GROUP ACTIONS ON RIGID COHOMOLOGY WITH COMPACT SUPPORT - ERRATUM TO "THE ZETA FUNCTION OF MONOMIAL DEFORMATIONS OF FERMAT HYPERSURFACES"

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1. Introduction

David Ouwehand, Steven Sperber and John Voight communicated me two issues with the paper mentioned in the title.

First, several results in Section 3 assume that for a smooth quasiprojective variety X over a finite field and a finite group $G \subset \operatorname{Aut}(X)$ we have $H^i_{\text{rig}}(X)^G = H^i_{\text{rig}}(X/G)$. This latter statement does not seem to be part of the literature. We strongly believe that this conjecture is true, but we will explain how one can prove the main results of the paper without this result. The main idea is that the corresponding statement for $H^i_{\text{rig,c}}$ holds true, and that for the calculations of zeta functions one is interested in the characteristic polynomial of Frobenius acting on $H^i_{\text{rig},c}$.

Second, one of the steps in the proof of Corollary 6.10 is incomplete and we discuss how to repair this. (See Proposition 4.4.)

2. Quotients by automorphisms

Let k be a perfect field.

Proposition 2.1. *Let X be a k-scheme of finite type. Let $\mathcal{G}$ be a finite étale group scheme acting on X such that X admits a cover of open affine $\mathcal{G}$-stable subsets. Then for all i we have*

$$H^i_{\text{rig},c}(X)^{\mathcal{G}} \cong H^i_{\text{rig},c}(X/\mathcal{G})$$

as vector spaces with Frobenius action.

Proof. It suffices to prove that the natural map

$$H^i_{\text{rig},c}(X/\mathcal{G}) \to H^i_{\text{rig},c}(X)^{\mathcal{G}}$$

is bijective, since this map respects the Frobenius action. In particular, we may assume that the base field is algebraically closed and that $\mathcal{G}$ is a group.

We prove this by induction on $\dim X$.

Suppose first that $\dim X = 0$ holds. In this case X is proper, hence $H^i = H^i_c$. Write $X = \{p_1, \dots, p_k\}$ where p_i is a geometric point of X. The geometric points of $\mathcal{G}$ permute the p_i. The dimension of $H^0((X/\mathcal{G})_{\text{red}})$ equals the number of $\mathcal{G}$ orbits in X. A straightforward calculation shows that $\dim H^0(X)^{\mathcal{G}}$ also equals the number of $\mathcal{G}$ orbits and that the natural map $H^0(X/\mathcal{G}) \to H^0(X)^{\mathcal{G}}$ is clearly an isomorphism.

In the induction step we make a further induction to the number of irreducible components of dimension equal to $\dim X$.

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If there is precisely one such a component X_0 then $\mathcal{G}$ maps this component to itself. Let N be subgroup of $\mathcal{G}$ which acts trivially on X_0. Let $\mathcal{H} = \mathcal{G}/N$. Then there is a nonempty smooth open subset U of $(X_0)_{\text{red}}$ such that $U \to U/\mathcal{H}$ is étale and $U/\mathcal{H}$ is smooth again.

In this case we can apply [2, Théorème 4.2] to U and $U/\mathcal{H}$ and we obtain that

$$H_c^\bullet(U)^{\mathcal{G}} = H_c^\bullet(U)^{\mathcal{H}} \cong H_c^\bullet(U/\mathcal{H}) = H_c^\bullet(U/\mathcal{G})$$

Since $Z := X \setminus U$ has dimension less than $\dim X$ (and can also be covered $\mathcal{G}$-invariant affine opens) we can use the induction hypothesis on the dimension to get $H_c^\bullet(Z/\mathcal{G}) \cong H^\bullet(Z)_c^{\mathcal{G}}$. The triple U, X, Z yields a distinguished triangle for H_c^i as does the triple $U/\mathcal{G}, X/\mathcal{G}, Z/\mathcal{G}$. Using these triangles we get $H^\bullet(X)^G \cong H^\bullet(X/G)$.

We prove now the induction step on the number of irreducible components. Using distinguished triangles as above and the first induction hypothesis we may assume that every irreducible component of X has dimension $\dim X$.

