L^p-ESTIMATES FOR THE $\bar{\partial}$-EQUATION ON A CLASS OF INFINITE TYPE DOMAINS

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Abstract. We prove L^p estimates for solutions to the Cauchy-Riemann equations $\bar{\partial} u = \phi$ on a class of infinite type domains in $\mathbb{C}^2$. The domains under consideration are a class of convex ellipsoids, and we show that if ϕ is a $\bar{\partial}$-closed $(0,1)$-form with coefficients in L^p and u is the Henkin kernel solution to $\bar{\partial} u = \phi$, then $\|u\|_p \leq C\|\phi\|_p$ where the constant C is independent of ϕ. In particular, we prove L^1 estimates and obtain L^p estimates by interpolation.

1. Introduction

A fundamental question in several complex variables is to establish L^p estimates for solutions of the Cauchy-Riemann equation

$$\bar{\partial} u = \phi$$

on domains $\Omega \subset \mathbb{C}^n$. In this paper, we provide the first examples of infinite type domains for which L^p bounds hold, $1 \leq p \leq \infty$. The domains under consideration are a class of convex ellipsoids in $\mathbb{C}^2$, and we show that if ϕ is a $\bar{\partial}$-closed $(0,1)$-form and u is the Henkin solution to $\bar{\partial} u = \phi$, then $\|u\|_p \leq C\|\phi\|_p$ where the constant C is independent of ϕ. Specifically, we prove L^1 estimates and use the Riesz-Thorin Interpolation Theorem to obtain L^p estimates by interpolating with the L^∞ estimates established by Khanh [Kha13] and (independently) Fornæss et. al. [FLZ11].

We investigate domains of the following form: $\Omega \subset \mathbb{C}^2$ is a smooth, bounded domain with the origin 0 in the boundary $b\Omega$. Moreover, there exists $\delta > 0$ so that $b\Omega \setminus B(0, \delta/2)$ is strictly convex and there exists a defining function ρ so that

$$\Omega \cap B(0,\delta) = \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = F(|z_1|^2) + r(z) < 0\} \tag{1.1}$$

or

$$\Omega \cap B(0,\delta) = \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = F(x_1^2) + r(z) < 0\} \tag{1.2}$$

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where $z_j = x_j + iy_j$, for $x_j, y_j \in \mathbb{R}$, $j = 1, 2$, and $i = \sqrt{-1}$. We also assume that the functions $F : \mathbb{R} \to \mathbb{R}$ and $r : \mathbb{C}^2 \to \mathbb{R}$ satisfy:

i. $F(0) = 0$;

ii. $F'(t), F''(t), F'''(t)$, and $\left(\dfrac{F(t)}{t}\right)'$ are nonnegative on $(0, \delta)$;

iii. $r(0) = 0$ and $\dfrac{\partial r}{\partial z_2} \neq 0$;

iv. r is convex and strictly convex away from 0.

This class of domains includes two well-known examples. If $F(t) = t^m$, with $m \geq 1$, then Ω is of finite type $2m$. On the other hand, if $F(t) = \exp(-1/t^\alpha)$, then Ω is of infinite type, and this is our main case of interest. We call our domains Ω ellipsoids because they are generalizations of real and complex ellipsoids in $\mathbb{C}^2$. Classically, a complex ellipsoid in $\mathbb{C}^n$ is a domain of the form $\{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2m_j} < 1\}$, and a real ellipsoid is a domain of the form $\{z = (x_1 + iy_1, \dots, x_n + iy_n) \in \mathbb{C}^n : \sum_{j=1}^n (x^{2n_j} + y^{2m_j}) < 1\}$ where $m_j, n_j \in \mathbb{N}$, $1 \leq j \leq n$.

There is a long history of proving L^p estimates for the $\bar\partial$-equation, dating back to the work of Kerzman [Ker71] and Øvrelid [Øvr71]. In [Kra76], Krantz proved essentially optimal Lipschitz and L^p estimates on strongly pseudoconvex domains. In the case that Ω is a real ellipsoid, et. al. obtained sharp Hölder estimates [DFW86] while Chen et. al. established optimal L^p estimates for complex ellipsoids [CKM93]. See also Range [Ran78] and Bruna and del Castillo [BdC84]. Both real and complex ellipsoids are domains of finite type, and the analysis in the referenced works depends in an essential fashion on the type. In $\mathbb{C}^2$, Chang et. al. [CNS92] proved L^p estimates for the $\bar\partial$-Neumann operator on weakly pseudoconvex domains of finite type. See [CKM93, FLZ11] and the references within for a more complete history.

More recently, there has been work on supnorm estimates for the Cauchy-Riemann equations on infinite type domains in $\mathbb{C}^2$. Fornæss et. al. provided the first examples in [FLZ11] and Khanh found the estimates hold when domains are of the type (1.1) or (1.2) [Kha13]. In particular, Khanh proved

Theorem 1.1 (Theorem 1.2, [Kha13])**.** *If*

i. Ω is defined by (1.1) and there exists $\delta > 0$ so that $\int_0^\delta |\log F(t^2)|\, dt < \infty$, or

ii. Ω is defined by (1.2) and there exists $\delta > 0$ so that $\int_0^\delta |\log(t) \log F(t^2)|\, dt < \infty$,

then for any bounded, $\bar\partial$-closed $(0,1)$-form ϕ on $\overline{\Omega}$, the Henkin solution u on Ω satisfies $\bar\partial u = \phi$ and

$$\|u\|_{L^\infty(\Omega)} \leq C \|\phi\|_{L^\infty(\Omega)},$$

where $C > 0$ is independent of ϕ.

