}' |u_{jK}|^2 \Bigg). \end{aligned} \tag{4.9}$$

We split (2.7) into (2.7)(i) and (2.7)(ii) as we did in the q-pseudoconvex case and write the coefficient in the right of (4.9) as $(A_j + B_j + C_j)$. The two first terms $A_j + B_j$ serve for getting (2.7)(ii), the third C_j for (2.7)(i). (This latter was discarded as $\geqslant 0$ in the case q-pseudoconvex; here it is essential because $\phi_j \neq 0$ for $j \leqslant q_o$.) Reasoning as in the first half of the proof we get, for any $j \leqslant k+1$

$$A_j + B_j \gtrsim \delta^{-\frac{1}{m_j}} \geqslant \delta^{-2\epsilon_k} \quad \text{on } S_\delta \cap V, \tag{4.10}$$

because $-\frac{1}{m_j} \leqslant -\frac{1}{m_{k+1}} = -2\epsilon_k$ for any $j \leqslant k+1$, along with

$$A_j + B_j \geqslant 0 \quad \text{on } \bar{D} \cap V. \tag{4.11}$$

We make the crucial remark for the case of concavity. If the degree of u is k, then

$$\sum_{j=1}^{k+1}\Bigg(|u|^2 - \sum_{|K|=k-1}{}' |u_{jK}|^2\Bigg) \geqslant |u|^2. \tag{4.12}$$

From (4.10) and (4.12) we get the second of (2.8). We now need to prove that on $\bar{D} \cap V$ and for a suitable ϵ we have

$$\begin{aligned}\sum_{j=1}^{k+1}\frac{1}{\log^2}\frac{|z_j|^2}{(|z_j|^2+\delta^{\frac{1}{m_j}})^2}\Bigg(|u|^2 - \sum_{|K|=k-1}{}' |u_{jK}|^2\Bigg) &\geqslant \epsilon\sum_{j=1}^{k+1}\frac{1}{\log^2}\frac{|z_j|^2}{(|z_j|^2+\delta^{\frac{1}{m_j}})^2}|u|^2 \\ &= \epsilon\sum_{j=1}^{k+1}|\psi_j|^2|u|^2. \end{aligned}\tag{4.13}$$

This would conclude the proof of (1.5). The last sum $\sum_{j=1}^{k+1}\cdot$ can be replaced by $\sum_{j=1}^{q_o}\cdot$ since $\psi_j = 0$ for $j = k+2, \dots, q_o$. Also, remember here that $\psi_j^I = 0$ for any j and $\psi_j^{II} = 0$ for any $j \geqslant k+2$; this justifies the last equality in (4.13) which is true. However, the first inequality is wrong. To make it true, we need a small perturbation of ψ. We take a vector v in the unit sphere S^k outside the first quadrant, set $\psi^{IIv} := \sum_{j=1}^{k+1}\log(-\log(|z_j|^2+\delta^{\frac{1}{m_j}})^{v_j})$, leave ψ^I unchanged and define a new ψ by

$$\psi := \psi^I + \frac{1}{2}\big(\psi^{II} + \psi^{IIv}\big).$$

Inequalities (4.10) and (4.11) are stable under perturbation and thus will remain true for this new ψ. As for the first of (4.13), we consider the vector field

$$w(z) := \left(\frac{1}{\log^2}\frac{|z_j|}{(|z_j|^2+\delta^{\frac{1}{m_j}})}\right)_{j=1,\dots,k+1}.$$

We also define

$$\mu(z) = \frac{w(z)}{|w(z)|}, \qquad \nu(z) = (\nu_j v_j)_{j=1,\dots,k+1};$$

thus $|\mu| = 1$ and $|\nu| \leqslant 1$. Finally, we set

$$u = \frac{(u_{jK})_{j=1,\dots,k+1}}{\sum_j |u_{jK}|^2}.$$

It suffices to prove that

$$\frac{1}{2}\big(\langle\mu, u\rangle^2 + \langle\nu, u\rangle^2\big) \leqslant 1 - \epsilon.$$

Now, we begin by noticing that

$$\begin{cases} \langle\mu, u\rangle \leqslant 1, \\ \langle\nu, u\rangle \leqslant 1, \end{cases} \tag{4.14}$$

by Cauchy–Schwartz inequality. Also, if the first of (4.14) happens to be equality, that is, μ is parallel to u, then

$$\begin{aligned} \langle\nu, u\rangle &= \sum_j v_j \mu_j u_j \\ &= \sum_j v_j \mu_j^2. \end{aligned}$$

But for this to be 1 we need both $\sum_j \mu_j^4 = 1$ and v parallel to $(\mu_j^2)_{j=1,\dots,k+1}$. If the first occurs then, since $\sum_j \mu_j^2 = 1$, we have $(\mu_j^2) = (\mu_j)$ (and both coincide with a Cartesian vector): thus (μ_j^2) is not parallel to v. In conclusion if the first of (4.14) is equality, the second is not. Therefore, the function $(u, \mu) \mapsto \frac{1}{2}(\langle u, \mu\rangle^2 + \langle u, \nu\rangle^2)$ has a minimum < 1, say $1 - \epsilon$, for $u \in S^k$ (and for $\nu = (\mu_j v_j)$).