Let $N \subset \mathcal{G}$ be the intersection of all stabilisers. Then N is a normal subgroup. Let $\mathcal{H} = \mathcal{G}/N$ then $(X/\mathcal{G})_{\text{red}} = (X/\mathcal{H})_{\text{red}}$. Hence their rigid cohomology groups are isomorphic. Similarly, the action of $\mathcal{H}$ and $\mathcal{G}$ on $H^\bullet(X)$ are the same. Hence we may assume that every element of $\mathcal{G}$ different from the identity acts nontrivially on X.

Hence there is a smooth affine open U in X such that $g(U) = U$ for all $g \in \mathcal{G}$, the quotient $U/\mathcal{G}$ is smooth and $U \to U/\mathcal{G}$ is étale. In this case we have $H_c^i(U)^{\mathcal{G}} = H_c^i(U/\mathcal{G})$ by [2, Théorème 4.2]. The number of irreducible components of $Z := X \setminus U$, whose dimension equal $\dim X$, is strictly less than the number of irreducible components of X of this dimension. Hence using both induction hypothesis we get $H_c^i(Z/\mathcal{G}) = H_c^i(Z)^{\mathcal{G}}$. Using distinguished triangles again we obtain $H_c^i(X)^{\mathcal{G}} = H_c^i(X/\mathcal{G})$. □

3. Changes in Section 3 through 5 of [3]

Suppose now that we are in the situation of [3]. I.e., we have a quasismooth hypersurface

$$X_\lambda : \sum x_i^{d_i} + \overline{\lambda} \prod x_i^{a_i}$$

in the weighted projective space $\mathbf{P} := \mathbf{P}(w_0, \dots, w_n)$. Denote with $U_{\overline{\lambda}} = \mathbf{P} \setminus X_{\overline{\lambda}}$.

This is a quotient of the hypersurface

$$Y_{\overline{\lambda}} : \sum x_i^d + \overline{\lambda} \prod x_i^{w_i a_i}$$

and from [3, Lemma 3.7] it follows that this latter hypersurface is smooth. Let $V_{\overline{\lambda}} := \mathbf{P}^n \setminus Y_{\overline{\lambda}}$. Now U_λ is V_λ/G where $G = \prod \mu_{w_i}$

In Theorem 3.8 from [3] one has to replace $H^i(U_{\lambda_0}, \mathbf{Q}_q)$ by $H^i_{\text{rig},c}(U_{\overline{\lambda}}, \mathbf{Q}_q)$. The proof for the case $\mathbf{P} = \mathbf{P}^n$ does not require any changes, the general case now follows directly from the case $\mathbf{P} = \mathbf{P}^n$ and Proposition 2.1 from the previous section.

Theorem 3.10 from [3] is not proven, and remains a conjecture.

Proposition 3.15 is true in the case of $\mathbf{P} = \mathbf{P}^n$ by the same argument as in the proof. If $\pi : \mathbf{P}^n \to \mathbf{P}$ is the quotient map by G then the argument of the proof shows that

$$\{\pi^*(\omega_{\mathbf{k}}) \colon \mathbf{k} \text{ an admissible monomial type}\}$$

is a basis for $H^n(V_\lambda)^G$. Using Poincaré dality we get that this latter space is isomorphic with $(H_c^n(V_\lambda)^*(-n))^G \cong H_c^n(U_\lambda)^*(-n)$.

The results in Section 4 are independent of Section 3. The results in Section 5 hold true in the case of $\mathbf{P} = \mathbf{P}^n$. In this case the formula of Proposition 5.3 hold true. If $\mathbf{P} \neq \mathbf{P}^n$ then using Poincaré duality we find that the formula of Proposition 5.3 give the deformation matrix of the operator $q^n \operatorname{Frob}_\lambda^{-1}$ acting on $H_c^n(U_\lambda)^G \cong H_c^n(V_\lambda)$.