In this paper, we will prove the L^p-version of Theorem 1.1. Our technique yields L^p-estimates both when in the case that Ω is of finite type as well as of infinite type.

Theorem 1.2. *If either of the following conditions hold:*

i. Ω *is defined by (1.1) and there exists* $\delta > 0$ *so that* $\int_0^\delta |\log F(t^2)|\,dt < \infty$,

ii. Ω *is defined by (1.2) and there exists* $\delta > 0$ *so that* $\int_0^\delta |\log(t) \log F(t^2)|\,dt < \infty$,

then for any $\bar{\partial}$*-closed* $(0,1)$*-form* ϕ *and in* $L^p(\Omega)$ *with* $1 \le p \le \infty$*, the Henkin kernel solution* u *on* Ω *satisfies* $\bar{\partial}u = \phi$ *and*

$$\|u\|_{L^p(\Omega)} \le C\|\phi\|_{L^p(\Omega)},$$

where $C > 0$ *is independent of* ϕ.

The following examples show that the L^p estimates in Theorem 1.2 are sharp in the case of infinite type case.

Example 1.1. *For* $0 < \alpha < 1$*, let* Ω *be defined by*

$$\Omega = \left\{(z_1, z_2) \in \mathbb{C}^2 : e^{1-\frac{1}{|z_1|^\alpha}} + |z_2|^2 < 1\right\}.$$

Then for any $\phi \in L^p(\Omega)$ *with* $1 \le p \le \infty$*, there is a solution* u *of the equation* $\bar{\partial}u = \phi$ *such that* $u \in L^p(\Omega)$*. Moreover, if* $p \ne \infty$*, there is no solution* $u \in L^q(\Omega)$ *with* $q > p$.

The organization of the paper is as follows: we recall the construction the Henkin solution via the Henkin kernel in Section 2. We prove Theorem 1.2 in Section 3 and discuss Example 1.1 in Section 4.

2. Henkin Solution

In this section, we recall the construction of the Henkin kernel and Henkin solution to $\bar{\partial}$. For complete details, see [Hen70, Ran86], or for a more modern treatment, see [CS01].

Definition 2.1. A $\mathbb{C}^2$-valued C^1 function $G(\zeta, z) = (g_1(\zeta, z), g_2(\zeta, z))$ is called a Leray map for Ω if $g_1(\zeta, z)(\zeta_1 - z_1) + g_2(\zeta, z)(\zeta_2 - z_2) \ne 0$ for every $(\zeta, z) \in \mathrm{b}\Omega \times \Omega$. A support function $\Phi(\zeta, z)$ for Ω is a smooth function defined near $\mathrm{b}\Omega \times \bar{\Omega}$ so that Φ admits a decomposition

$$\Phi(\zeta, z) = 2\sum_{j=1}^{2} \Phi_j(\zeta, z)(\zeta_j - z_j)$$

where $\Phi_j(\zeta, z)$ are smooth near $\mathrm{b}\Omega \times \bar{\Omega}$, holomorphic in z, and vanish only on the diagonal $\{\zeta = z\}$.

For a convex domain, it is well known that $G(\zeta, z) = \frac{\partial\rho}{\partial\zeta} = (\frac{\partial\rho}{\partial\zeta_1}, \frac{\partial\rho}{\partial\zeta_2})$ is a Leray map [CS01, Lemma 11.2.6], and Φ defined by Leray map

$$\Phi_j(\zeta, z) = \frac{\partial\rho(\zeta)}{\partial\zeta_j},\ j = 1, 2,$$

is a support function for Ω.

Taylor's Theorem and the convexity of F imply a lower bound for $\operatorname{Re}\Phi(\zeta,z)$ on $S_{0,2\delta} := \{z\in\bar\Omega : \rho(z)\ge -2\delta\}$.

Lemma 2.2. *Let $\Omega\subset\mathbb{C}^2$ be as in Section 1 with Φ as above. Then there exist $\epsilon, c>0$ so that*

$$\operatorname{Re}\Phi(\zeta,z)\ge\rho(\zeta)-\rho(z)+\begin{cases} c|z-\zeta|^2 & \zeta\in S_{0,2\delta}\setminus B(0,\delta),\\ P(z_1)-P(\zeta_1)-2\operatorname{Re}\left\{\frac{\partial P}{\partial\zeta_1}(\zeta_1)(z_1-\zeta_1)\right\} & \zeta\in S_{0,2\delta}\cap B(0,\delta).\end{cases} \tag{2.1}$$

for all $z\in\bar\Omega$ with $|z-\zeta|\le\epsilon$,where $P(z_1)=F(|z_1|^2)$ or $P(z_1)=F(x_1^2)$.