5. Proof of Theorem 1.13

Let $\epsilon \leqslant \inf_{j \leqslant p} \frac{1}{2m_j}$.

Lemma 5.1. *We have for large t*

$$\int_0^\delta \dots \int_0^\delta \frac{dx_1\, dy_1 \dots dx_p\, dy_p}{(t \sum_{j=1}^p |t^{-\varepsilon} z_j|^{2m_j} + 1)^s} \cong t^{-\sum_1^p \frac{1}{m_j} + 2p\varepsilon}, \tag{5.1}$$

provided that $s > \frac{1}{m_1} + \dots + \frac{1}{m_p} + 1$.

Proof. We can assume that $m_1 \leqslant m_2 \leqslant \dots \leqslant m_p$. Put $a(t) = t \sum_{j=2}^p |t^{-\varepsilon} z_j|^{2m_j} + 1$. First, we perform integration

$$M(z_2, \dots, z_p) = \int_0^\delta \int_0^\delta \frac{dx_1\, dy_1}{(t|t^{-\varepsilon} z_1|^{2m_1} + a(t))^s}.$$

We also make a change of variables $z_1' = t^{\frac{1}{2m_1}-\varepsilon} a(t)^{-\frac{1}{2m_1}} z_1$ and get

$$M(z_2,\dots,z_p) = a(t)^{-s+\frac{1}{m_1}} t^{-\frac{1}{m_1}+2\varepsilon} \int_0^{t^{\frac{1}{2m_1}-\varepsilon} a(t)^{-\frac{1}{2m_1}}\delta} \int_0^{t^{\frac{1}{2m_1}-\varepsilon} a(t)^{-\frac{1}{2m_1}}\delta} \frac{dx_1'\,dy_1'}{(|z_1'|^{2m_1}+1)^s}.$$

Since

$$t^{\frac{1}{2m_1}-\varepsilon} a(t)^{-\frac{1}{2m_1}}\delta = \left(\frac{t^{1-2\varepsilon m_1}\delta^{2m_1}}{\sum_{j=2}^p t^{1-2\varepsilon m_j}|z_j|^{2m_j}+1}\right)^{\frac{1}{2m_1}} \geqslant C > 0,$$

then

$$M(z_2,\dots,z_p) \cong a(t)^{-s+\frac{1}{m_1}} t^{-\frac{1}{m_1}+2\varepsilon}.$$

In conclusion, the left-hand side of (5.1) is equivalent to

$$t^{-\frac{1}{m_1}+2\varepsilon} \int_0^\delta \dots \int_0^\delta \frac{dx_2\,dy_2\dots dx_p\,dy_p}{(t\sum_{j=2}^p |t^{-\varepsilon}z_j|^{2m_j}+1)^{s-\frac{1}{m_1}}}.$$

Repetition of this argument for $z_2,\dots,z_p$ yields the proof of the lemma. □

Proof of Theorem 1.13(i). As in the proof of Theorem 1.9, we start from

$$\begin{cases} L_j = \partial_{z_j} - r_{z_j}\partial_{z_n}, \\ L_n = \sum_{j=1}^n r_{\bar z_j}\partial_{z_j} \end{cases}$$

as an adapted basis of $(1,0)$ vector fields. It is not yet an orthonormal system but it has the property that $\langle L_j, L_n\rangle = 0$; by a perturbation within the plane of $L_1,\dots,L_{n-1}$ we can make it orthogonal. Our calculations will be performed in this orthonormal system. Since the standard metric is preserved, the hypothesis of existence of subelliptic estimates is kept. Note that much simpler calculations could be performed, instead, in the system

$$\begin{cases} L_j = \partial_{z_j} - r_{z_j}\partial_{z_n}, \\ L_n = \partial_{z_n} \end{cases} \tag{5.2}$$

(cf. e.g. Ho [9]). This is still a basis adapted to the boundary but does not enjoy any orthogonality relation. Since we are assuming subelliptic estimates in the standard basis, we do not use the system (5.2).