Summarizing everything we now find that

$$Z(X_\lambda) = Z(\mathbf{P}(w_0, \dots w_n))/Z(U_\lambda)$$

and

$$Z(U_{\overline{\lambda}}) = \prod_i (\det((1 - TF^*)|H_c^i(V_\lambda)^G))^{(-1)^{i+1}}$$

By the modified version of Theorem 3.8 and the above remarks we find

$$Z(X_\lambda) = \det((1 - TF^*)|H_c^n(V_\lambda)^G)^{(-1)^n} \prod_{i=0}^{n-1} \frac{1}{1 - q^i T}$$

and that the action of F^* on $H_c^n(V_\lambda)^G$ equals the action of $q^n F^{-1*}$ on $H^n(V_\lambda)^G$. The latter action can be calculated by the methods of [3, Section 5].

4. Changes in Section 6

A second proof which is incomplete is the proof of Corollary 6.10. This Corollary gives a factorization of the zeta function. We present a result here which implies Corollary 6.10. We also (as an addendum) give a strategy to find a finer factorization. This latter factorization, when applied to the case of $X_0^5 + \cdots + X_4^5 + \overline{\lambda} X_0 \dots X_5 = 0$, is the factorization as found by Candelas et. al in [1].

4.1. **Factorization in $\mathbf{Q}_q[T]$.** Recall that we have three equivalence relations on monomial types.

For the notation used in the following we refer to [3, Section 2]

Definition 4.1. *We say that two monomial types* $\mathbf{k}, \mathbf{m}$ *are* strongly equivalent*, if* $\mathbf{k} - \mathbf{m}$ *is a multiple of the deformation vector* $\mathbf{a}$.

We say that they are weakly equivalent *if for some* $r \in \mathbf{Z}/d\mathbf{Z}^*$ *we have* $\mathbf{k} - r\mathbf{m}$ *is a multiple of the deformation vector.*

We say that they are indistinguishable by automorphisms *if the stabilisers of* $\omega_{\mathbf{k}}$ *and of* $\omega_{\mathbf{m}}$ *(in* $\times w_i \mathbf{Z}/d\mathbf{Z}$*) coincide.*

Note that the stabiliser of $\omega_{\mathbf{k}}$ consists of $(w_0 g_0, \dots, w_n g_n) \in \times w_i \mathbf{Z}/d\mathbf{Z}$ such that

$$\sum g_i a_i w_i \equiv 0 \bmod d \text{ and} \sum g_i (k_i + 1) w_i \equiv 0 \bmod d$$

holds. In particular, strongly equivalent implies weakly equivalent and weakly equivalent implies indistinguishable by automorphisms.

Lemma 4.2. *Two monomial types are weakly equivalent if and only if they are indistinguishable by automorphisms.*

Proof. We have to show that indistinguishable by automorphisms implies weakly equivalent.

Consider the bilinear form $(\prod w_i \mathbf{Z}/d\mathbf{Z}) \times (\prod w_i \mathbf{Z}/d\mathbf{Z}) \to \mathbf{Z}/d\mathbf{Z}$ defined by

$$(\dots, w_i c_i, \dots), (\dots, w_i d_i, \dots) \to \sum w_i c_i d_i.$$

Then the stabiliser of $\omega_{\mathbf{k}}$ is the orthogonal complement of the $\mathbf{Z}/d\mathbf{Z}$-submodule of $\prod w_i\mathbf{Z}/d\mathbf{Z}$ spanned by $\mathbf{k}$ and $\mathbf{a}$.

If R is a $\mathbf{Z}/d\mathbf{Z}$-submodule of $\prod w_i\mathbf{Z}/d\mathbf{Z}$ then a straightforward calculation shows that $\#R\#R^\perp = \#\prod w_i\mathbf{Z}/d\mathbf{Z}$. In particular $\#(R^\perp)^\perp = \#R$. Since $R \subset (R^\perp)^\perp$ we find that $R = (R^\perp)^\perp$.

Hence if $\mathbf{k}$ and $\mathbf{m}$ are indistinguishable by automorphisms then $\{\mathbf{a},\mathbf{k}\}$ and $\{\mathbf{a},\mathbf{m}\}$ generate the same submodule of $\prod w_i\mathbf{Z}/d\mathbf{Z}$.