Proof. Let h be a $\mathbb{R}$-valued smooth function in $\mathbb{C}^2=\mathbb{R}^4$ and $x,y\in\mathbb{R}^4$. If $\alpha(t)=tx+(1-t)y$, and $\varphi(t)=h(\alpha(t))$, then it follows from Taylor's Theorem applied to $\varphi(t)$ that there exists $\tilde y\in\alpha([0,1])$ so that

$$h(x)=h(y)+\sum_{j=1}^4\frac{\partial h(y)}{\partial y_j}(x-y)+\frac12\sum_{j,k=1}^4\frac{\partial^2 h(\tilde y)}{\partial y_j\partial y_k}(x_j-y_j)(x_k-y_k).$$

Set $z=(z_1,z_2)=(x_1+ix_2,x_3+ix_4)$ and $\zeta=(\zeta_1,\zeta_2)=(y_1+iy_2,y_3+iy_4)$. Translating the first order component of the Taylor series expansion to complex coordinates, we compute

$$\begin{aligned}2\operatorname{Re}\left\{\frac{\partial h(\zeta)}{\partial\zeta_j}(z_j-\zeta_j)\right\}&=\operatorname{Re}\left\{\left(\frac{\partial h(\zeta)}{\partial y_{2j-1}}-i\frac{\partial h(\zeta)}{\partial y_{2j}}\right)\big((x_{2j-1}-y_{2j-1})+i(x_{2j}-y_{2j})\big)\right\}\\&=\frac{\partial h(\zeta)}{\partial y_{2j-1}}(x_{2j-1}-y_{2j-1})+\frac{\partial h(\zeta)}{\partial y_{2j}}(x_{2j}-y_{2j}),\end{aligned}$$

$j=1,2$. Consequently, if $[\zeta,z]$ is the line segment connecting ζ and z, then

$$h(z)\ge h(\zeta)+2\sum_{j=1}^2\operatorname{Re}\left\{\frac{\partial h(\zeta)}{\partial\zeta_j}(z_j-\zeta_j)\right\}+\min_{\tilde y\in[\zeta,z]}\frac12\sum_{j,k=1}^4\frac{\partial^2 h(\tilde y)}{\partial y_j\partial y_k}(x_j-y_j)(x_k-y_k). \tag{2.2}$$

Applying (2.2) to the defining function ρ yields

$$\rho(z)\ge\rho(\zeta)-\operatorname{Re}\Phi(\zeta,z)+\min_{\tilde y\in[\zeta,z]}\frac12\sum_{j,k=1}^4\frac{\partial^2\rho(\tilde y)}{\partial y_j\partial y_k}(x_j-y_j)(x_k-y_k).$$

Since ρ is strictly convex on $\mathrm{b}\Omega\setminus B(0,\delta)$, there exists $c>0$ so that $|\sum_{j,k=1}^4\frac{\partial^2\rho(\tilde y}{\partial y_j\partial y_k}(x_j-y_j)(x_k-y_k)|\ge c|x-y|^2$ if $y\in S_{0,2\delta}\setminus B(0,\delta)$ and $\epsilon>0$ is sufficiently small. The first case of (2.1) now follows.

For the remaining case, we use (2.2) and the convexity of r to observe that

$$\begin{aligned}
\rho(\zeta) - \rho(z) + P(z_1) - P(\zeta_1) - 2\operatorname{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1-\zeta_1)\right\}& \\
= r(\zeta) - r(z) + 2\operatorname{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(\zeta_1 - z_1)\right\}& \\
\leq 2\sum_{j=1}^{2}\operatorname{Re}\left\{\frac{\partial r(\zeta)}{\partial \zeta_j}(\zeta_j - z_j)\right\} + 2\operatorname{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(\zeta_1 - z_1)\right\}& \\
= \operatorname{Re}\Phi(\zeta, z).&
\end{aligned}$$

This completes the proof. □

We take the ϵ constructed in Lemma 2.2 to be a global constant in the paper, though we reserve the right to decrease it.

Let $\phi = \sum_{j=1}^{2}\phi_j d\bar{z}_j$ be a bounded, C^1, $\bar{\partial}$-closed $(0,1)$-form on $\bar{\Omega}$. The solution u of the $\bar{\partial}$-equation, $\bar{\partial}u = \phi$, provided by the Henkin kernel is given by

$$u = T\phi(z) = H\phi(z) + K\phi(z). \tag{2.3}$$

where

$$\begin{aligned}
H\phi(z) =& \frac{1}{2\pi^2}\int_{\zeta\in b\Omega}\frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \frac{\partial\rho(\zeta)}{\partial\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{\Phi(\zeta,z)|\zeta - z|^2}\phi(\zeta)\wedge\omega(\zeta);\\
K\phi(z) =& \frac{1}{4\pi^2}\int_{\Omega}\frac{\phi_1(\zeta)(\bar{\zeta}_1 - \bar{z}_1) - \phi_2(\zeta)(\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4}\omega(\bar{\zeta})\wedge\omega(\zeta)
\end{aligned} \tag{2.4}$$

where $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$. See, for example, [FLZ11, DFW86]. To understand the L^p-norm of u, it suffices to investigate the L^p mapping properties of integral operators H and K.