We remark that for a k-form u we have $u \in D_{\bar\partial^*}$ if and only if its coefficients satisfy $u_{nK}|_{\partial D} \equiv 0$ for any $|K| = k-1$. Let L_j be the dual basis of $(1,0)$ vector fields; these are a perturbation of $\partial_{z_j} - r_{z_j}\partial_{z_n}$, $j = 1,\dots,n-1$, and $\sum_{j=1}^n r_{z_j}\partial_{z_j}$. We have

- $(\omega_{iK}, \omega_{jK}) = \kappa_{ij} + r_{z_i} r_{\bar{z}_j}$ for any $i, j \leqslant n-1$,
- $(\omega_{jK}, \omega_{nK}) = 0$ for any $j \leqslant n-1$,
- $(\omega_I, \omega_J) = 0$ if $|I \cap J| \leqslant k-2$,
- $(\bar{\partial}^* u)_K = \sum_{j=1}^{n} \sum_{\{J:\ |J \cap jK| = k\}} L_j(u_J) + \sum_{j=1}^{n-1} \sum_{\{J:\ |J \cap jK| = k-1\}} L_j(u_J)(O(r_{z_j}) + \sum_{i \in J} O(r_{z_i})) + \text{error}$,

where “error” denotes a term where no derivatives of u occur. We will deal with the form

$$u_t = U_t \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k,$$

where U_t is a function which will be specified later. We have for this form

$$\begin{cases} \bar{\partial} u \simeq \displaystyle\sum_{j=k+1}^{n} \bar{L}_j(U_t) \bar{\omega}_j \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k + \text{error}, \\ \bar{\partial}^* u = \displaystyle\sum_{j=1}^{k} L_j(U_t) \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_{j-1} \wedge \bar{\omega}_{j+1} \wedge \cdots \wedge \bar{\omega}_k \\ \qquad + \displaystyle\sum_{j=1}^{n-1} \sum_{H \cap \{1,\dots,k,j\} \neq \emptyset} L_j(U_t) \Big(O(r_{z_j}) + \sum_{i \leqslant k} O(r_{z_i}) \Big) \bar{\omega}_H + \text{error}. \end{cases}$$

In particular

$$\begin{aligned} \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 \lesssim{} & \sum_{j=k+1}^{n} \|\bar{L}_j U_t\|^2 + \sum_{j=1}^{k} \|L_j U_t\|^2 \\ & + \sum_{i=k+1,\dots,n-1} \Big\| \Big(O(|r_{z_i}|) + \sum_{i \leqslant k} O(|r_{z_j}|) \Big) L_i U_t \Big\|^2 + \|U_t\|^2. \end{aligned} \tag{5.3}$$

Let $\epsilon_k = \frac{1}{2m_k}$ and define $U_t = f_t(z', z_n) \Phi_t(z)$ by

$$\begin{cases} f_t(z', z_n) = \Big(z_n - Q(z') - \dfrac{1}{t} \Big)^{-p}, \\ \Phi_t(z) = \Big(\displaystyle\prod_{j=1}^{n-1} \phi(t^{\epsilon_k} x_j) \phi(t^{\epsilon_k} y_j) \Big) \lambda(x_n) \phi(y_n). \end{cases}$$

Here $\phi \in C_0^\infty(\mathbb{R})$ satisfies

$$\phi(x) = \begin{cases} 1, & x \leqslant \delta, \\ 0, & x \geqslant 2\delta, \end{cases}$$

where δ is a small parameter, and $\lambda \in C_0^\infty(\mathbb{R})$ will be chosen later.

Since

$$\begin{cases} L_j(f_t)=0 & \text{for any } j\leqslant q_o,\\ \bar L_j(f_t)=0 & \text{for } q_o+1\leqslant j\leqslant n-1,\\ \partial_{z_j}(f_t)=\partial_{\bar z_j}(f_t)=0 & \text{for } j=q_o+1,\dots,n-1,\end{cases}$$

we can restrict the first sum in (5.3) to $j=n$ and the second to $j=q_o+1,\dots,k;$ thus we get

$$\begin{aligned}Q(u_t,u_t)\lesssim{}&\sum_{ij=1}^{q_o}\big\|r_{z_i}\partial_{\bar z_j}(f_t)\Phi_t\big\|^2+\sum_{j=q_o+1}^{k}\big\|r_{z_j}\partial_{z_n}(f_t)\Phi_t\big\|^2\\&+\sum_{\substack{i=k+1,\dots,n-1\\ j=1,\dots,k,i}}\big\||r_{z_j}||r_{z_i}|\partial_{z_n}(f_t)\Phi_t\big\|^2\\&+\sum_{ij=1}^{n-1}\big\|O^2\big(|r_{z_i}|\big)\partial_{z_j}f_t\Phi_t\big\|^2+\sum_{j=1}^{n}\|f_t\partial_{z_j}\Phi_t\|^2+\|U_t\|^2.\end{aligned}\tag{5.4}$$