In particular there exists $r_1, r_2, s_1, s_2 \in \mathbf{Z}/d\mathbf{Z}$ such that $\mathbf{k} = r_1\mathbf{a} + s_1\mathbf{m}$ and $\mathbf{m} = r_2\mathbf{a}+s_2\mathbf{k}$. It remains to show that $s_1, s_2 \in (\mathbf{Z}/d\mathbf{Z})^*$. By the Chinese remainder theorem we may assume that d is a prime power. Moreover we may multiply the entries of $\mathbf{a}$, $\mathbf{k}$, $\mathbf{m}$ by w_i and assume that each $w_i = 1$.

If one of the entries of $\mathbf{a}$, say a_i, has valuation zero then consider $\tilde{\mathbf{k}} = \mathbf{k} - \frac{k_i}{a_i}\mathbf{a}$ and $\tilde{\mathbf{m}} = \mathbf{m} - \frac{m_i}{a_i}\mathbf{m}$. Now $\{\mathbf{a},\tilde{\mathbf{k}}\}$ and $\{\mathbf{a},\tilde{\mathbf{m}}\}$ generate the same submodule. Since a_i is invertible, $\tilde{\mathbf{k}}_i = 0 = \tilde{\mathbf{m}}_i$ and $\tilde{\mathbf{m}}$ is in the submodule generated by $\tilde{\mathbf{k}}$ and $\mathbf{a}$, we find that $\tilde{\mathbf{k}}$ is a multiple of $\tilde{\mathbf{m}}$. More precisely, we have that

$$\tilde{\mathbf{k}} = \lambda\tilde{\mathbf{m}} \text{ and } \tilde{\mathbf{m}} = \nu\tilde{\mathbf{k}}$$

In particular $v(\tilde{\mathbf{k}}_j) = v(\tilde{\mathbf{m}}_j)$ for every j and therefore $v(\lambda) = v(\nu) = 0$. Now

$$\mathbf{k} = \lambda\mathbf{m} + ((-\lambda m_i + k_i)/a_i)\mathbf{a}.$$

and hence we are done.

If all entries of $\mathbf{a}, \mathbf{k}, \mathbf{m}$ are divisible by ℓ then we can divide them and d by ℓ. Hence we may assume in the remaining case that for all i we have $v(\mathbf{a}_i) > 0$ and that for some j we have that $v(\mathbf{k}_j) = 0$. Then also $v(\mathbf{m}_j) = 0$.

If we now write $\mathbf{k} = \alpha\mathbf{a} + \beta\mathbf{m}$ and consider this modulo ℓ then we find $\beta \equiv k_j/m_j \bmod \ell$. In particular $v(\beta) = 0$ and we find $\beta \in (\mathbf{Z}/d\mathbf{Z})^*$. □

The aim is to describe a factorisation of the zeta function of the hypersurface $X_{\overline{\lambda}}$, provided $X_{\overline{\lambda}}$ is quasismooth. In [3, Section 6] we started with a factorisation in $\mathbf{Q}_q[T]$.

Let $\mathbf{k}$ be an admissible monomial type. Let $e = q \bmod d$. From [3, Lemma 6.1] it follows that Frobenius at $\overline{\lambda} = 0$ sends $\omega_{\mathbf{k}}$ to $\omega_{e\mathbf{k}}$. By the results of [3, Section 5] we have that the deformation matrix leaves the subspace spanned by

$$\{\omega_{\mathbf{k}+t\mathbf{a}} \mid \mathbf{k} + t\mathbf{a} \text{ admissible.}\}$$

invariant. Combining this we see that Frobenius leaves the space spanned by

$$\{\omega_{e^s\mathbf{k}+t\mathbf{a}} \mid e^s\mathbf{k} + t\mathbf{a} \text{ admissible.}\}$$

invariant. In particular, if $e \equiv 1 \bmod d$ then we find for each strongly equivalence class (see [3, Section 6]) a factor $P_{[\omega_{\mathbf{k}}]}(T) \in \mathbf{Q}_q[T]$, such that the characteristic polynomial on $H^n(U_{\overline{\lambda}})$ equals

$$\prod_{[\mathbf{k}]} P_{[\mathbf{k}]}(T)$$

This is not a factorisation in $\mathbf{Q}[T]$. However, we need this factorisation later on.