3. Proof of the theorem 1.2

As a consequence of the Riesz-Thorin Interpolation Theorem and Theorem 1.1, proving that T is a bounded, linear operator on $L^1(\Omega)$ suffices to establish that T is a bounded linear operator on $L^p(\Omega)$, $1 \leq p \leq \infty$.

The L^1-estimate of $|K\phi(z)|$ is standard and does not require interpolation. Indeed, since $|\zeta - z|^{-3} \in L^1(\Omega)$ in both ζ and z (separately), L^p boundedness of K, $1 \leq p \leq \infty$, follows from [Fol99, Theorem 6.18].

For the boundedness of H, we first begin the analysis of $H\phi(z)$ by using Stokes' Theorem. Using the assumption that ϕ is $\bar{\partial}$-closed, we observe

$$H\phi(z) = \frac{1}{2\pi^2}\int_{\Omega}\bar{\partial}_\zeta\left(\frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \frac{\partial\rho(\zeta)}{\partial\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{(\Phi(\zeta,z) - \rho(\zeta))(|\zeta - z|^2 + \rho(\zeta)\rho(z))}\right)\wedge\phi(\zeta)\wedge\omega(\zeta).$$

We abuse notation slightly and let $H(\zeta,z)$ be the integral kernel of H. Direct calculation shows that we can decompose

$$
\begin{aligned}
|H(\zeta,z)| \leq & \left|\bar\partial_\zeta\left(\frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\bar\zeta_2-\bar z_2)-\frac{\partial\rho(\zeta)}{\partial\zeta_2}(\bar\zeta_1-\bar z_1)}{(\Phi(\zeta,z)-\rho(\zeta))(|\zeta-z|^2+\rho(\zeta)\rho(z))}\right)\right| \\
\lesssim & \frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^2(|\zeta-z|^2+\rho(\zeta)\rho(z))^{1/2}}+\frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|(|\zeta-z|^2+\rho(\zeta)\rho(z))}.
\end{aligned}
\tag{3.1}
$$

Since ρ is smooth, ρ is Lipschitz, so $(\rho(\zeta)-\rho(z))^2 \lesssim |\zeta-z|^2$. Therefore, $\rho(\zeta)^2 \lesssim |\zeta-z|^2+\rho(\zeta)\rho(z)$, hence $|\zeta-z|+|\rho(\zeta)| \lesssim (|\zeta-z|^2+\rho(\zeta)\rho(z))^{1/2}$. Thus

$$
|\Phi(\zeta,z)-\rho(\zeta)| \lesssim |\zeta-z|+|\rho(\zeta)| \lesssim (|\zeta-z|^2+\rho(\zeta)\rho(z))^{1/2}. \tag{3.2}
$$

Combining (3.1) and (3.2), we obtain

$$
\begin{aligned}
|H(\zeta,z)| \lesssim & \frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^2(|\zeta-z|^2+\rho(\zeta)\rho(z))^{1/2}} \\
\leq & \frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^2|\zeta-z|} \\
\leq & \frac{1}{(|\operatorname{Re}\Phi(\zeta,z)-\rho(\zeta)|^2+|\operatorname{Im}\Phi(\zeta,z)|^2)|\zeta_1-z_1|}.
\end{aligned}
\tag{3.3}
$$

We will show that

$$
\iint_{(\zeta,z)\in\Omega\times\Omega}\left|H(\zeta,z)\phi(\zeta)\right| dV(\zeta,z) \lesssim \|\phi\|_{L^1(\Omega)} < \infty. \tag{3.4}
$$

By Tonelli's Theorem, it then follows that

$$
\begin{aligned}
\|H\phi\|_{L^1(\Omega)} =& \left|\int_{z\in\Omega} H\phi(z)\,dV(z)\right| \\
=& \left|\int_{z\in\Omega}\int_{\zeta\in\Omega} H(\zeta,z)\phi(\zeta)\,dV(\zeta)\,dV(z)\right| \\
\leq & \iint_{(\zeta,z)\in\Omega\times\Omega}\left|H(\zeta,z)\phi(\zeta)\right| dV(\zeta,z) \lesssim \|\phi\|_{L^1(\Omega)}.
\end{aligned}
\tag{3.5}
$$

In order to prove (3.4), we remark that it is enough to assume that $z,\zeta\in\Omega\cap B(0,\delta)=\{\rho(z)=P(z_1)+r(z)<0\}$ because if $\zeta,z\in\bar\Omega\setminus B(0,\delta/2)$, then the estimates following classically using the strict convexity of r. If one of $\{z,\zeta\}$ is in $B(0,\delta/2)$ and other is an element of $B(0,\delta)^c$, then the integrand of H is bounded and bounded away from 0, and the estimate is trivial. We will investigate the complex and real ellipsoid cases separately to show

$$
\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} H(\zeta,z)\phi(\zeta)dV(\zeta,z) \lesssim \|\phi\|_{L^1(\Omega)}. \tag{3.6}
$$

First, however, we recall the following facts for a class of real functions F in the first part with the additional assumption that $F'(0) = 0$.