To estimate the first three sums in (5.4) we need to evaluate r_{z_i} for $i=1,\dots,q_o$, next r_{z_j} for $j=q_o+1,\dots,k$ and finally $r_{\bar z_i}r_{z_j}r_{z_i}$ for $i=k+1,\dots,n-1,\ j=1,\dots,k$. We perform the change of variables

$$\begin{cases}\tilde z_j=t^{\epsilon_k}z_j, & j\leqslant n-1,\\ \tilde z_n=tz_n.\end{cases}$$

For $|\tilde z|\leqslant 1$ we have for the first terms

$$\begin{aligned}\big|r_{z_j}(z)\big|&=|z_j|^{4m_j-2}\\&=t^{-\epsilon_k(4m_j-2)}\leqslant t^{-2+2\epsilon_k},\quad j=q_o+1,\dots,k,\end{aligned}\tag{5.5}$$

where the last inequality follows from $m_j\geqslant m_k$. For the second terms we have

$$\begin{aligned}\big|r_{z_i}\big(z'\big)\big|^2&\leqslant\big|z'\big|^{4m-2}\\&=t^{-\epsilon_k(4m-2)}\leqslant t^{-2+2\epsilon_k},\quad i=1,\dots,q_o,\end{aligned}\tag{5.6}$$

where the last estimate follows from $m\geqslant m_{q_o+1}\geqslant m_k$. For the third terms we extend the definition of m_j to $j\leqslant q_o$ by putting $m_j=m$. We have, for $i\geqslant k+1,\ j\leqslant k$ or $j=i$

$$\begin{aligned}\big|r_{z_j}(z)\big|^2\big|r_{z_i}(z)\big|^2&\leqslant t^{-\epsilon_k(4m_i+4m_j-4)}\\&\leqslant t^{-2-\epsilon_k(4m_j-4)}\leqslant t^{-2},\end{aligned}\tag{5.7}$$

where the second inequality follows from $m_i\leqslant m_k$. If we pass to estimate the terms in the second sum of (5.4), we then have

$$\begin{aligned}\left\| r_{z_j}\frac{\partial f_t}{\partial z_n}\Phi_t\right\|^2 &\cong \int |r_{z_j}|^2\left|z_n - Q\big(z'\big)-\frac{1}{t}\right|^{-2p-2}\Phi_t^2(z)\,dx_1\,dy_1\ldots dx_n\,dy_n\\ &\lesssim \int \frac{|z_j|^{4m_j-2}}{((\frac{1}{t}+Q(z')-x_n)^2+y_n^2)^{p+1}}\Phi_t^2(z)\,dx_1\,dy_1\ldots dx_n\,dy_n\\ &\lesssim t^{2p-2+2\epsilon_k-2(n-1)\epsilon_k} I_t,\end{aligned}$$

where

$$I_t = \int \frac{(\prod_{j=1}^{n-1}\phi(\tilde{x}_j)\phi(\tilde{y}_j))^2\lambda(t^{-1}\tilde{x}_n)^2\phi(t^{-1}\tilde{y}_n)^2}{((1+tQ(t^{-\epsilon_k}\tilde{z}')-\tilde{x}_n)^2+\tilde{y}_n^2)^{p+1}}\,d\tilde{x}_1\,d\tilde{y}_1\ldots d\tilde{x}_n\,d\tilde{y}_n.$$

We now perform integration in $\tilde{y}_n$ from $-\infty$ to $+\infty$ and get

$$I_t \lesssim \int \frac{(\prod_{j=1}^{n-1}\phi(\tilde{x}_j)\phi(\tilde{y}_j))^2\lambda(t^{-1}\tilde{x}_n)^2}{(1+tQ(t^{-\epsilon_k}\tilde{z}')-\tilde{x}_n)^{2p+1}}\,d\tilde{x}_1\,d\tilde{y}_1\ldots d\tilde{x}_{n-1}\,d\tilde{y}_{n-1}\,d\tilde{x}_n.$$

Next, we integrate in $\tilde{x}_n$ from

$$-\infty \quad \text{to} \quad \left(tQ\big(t^{-\epsilon_k}\tilde{z}'\big) - t\sum_{j=q_0+1}^{n-1}\left|t^{-\epsilon_k}\tilde{z}_j\right|^{2m_j}\right)\Big/2,$$

and get

$$\begin{aligned}I_t &\lesssim \int \frac{(\prod_{j=1}^{n-1}\phi(\tilde{x}_j)\phi(\tilde{y}_j))^2}{(tQ(t^{-\epsilon_k}\tilde{z}')+t\sum_{j=q_0+1}^{n-1}|t^{-\epsilon_k}\tilde{z}_j|^{2m_j}+2)^{2p}}\,d\tilde{x}_1\,d\tilde{y}_1\ldots d\tilde{x}_{n-1}\,d\tilde{y}_{n-1}\\ &\lesssim \int_0^{2\delta}\ldots\int_0^{2\delta}\frac{d\tilde{x}_{k+1}\,d\tilde{y}_{k+1}\ldots d\tilde{x}_{n-1}\,d\tilde{y}_{n-1}}{(t\sum_{j=k+1}^{n-1}|t^{-\epsilon_k}z_j|^{2m_j}+1)^{2p}}\\ &\lesssim t^{-\sum_{j=k+1}^{n-1}\frac{1}{m_j}+2(n-k-1)\epsilon_k}\end{aligned}$$

where the last inequality follows by Lemma 5.1.

In conclusion we have obtained

$$\left\| r_{z_j}\frac{\partial f_t}{\partial z_n}\Phi_t\right\|^2 \lesssim t^{2p-2+2\epsilon_k-2k\epsilon_k-\sum_{j=k+1}^{n-1}\frac{1}{m_j}}. \tag{5.8}$$

The same integration combined with (5.7) yields the same estimate as (5.8) also for the terms $\| |r_{z_j}||r_{z_i}|\partial_{z_n}(f_t)\Phi_t\|^2$ for $i \geqslant k+1$ and $j \leqslant k$. As for the terms in the first sum in (5.4) with $i, j = 1, \ldots, q_0$, we have

$$
\begin{aligned}
\left\| r_{z_j} \frac{\partial f_t}{\partial \bar z_j} \right\|^2 &\cong \int |r_{z_j}|^4 \left| z_n - Q(z') - \frac{1}{t} \right|^{-2p-2} \Phi_t^2(z)\, dx_1\, dy_1 \dots dx_n\, dy_n \\
&\lesssim \int \frac{|z'|^{4m_k-2}}{((\frac{1}{t} + Q(z') - x_n)^2 + y_n^2)^{p+1}} \Phi_t^2(z)\, dx_1\, dy_1 \dots dx_n\, dy_n \\
&\lesssim t^{2p-2+2\epsilon_k - 2k\epsilon_k - \sum_{j=k+1}^{n-1} \frac{1}{m_j}}
\end{aligned}
$$

where the last inequality follows by the same technique as above.

By the same argument all the sums $\sum_{ij=1}^{n-1} \| O^2(|r_{z_i}|)\partial_{z_j} f_t \Phi_t\|^2$, the terms $\| f_t \frac{\partial \Phi_t}{\partial z_j}\|^2$, $j = 1, \dots, n$, and $\|U_t\|^2$ have the same estimate in terms of t. Combining all these estimates, we get the basic estimate from above for $Q(u_t, u_t)$

$$
Q(u_t, u_t) \lesssim t^{2p-2+2\epsilon_k - 2k\epsilon_k - \sum_{j=k+1}^{n-1} \frac{1}{m_j}}. \tag{5.9}
$$

To calculate $|||u|||_\epsilon$ we use the boundary coordinates $(x_1, \dots, x_{n-1}, y_1, \dots, y_n, r)$ and dual coordinates $(\xi, r) = (\xi_1, \dots, \xi_{2n-1}, r)$. We have

$$
\begin{aligned}
|||u_t|||_\epsilon &= |||U_t|||_\epsilon^2 + \sum_{j=1}^{k} |||r_i U_t|||_\epsilon^2 \geqslant |||U_t|||_\epsilon^2 \\
&\geqslant \int \left(1 + |\xi|^2\right)^\epsilon \left| \hat U_t(x_1, \dots, x_{n-1}, y_1, \dots, y_n, r) \right|^2 d\xi\, dr \\
&\geqslant \int \left| \xi_{2n-1}^{2\epsilon} \right| \left| \int \frac{\phi(y_n)\lambda(x_n) e^{-i\xi_{2n-1} y_n}\, dy_n}{(x_n - Q(z') - 1/t + iy_n)^p} \right|^2 \\
&\quad \cdot \left(\prod_{j=1}^{n-1} \phi\left(t_k^\epsilon x_j\right) \phi\left(t^{\epsilon_k} y_j\right) \right)^2 dx_1\, dy_1 \dots dx_{n-1}\, dy_{n-1}\, d\xi_{2n-1}\, dr,
\end{aligned}
$$