Lemma 3.1. *Let F be a C^2 convex function on $[0, \delta]$. Then*

$$F(p) - F(q) - F'(q)(p-q) \geq 0 \tag{3.7}$$

for any $p, q \in [0, \delta]$. If, in addition, $F'(0) = 0$ and F'' is nondecreasing, then

$$F(p) - F(q) - F'(q)(p-q) \geq F(p-q), \tag{3.8}$$

for any $0 \leq q \leq p \leq \delta$.

Proof. The proof of (3.7) is simple and is omitted here (see, e.g., (2.2)). For (3.8), let $s := p-q \geq 0$ and $g(s) := F(s+q)-F(q)-sF'(q)-F(s)$. Hence, $g'(s) = F'(s+q)-F'(q)-F'(s)$ and $g''(s) = F''(s+q) - F''(s)$. Using the assumption $F''(t)$ is nondecreasing, we have $g''(s) \geq 0$, thus $g'(s)$ is nondecreasing. This implies $g'(s) \geq g'(0) = 0$ (since $F'(0) = 0$) and consequently that $g(s)$ is increasing. We thus obtain $g(s) \geq g(0) = 0$ (since F(0)=0). This completes the proof of (3.8). ☐

3.1. **Complex Ellipsoid Case.** In this subsection, Ω is defined by (1.1). Since the argument of F is $|\zeta_1|^2$, the chain rule shows that $\dfrac{\partial}{\partial \zeta_1} F(|\zeta_1|^2) = \bar{\zeta}_1 F'(|\zeta_1|^2)$. Similarly to Khanh [Kha13, (4.1)], Lemma 2.2 shows that

$$\begin{aligned} &\operatorname{Re}\left\{\Phi(\zeta,z)\right\} - \rho(\zeta) \geq -\rho(z) + F(|z_1|^2) - F(|\zeta_1|^2) - 2F'(|\zeta_1|^2)\operatorname{Re}\left\{\bar{\zeta}_1(z_1-\zeta_1)\right\} \\ &= -\rho(z) + F'(|\zeta_1|^2)|z_1-\zeta_1|^2 + \Big(F(|z_1|^2) - F(|\zeta_1|^2) - F'(|\zeta_1|^2)\big(|z_1|^2 - |\zeta_1|^2\big)\Big). \end{aligned} \tag{3.9}$$

The analysis splits into two cases: i) $F'(0) \neq 0$ and ii) $F'(0) = 0$. In the first case, the hypotheses on F guarantee the existence of a $\delta > 0$ such that $F'(|\zeta_1|^2) > 0$ for any $|\zeta_1| < \delta$. Hence,

$$\operatorname{Re}\left\{\Phi(\zeta,z)\right\} - \rho(\zeta) \gtrsim -\rho(z) + |z_1-\zeta_1|^2$$

and

$$|H(\zeta,z)| \leq \frac{1}{(|\rho(z)|^2 + |\operatorname{Im}\Phi(\zeta,z)|^2 + |\zeta_1 - z_1|^4)|\zeta_1 - z_1|}.$$

The estimate in this case is the estimate for the case of a strongly pseudoconvex domain, and the result is classical and well-known. Thus, we may assume that $F'(0) = 0$.

Lemma 3.2. *Let F be defined in Section 1 with the additional assumption $F'(0) = 0$. Then*

$$|H(\zeta,z)| \lesssim \begin{cases} \dfrac{1}{(|\rho(z) + i\operatorname{Im}\Phi(\zeta,z)|^2 + F^2(|z_1-\zeta_1|^2))|z_1-\zeta_1|} & \text{if } |\zeta_1| \geq |z_1-\zeta_1|, \\ \dfrac{1}{(|\rho(z) + i\operatorname{Im}\Phi(\zeta,z)|^2 + F^2(\frac{1}{2}|z_1|^2))|z_1|} & \text{if } |\zeta_1| \leq |z_1-\zeta_1|. \end{cases} \tag{3.10}$$

Proof. Applying Lemma 3.1 to (3.9), we obtain

$$\operatorname{Re}\left\{\Phi(\zeta,z)\right\}-\rho(\zeta)\geq-\rho(z)+\begin{cases}F'(|\zeta_1|^2)|z_1-\zeta_1|^2 & \text{if } 0<|z_1|,|\zeta_1|<\delta,\\ F(|z_1|^2-|\zeta_1|^2) & \text{if } |\zeta_1|\leq|z_1|\leq\delta.\end{cases} \tag{3.11}$$

We compare $|\zeta_1|$ and $|z_1-\zeta_1|^2$.

Case 1: $|\zeta_1|\geq|z_1-\zeta_1|$. Combining the first inequality from (3.11) with the facts that F' is increasing and $F'(t)t\geq F(t)$ (since $\frac{F(t)}{t}$ is nondecreasing), we obtain

$$\operatorname{Re}\left\{\Phi(\zeta,z)\right\}-\rho(\zeta)\geq-\rho(z)+F(|z_1-\zeta_1|^2).$$

The first line of (3.10) follows by this inequality and (3.3).