where we use Plancherel's theorem on $\xi_1, \dots, \xi_{2n-2}$ in the second line. Similarly as before, we use transformations

$$
\begin{cases}
\tilde x_j = t^{\epsilon_k} x_j, \quad \tilde y_j = t^{\epsilon_k} y_j, \quad j = 1, \dots, n-1, \\
\tilde y_n = t y_n, \quad \tilde \xi_{2n-1} = 1/t \xi_{2n-1}, \quad \tilde r = tr,
\end{cases}
$$

and obtain

$$
|||u_t|||_\epsilon^2 \geqslant t^{2p-2+2\epsilon - 2(n-1)\epsilon_k} J_t,
$$

where

$$
\begin{aligned}
J_t = \int |\tilde\xi_{2n-1}|^{2\epsilon} &\left| \int \frac{\phi(t^{-1}\tilde y_n)\lambda(x_n(t^{-\epsilon_k}\tilde x_1, \dots, t^{-1}\tilde r)) e^{-i\tilde\xi_{2n-1}\tilde y_n}\, d\tilde y_n}{(-g + i\tilde y_n)^p} \right|^2 \\
&\cdot \left(\prod_{j=1}^{n-1} \phi(\tilde x_j)\phi(\tilde y_j) \right)^2 d\tilde x_1\, d\tilde y_1 \dots d\tilde x_{n-1}\, d\tilde y_{n-1}\, d\tilde\xi_{2n-1}\, d\tilde r.
\end{aligned}
$$

Here

$$g=-\left(\frac{\tilde r-tQ(t^{-\epsilon_k}\tilde z')-t\sum_{j=q_0+1}^{j=n-1}|t^{-\epsilon_k}\tilde z_j|^{2m_j}}{2}-1\right).$$

Since $tQ(t^{-\epsilon_k}\tilde z')+t\sum_{j=q_0+1}^{k}|t^{-\epsilon_k}\tilde z_j|^{2m_j}\lesssim\sum_{j=1}^{q_0}|\tilde z_j|^{2m}+\sum_{j=q_0+1}^{k}|\tilde z_j|^{2m_j}$, then if the support of ϕ is small enough we can assume

$$tQ\big(t^{-\epsilon_k}\tilde z'\big)+t\sum_{j=n-k+q_0}^{n-1}\big|t^{-\epsilon_k}\tilde z_j\big|^{2m_j}\leqslant 1.$$

This implies $0<g\leqslant\frac{-\tilde r+t\sum_{j=k+1}^{n-1}|t^{-\epsilon_k}\tilde z_j|^{2m_j}+3}{2}$. Using a further substitution

$$y_n'=g\tilde y_n,\qquad \xi_{2n-1}'=\frac{1}{g}\tilde\xi_{2n-1},$$

we get

$$\begin{aligned}
J_t&=\int\frac{|\xi_{2n-1}'|^{2\epsilon}}{g^{2p+1-2\epsilon}}\left|\int\frac{\phi(\frac{y_n'}{tg})\lambda(x_n)e^{-i\xi_{2n-1}'y_n'}\,dy_n'}{(-1+i\tilde y_n)^p}\right|^2\\
&\quad\cdot\left(\prod_{j=1}^{n-1}\phi(\tilde x_j)\phi(\tilde y_j)\right)^2 d\tilde x_1\,d\tilde y_1\ldots d\tilde x_{n-1}\,d\tilde y_{n-1}\,d\tilde\xi_{2n-1}\,d\tilde r\\
&=J_1+J_2
\end{aligned}$$

where J_1 is the integration from $-\infty$ to $-tK$, J_2 from $-tK$ to 0 and where K is suitably chosen. Note that $J_1\geqslant 0$. Now, we consider J_2.