Case 2: $|\zeta_1|\leq|z_1-\zeta_1|$. In this case, the estimate depends on the relative sizes of $|\zeta_1|$ and $\frac{1}{\sqrt{2}}|z_1|$. If $|\zeta_1|\geq\frac{1}{\sqrt{2}}|z_1|$, then the argument from Case 1 proves that

$$\operatorname{Re}\left\{\Phi(\zeta,z)\right\}-\rho(\zeta)\geq-\rho(z)+F(\frac{1}{2}|z_1|^2),$$

and we obtain the second estimate in (3.10). Otherwise, $|\zeta_1|\leq\frac{1}{\sqrt{2}}|z_1|$, and this implies both $|z_1|\geq|\zeta_1|$ and $|z_1-\zeta_1|\geq(1-\frac{1}{\sqrt{2}})|z_1|$. By the second case of (3.11), we observe that

$$\operatorname{Re}\left\{\Phi(\zeta,z)\right\}-\rho(\zeta)\geq-\rho(z)+F(|z_1|^2-|\zeta_1|^2)\geq-\rho(z)+F(\frac{1}{2}|z_1|^2),$$

and

$$(|\operatorname{Re}\Phi(\zeta,z)-\rho(\zeta)|^2+|\operatorname{Im}\Phi(\zeta,z)|^2)|\zeta_1-z_1|\gtrsim(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(\frac{1}{2}|z_1|^2))|z_1|.$$

This completes the proof. $\square$

Proof of the Theorem 1.2.i. By Lemma 3.2, we have

$$\begin{aligned}&\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2}\left|H(\zeta,z)\phi(\zeta)\right|dV(\zeta,z)\\ &=\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2\text{ and }|\zeta_1|\geq|z_1-\zeta_1|}\cdots+\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2\text{ and }|\zeta_1|\leq|z_1-\zeta_1|}\cdots\\ &\lesssim(I)+(II),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}(I)&:=\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2}\frac{|\phi(\zeta)|dV(\zeta,z)}{(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(|z_1-\zeta_1|^2))|z_1-\zeta_1|};\\ (II)&:=\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2}\frac{|\phi(\zeta)|dV(\zeta,z)}{(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(\frac{1}{2}|z_1|^2))|z_1|}.\end{aligned} \tag{3.13}$$

For the integral (I), we make the change variables $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, z_1 - \zeta_1, \rho(z) + i \operatorname{Im} \Phi(\zeta, z))$. Direct calculus the Jacobian of this transformation is the matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \rho(z)}{\partial(\operatorname{Re} z_1)} & \frac{\partial \rho(z)}{\partial(\operatorname{Im} z_1)} & \frac{\partial \rho(z)}{\partial(\operatorname{Re} z_2)} & \frac{\partial \rho(z)}{\partial(\operatorname{Im} z_2)} \\ \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Re}\zeta_1)} & \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Im}\zeta_1)} & \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Re}\zeta_2)} & \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Im}\zeta_2)} & \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Re} z_1)} & \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Im} z_1)} & \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Re} z_2)} & \frac{\partial \operatorname{Im}\Phi(\zeta,z)}{\partial(\operatorname{Im} z_2)} \end{pmatrix}.$$

To justify this coordinate change, we write $z_j = x_j + iy_j$ and compute

$$\det(J) = \frac{\partial \operatorname{Im}(\Phi(\zeta, z))}{\partial y_2} \frac{\partial \rho(z)}{\partial x_2} - \frac{\partial \operatorname{Im}(\Phi(\zeta, z))}{\partial x_2} \frac{\partial \rho(z)}{\partial y_2}.$$

By a possible rotation and dilation of Ω, we can assume that $\nabla \rho(0) = (0, 0, 0, -1)$. Direct calculation then establishes that if δ is chosen sufficiently small (so that $\frac{\partial \rho(z)}{\partial y_2}$ dominates the other partials of ρ and $|\zeta - z|$ is small), then $\det(J) \neq 0$. Since Φ is smooth, we can assume that there exists $\delta' > 0$ that depends on Ω and ρ so that

$$\begin{aligned} (I) &\lesssim \iint_{(\psi,w)\in(\Omega\cap B(0,\delta))\times B(0,\delta')} \frac{|\phi(\psi)|}{(|w_2|^2 + F^2(|w_1|^2)|w_1|} \, dV(\psi, w) \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \int_0^{\delta'} \frac{r_1 r_2}{(r_2^2 + F^2(r_1^2))r_1} \, dr_2 \, dr_1 \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \log F(r_1^2) \, dr_1 < \infty. \end{aligned}$$

That the integral is finite follows by the hypotheses on ϕ and F.

Repeating this argument with the change of coordinates $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, \frac{1}{\sqrt{2}} z_1, \rho(z) + i \operatorname{Im} \Phi(\zeta, z))$ for the integral (II), we can obtain the same conclusion. Therefore, the estimate in complex case is complete.