For $\tilde r\in[-tK,0]$, we see that

$$|x_n|=\left|\frac{\tilde r/t+Q(t^{-\epsilon_k}\tilde z')-\sum_{j=q_0+1}^{n-1}|t^{-\epsilon_k}\tilde z_j|^2}{2m_j}\right|\leqslant C.$$

We may choose $\lambda\in C_0(\mathbb{R})$ such that $\lambda(x)=1$ for $|x|\leqslant C$. Then

$$\int\big|\xi_{2n-1}'\big|^{2\epsilon}\left|\int\frac{\phi(\frac{y_n'}{tg})\lambda(x_n)e^{-i\xi_{2n-1}'y_n'}\,dy_n'}{(-1+i\tilde y_n)^p}\right|^2 d\xi_{2n-1}'\geqslant\text{const}>0.$$

It follows

$$J_2\gtrsim\int\int_{\tilde r=-tK}^{0}\frac{(\prod_{j=1}^{n-1}\phi(\tilde x_j)\phi(\tilde y_j))^2\,d\tilde x_1\,d\tilde y_1\ldots d\tilde x_{n-1}\,d\tilde y_{n-1}}{(-\tilde r+t\sum_{j=k+1}^{n-1}|t^{-\epsilon_k}\tilde z_j|^{2m_j}+3)^{2p+1-2\epsilon}}\,d\tilde r$$

$$\gtrsim \int \frac{(\prod_{j=1}^{n-1} \phi(\tilde{x}_j)\phi(\tilde{y}_j))^2 \, d\tilde{x}_1 \, d\tilde{y}_1 \ldots d\tilde{x}_{n-1} \, d\tilde{y}_{n-1}}{(t \sum_{j=k+1}^{n-1} |t^{-\epsilon_k} \tilde{z}_j|^{2m_j} + 3)^{2p-2\epsilon}} \, d\tilde{r}$$

$$- \int \frac{(\prod_{j=1}^{n-1} \phi(\tilde{x}_j)\phi(\tilde{y}_j))^2 \, d\tilde{x}_1 \, d\tilde{y}_1 \ldots d\tilde{x}_{n-1} \, d\tilde{y}_{n-1}}{(tK + t \sum_{j=k+1}^{n-1} |t^{-\epsilon_k} \tilde{z}_j|^{2m_j} + 3)^{2p-2\epsilon}} \, d\tilde{r}$$

$$\gtrsim \int \frac{(\prod_{j=1}^{n-1} \phi(\tilde{x}_j)\phi(\tilde{y}_j))^2 \, d\tilde{x}_1 \, d\tilde{y}_1 \ldots d\tilde{x}_{n-1} \, d\tilde{y}_{n-1}}{(t \sum_{j=k+1}^{n-1} |t^{-\epsilon_k} \tilde{z}_j|^{2m_j} + 3)^{2p-2\epsilon}} \, d\tilde{r}.$$

The last inequality follows from the fact that we can choose K and t such that

$$tK + t \sum_{j=k+1}^{n-1} \left|t^{-\epsilon_k} \tilde{z}_j\right|^{2m_j} + 3 \geqslant 2\left(t \sum_{j=k+1}^{n-1} \left|t^{-\epsilon_k} \tilde{z}_j\right|^{2m_j} + 3\right).$$

Then

$$J_t \gtrsim \int_0^\delta \ldots \int_0^\delta \frac{d\tilde{x}_{k+1} \, d\tilde{y}_{k+1} \ldots d\tilde{x}_{n-1} \, d\tilde{y}_{n-1}}{(t \sum_{j=k+1}^{n-1} |t^{-\epsilon_k} z_j|^{2m_j} + 1)^{2p-2\epsilon}} \cong t^{-\sum_{j=k+1}^{n-1} \frac{1}{m_j} + 2(n-k-1)\epsilon_k},$$

where the last inequality follows by Lemma 5.1. So we have

$$|||u_t|||_\epsilon^2 \gtrsim t^{2p-2+2\epsilon-2k\epsilon_k - \sum_{j=k+1}^{n-1} \frac{1}{m_j}}. \tag{5.10}$$

Since subelliptic estimates hold with order ϵ for any k-form ($q_0 + 1 \leqslant k \leqslant n-1$), then

$$|||u_t|||_\epsilon^2 \lesssim Q(u_t, u_t). \tag{5.11}$$

Combining (5.9), (5.10) and (5.11), we get $\epsilon \leqslant \epsilon_k$.

The proof of Theorem 1.13(i) is complete. □

Proof of Theorem 1.13(ii). We proceed in similar way as in the proof of Theorem 1.13(i) and choose the coefficient of our form by setting

$$\begin{cases} f_t(z', z_n) = \left(z_n - \sum_{j=1}^{q_0} |z_j|^{2m_j} - 1/t\right)^{-p}, \\ \Phi_t(z) = \left(\prod_{j=1}^{n-1} \phi(t^{\epsilon_k} x_j)\phi(t^{\epsilon_k} y_j)\right) \lambda(x_n)\phi(y_n). \end{cases}$$

Then

$$Q(u_t,u_t)\lesssim\sum_{j=1}^{q_o}\left\|r_{z_j}\frac{\partial f_t}{\partial\bar z_j}\Phi_t\right\|^2+\sum_{j=k+1}^{q_o}\left\|\frac{\partial f_t}{\partial\bar z_j}\Phi_t\right\|^2$$
$$+\sum_{\substack{i=k+1,\dots,n-1\\ j=1,\dots,k,i}}\left\||r_{z_j}|,|r_{z_i}|\partial_{z_n}(f_t)\Phi_t\right\|^2+\sum_{j=1}^{n}\left\|f_t\frac{\partial\Phi_t}{\partial z_j}\right\|^2+\|U_t\|^2.$$

We can show that $Q(u_t,u_t)\lesssim t^{2p-2+2\epsilon_k-2(n-1)\epsilon_k}I_t$ where

$$I_t=\int_0^\delta\dots\int_0^\delta\frac{dx_1\,dy_1\dots dx_k\,dy_k}{(t\sum_{j=1}^k|t^{-\epsilon_k}z_j|^{2m_j}+1)^{2p-2}}.$$

Owing to Lemma 5.1 we have $I_t\lesssim t^{-\sum_{j=1}^k\frac{1}{m_j}+2k\epsilon_k}$ which yields

$$Q(u_t,u_t)\lesssim t^{2p-2+2\epsilon_k-2(n-k-1)\epsilon_k-\sum_{j=1}^k\frac{1}{m_j}}.$$

Similarly, we have

$$|||u_t|||_\epsilon^2\gtrsim t^{2p-2+2\epsilon_k-2(n-k-1)\epsilon_k-\sum_{j=1}^k\frac{1}{m_j}},$$

which yields the conclusion of the proof of Theorem 1.13(ii). □

Acknowledgment

The authors are grateful to the referee for important work.

References

[1] H. Ahn, Global boundary regularity of the $\bar\partial$-equation on the q-pseudoconvex domains, Math. Nachr. 280 (2007) 343–350.
[2] H. Ahn, L. Baracco, G. Zampieri, Subelliptic estimates and regularity of $\bar\partial$ at the boundary of Q-pseudoconvex domain of finite type, Math. Nachr. (2011).
[3] D. Catlin, Subelliptic estimates for the $\bar\partial$-Neumann problem on pseudoconvex domains, Ann. of Math. 126 (1987) 131–191.
[4] S.C. Chen, M.C. Shaw, Partial Differential Equations in Several Complex Variables, Stud. Adv. Math., vol. 19, AMS Int. Press, 2001.
[5] J. D'Angelo, Real hypersurfaces, order of contact, and applications, Ann. of Math. 115 (1982) 615–637.
[6] G.B. Folland, J.J. Kohn, The Neumann Problem for the Cauchy–Riemann Complex, Ann. of Math. Stud., vol. 75, Princeton Univ. Press, Princeton, NJ, 1972.
[7] L. Hörmander, L^2 estimates and existence theorems for the $\bar\partial$ operator, Acta Math. 113 (1965) 89–152.
[8] L. Hörmander, An Introduction to Complex Analysis in Several Complex Variables, Van Nostrand, Princeton, NJ, 1973.
[9] L.H. Ho, Subellipticity of the $\bar\partial$-Neumann problem on the nonpseudoconvex, Trans. Amer. Math. Soc. 291 (1985) 43–73.
[10] L.H. Ho, Subellipticity of the $\bar\partial$-Neumann problem for $n-1$ forms, Trans. Amer. Math. Soc. 325 (1991) 171–185.

[11] J.J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds, I, Ann. of Math. 78 (1963) 112–148; J.J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds, II, Ann. of Math. 79 (1964) 450–472.
[12] J.J. Kohn, Boundary behavior of $\bar\partial$ on weakly pseudo-convex manifolds of dimension two, J. Differential Geom. 6 (1972) 523–542.
[13] J.J. Kohn, Methods of partial differential equations in complex analysis, Proc. Sympos. Pure Math. 30 (1977) 215–237.
[14] J.J. Kohn, Subellipticity of the $\bar\partial$-Neumann problem on pseudoconvex domains: sufficient conditions, Acta Math. 142 (1979) 79–122.
[15] J.J. Kohn, L. Nirenberg, Non-coercive boundary value problems, Comm. Pure Appl. Math. 18 (1965) 443–492.
[16] G. Zampieri, q-Pseudoconvexity and regularity at the boundary for solutions of the $\bar\partial$-problem, Compos. Math. 121 (2) (2000) 155–162.
[17] G. Zampieri, Complex Analysis and CR Geometry, Univ. Lecture Ser., vol. 43, AMS, 2008.