□

3.2. **Real Ellipsoid Case.** In this subsection, Ω is defined by (1.2). The analogous argument from (3.9) case yields

$$\operatorname{Re}\{\Phi(\zeta, z)\} - \rho(\zeta) \geq -\rho(z) + F'(\xi_1^2)(x_1 - \xi_1)^2 + \left(F(x_1^2) - F(\xi_1^2) - F'(\xi_1^2)(x_1^2 - \xi_1^2)\right),$$

where $z_1 = x_1 + iy_1$, $\zeta_1 = \xi_1 + i\eta_1$. Following the setup in the complex case, with the same proof, one also have

Lemma 3.3. *Let F be defined in Section 1 with the extra assumption $F'(0) = 0$. Then*

$$|H(\zeta, z)| \lesssim \begin{cases} \dfrac{1}{(|\rho(z) + i\,\mathrm{Im}\,\Phi(\zeta, z)|^2 + F^2((x_1-\xi_1)^2))(|x_1-\xi_1| + |y_1-\eta_1|)} & \text{if } |\xi_1| \geq |x_1-\xi_1|, \\ \dfrac{1}{(|\rho(z) + i\,\mathrm{Im}\,\Phi(\zeta, z)|^2 + F^2(\frac{1}{2}x_1^2))(\frac{1}{\sqrt{2}}|x_1| + |y_1-\eta_1|)} & \text{if } |\xi_1| \leq |x_1-\xi_1|. \end{cases} \tag{3.14}$$

Proof of Theorem 1.2.ii. Using Lemma 3.3, we have

$$\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} \left|H(\zeta, z)\phi(\zeta)\right| dV(\zeta, z) \lesssim (I) + (II) \tag{3.15}$$

where

$$\begin{aligned} (I) &:= \iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} \frac{|\phi(\zeta)|\, dV(\zeta, z)}{(|\rho(z) + i\,\mathrm{Im}\,\Phi(\zeta, z)|^2 + F^2((x_1-\xi_1)^2))(|x_1-\xi_1| + |y_1-\eta_1|)}; \\ (II) &:= \iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} \frac{|\phi(\zeta)|\, dV(\zeta, z)}{(|\rho(z) + i\,\mathrm{Im}\,\Phi(\zeta, z)|^2 + F^2(\frac{1}{2}x_1^2))(\frac{1}{\sqrt{2}}|x_1| + |y_1-\eta_1|)}. \end{aligned} \tag{3.16}$$

We make the change of variables $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, z_1 - \xi_1, \rho(z) + i\,\mathrm{Im}\,\Phi(\zeta, z))$ for (I) and $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, \frac{1}{\sqrt{2}}x_1 - i(y_1 - \eta_1), \rho(z) + i\,\mathrm{Im}\,\Phi(\zeta, z))$ for (II). Similarly to the argument above, we can check that $\det(J) \neq 0$. Therefore

$$\begin{aligned} (I) + (II) &\lesssim \iint_{(\psi,w)\in(\Omega\cap B(0,\delta))\times B(0,\delta')} \frac{|\phi(\psi)|}{(|w_2|^2 + F^2((\mathrm{Re}\,w_1)^2)(|\mathrm{Re}\,w_1| + |\mathrm{Im}\,w_1|)} dV(\psi, w) \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'}\int_0^{\delta'}\int_0^{\delta'} \frac{r_2\, dr_2\, d(\mathrm{Im}\,w_1)\, d(\mathrm{Re}\,w_1)}{(r_2^2 + F^2((\mathrm{Re}\,w_1)^2))(|\mathrm{Re}\,w_1| + |\mathrm{Im}\,w_1|)} \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'}\int_0^{\delta'} \frac{\log(F((\mathrm{Re}\,w_1)^2)\, d(\mathrm{Im}\,w_1)\, d(\mathrm{Re}\,w_1)}{|\mathrm{Re}\,w_1| + |\mathrm{Im}\,w_1|} \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \log(|\mathrm{Re}\,w_1|)\log(F((\mathrm{Re}\,w_1)^2)\, d(\mathrm{Re}\,w_1) < \infty. \end{aligned}$$

That the integral is finite follows by the hypotheses on ϕ and F. This completes the proof of Theorem 1.2.

□

4. Examples

In this section, we present an example to show that our estimates are optimal in the sense that the inequality $\|u\|_{L^q(\Omega)} \lesssim \|\phi\|_{L^p(\Omega)}$ cannot hold if $1 \le p < q \le \infty$. Specifically, let $0 < \alpha < 1$, fix $1 \le p < q \le \infty$, and set

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : e^{1-\frac{1}{|z_1|^\alpha}} + |z_2|^2 < 1\}. \tag{4.1}$$

We will show that there is a $\bar\partial$-closed $(0,1)$-form $\phi \in L^p_{0,1}(\Omega)$ for which there does not exist a function $u \in L^q(\Omega)$ so that $\bar\partial u = \phi$ in Ω. Indeed, let

$$\phi(z) = \frac{(1-\log(1-z_2))^k}{(1-z_2)^{2/q}} d\bar z_1 \quad \text{and} \quad v(z) = \frac{(1-\log(1-z_2))^k}{(1-z_2)^{2/q}} \bar z_1 \tag{4.2}$$

where $k := \lfloor \frac{q+2}{q\alpha} \rfloor + 1 \in \mathbb{N}$. The function $\frac{(1-\log(1-z_2))^k}{(1-z_2)^{1/q}}$ is holomorphic on Ω with the principle branch of the logarithm $0 < \arg(1-z_2) < 2\pi$. The form ϕ is a $\bar\partial$-closed $(0,1)$-form on Ω and function v is a solution of the equation $\bar\partial v = \phi$. Moreover, we observe that v is L^2-orthogonal to all holomorphic functions on Ω (by Mean Value Theorem). By direct calculation (Lemma 4.1 below), we obtain $\phi \in L^p_{0,1}(\Omega)$, $v \in L^p(\Omega)$, and $v \notin L^q(\Omega)$. Let P be the Bergman projection on Ω, i.e., the L^2-orthogonal projection onto all holomorphic functions on Ω. Recently, Khanh and Thu [KT] have proven that P is a bounded operator form $L^{q'}(\Omega)$ to $L^{q'}(\Omega)$ for any $q' > 1$. Therefore if $u \in L^q(\Omega)$ is a solution to $\bar\partial u = \phi$, then $v = u - P(u)$ is in $L^q(\Omega)$. This is impossible. Therefore, there is no solution $u \in L^q(\Omega)$.

Lemma 4.1. *Let ϕ and v be defined in (4.2). Then, $\phi \in L^p_{1,0}(\Omega)$, $v \in L^p(\Omega)$ and $v \notin L^q(\Omega)$.*

Proof. We now show that $\phi \in L^p_{0,1}(\Omega)$. We have

$$\begin{aligned}
\int_\Omega |\phi(z)|^p dV(v) &= \int_\Omega \frac{|1-\log|1-z_2| + i\arg(1-z_2)|^{kp}}{|1-z_2|^{2p/q}} dV(z) \\
&\le \int_{|z_2|<1} \frac{\left((1-\log|1-z_2|)^2 + 4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}} \int_{|z_1|<(1-\log(1-|z_2|^2))^{-1/\alpha}} 1\, dV(z_1)\, dV(z_2) \\
&\lesssim \int_{|z_2|<1} \frac{\left((1-\log|1-z_2|)^2 + 4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}((1-\log(1-|z_2|^2))^{2/\alpha}} dV(z_2) \\
&\lesssim \int_{|z_2|<1} \frac{\left((1-\log|1-z_2|)^2 + 4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}} dV(z_2) \\
&\lesssim \int_{|z_2|<1, |z_2-1|\ge 1} \cdots + \int_{|z_2|<1, |z_2-1|<1} \cdots .
\end{aligned}$$

Since the function $\frac{\left((1-\log|1-z_2|)^2+4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}}$ is bounded on $\{|z_2|<1, |z_2-1|\geq 1\}$, the first integral $\int_{|z_2|<1,|z_2-1|\geq 1}\cdots$ is bounded. For the second integral, we have

$$\begin{aligned}\int_{|z_2|<1,|z_2-1|<1}\cdots &\leq \int_{|z_2-1|<1}\cdots\\ &= \int_0^1 \frac{((1-\log t)^2+4\pi^2)^{kp/2}}{t^{2p/q-1}}dt < \infty,\end{aligned}$$

since $2p/q-1<1$. The proof that $v\in L^p(\Omega)$ follows by our computation that $\phi\in L^p_{0,1}(\Omega)$ since $|z_1|$ is bounded. Now, we prove that $v\notin L^q(\Omega)$. We have

$$\begin{aligned}\int_\Omega |v(z)|^q dV(z) &= \int_\Omega \frac{|1-\log(1-z_2)|^{kq}\,|z_1|^q}{|1-z_2|^2}\,dV(z)\\ &= \int_{|z_2|<1}\frac{|1-\log|1-z_2|+i\arg(1-z_2)|^{kq}}{|1-z_2|^2}\int_{|z_1|<(1-\log(1-|z_2|^2))^{-1/\alpha}}|z_1|^q\,dV(z_1)\,dV(z_2)\\ &\geq \frac{2\pi\alpha}{q+2}\int_{|z_2|<1}\frac{|1-\log|1-z_2||^{kq}}{|1-z_2|^2(1-\log(1-|z_2|^2))^{\frac{q+2}{\alpha}}}\,dV(z_2)\\ &\gtrsim \int_{z_2\in D}\frac{|1-\log|1-z_2||^{kq}}{|1-z_2|^2(1-\log(1-|z_2|^2))^{\frac{q+2}{\alpha}}}\,dV(z_2),\end{aligned}$$

where

$$D=\{z_2=1+re^{i\theta}\in\mathbb{C}: 0<r<\frac{1}{3}, \frac{3\pi}{4}<\theta<\frac{5\pi}{4}\}\subset\{|z_2|<1, |z_2-1|<\frac{1}{3}\}\subset\{|z_2|<1\}.$$

The domain of the integral forces $1-\log(1-|z_2|^2)\sim 1-\log|1-z_2|$, and we obtain

$$\int_\Omega |v(z)|^q dV(z)\gtrsim \int_0^{\frac{1}{3}}\frac{(1-\log r)^{kq-\frac{q+2}{\alpha}}}{r}dr \geq \int_0^{\frac{1}{3}}\frac{dr}{r}\quad\text{(diverges)}.$$

Here, the last inequality holds because we chose k so that $kq-\frac{q+2}{\alpha}>0$ by the choice of k. $\square$

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