4.2. **Factorisation in $\mathbf{Q}[T]$.** Assume that $X_{\overline{\lambda}}$ is quasismooth. Consider now $H^n_c(U_{\overline{\lambda}}, \mathbf{Q}_q)$. Then the characteristic polynomial of Frobenius is an element of $\mathbf{Q}[T]$.

Let K be the subgroup of $\times w_i\mathbf{Z}/d\mathbf{Z}$ consisting of all $(w_0g_0, \dots, w_ng_n)$ such that $\sum g_ia_iw_i \equiv 0 \bmod d$. Then K is a subgroupscheme of $\mathrm{Aut}(U_{\overline{\lambda}})$ and of $\mathrm{Aut}(X_{\overline{\lambda}})$.:

For an admissible monomial type $\mathbf{k}$, let $G_{\mathbf{k}} \subset K$ the stabilizer of $\omega_{\mathbf{k}}$.

Lemma 4.3. *The group scheme associated with $G_{\mathbf{k}}$ is defined over $\mathbf{F}_p$. In particular, the characteristic polynomial of Frobenius on $H^n_c(V_{\overline{\lambda}})^{G_{\mathbf{k}}}$ is an element of $\mathbf{Q}[T]$.*

Proof. Note that the stabiliser of $\omega_{\mathbf{k}}$ consists of automorphisms which are defined over $\mathbf{F}_q(\zeta_d)$. However, an element of $\mathrm{Gal}(\mathbf{F}_q(\zeta_d)/\mathbf{F}_q)$ sends $\zeta_d \to \zeta_d^e$ for some e coprime to d. Hence it maps $(w_0g_0, \dots, w_ng_n) \to (ew_0g_0, \dots, ew_ng_n)$. If the former is in $G_{\mathbf{k}}$ then so is the latter, and we have that the group scheme $G_{\mathbf{k}}$ is defined over $\mathbf{F}_q$.

Moreover, we have that $H^i_c(U_{\overline{\lambda}}/G_{\mathbf{k}}) = H^i_c(U_{\overline{\lambda}})^{G_{\mathbf{k}}} = H^i_c(V_{\overline{\lambda}})^{G_{\mathbf{k}}}$ is zero for $i \neq n, 2n$, one-dimensional for $i = 2n$. Hence the zeta function is either the quotient of the characteristic polynomial of Frobenius on H^n divided by $(1 - q^nT)$ or it is the reciprocal of $(1 - q^nT)$ times the the characteristic polynomial of Frobenius on H^n. Since the zeta function is in $\mathbf{Q}(T)$ we have that the characteristic polynomial is in $\mathbf{Q}[T]$. □

We can now fill the gap in the proof of [3, Corollary 6.10] (with the modification indicated in the previous section):

Proposition 4.4. *Let $\overline{\lambda} \in \mathbf{F}_q$ be such that $X_{\overline{\lambda}}$ is quasismooth. Let $P(t)$ be the characteristic polynomial of Frobenius acting on $H^n_c(U_{\overline{\lambda}})$. Then*

$$P(t) = \prod_{[\mathbf{k}]} P_{[\mathbf{k}]}(t)$$

where the product is taken over all weakly equivalence classes of admissible monomial types, such that each $P_{[\mathbf{k}]}(t)$ is in $\mathbf{Q}[t]$ and the degree of $P_{[\mathbf{k}]}(t)$ equals the number of admissible monomial types in the equivalence class of $\mathbf{k}$.

Proof. Let $V = H^n_c(U_{\overline{\lambda}}, \mathbf{Q}_q)$. Fix an admissible monomial type $\mathbf{k}$. Since every $\omega_{\mathbf{m}}$ is an eigenvector for every element of K we can write

$$V^{G_{\mathbf{k}}} = V_{\mathbf{k}} \oplus V'$$

where V' is spanned by all $\omega_{\mathbf{m}}$ such that $G_{\mathbf{k}}$ is a proper subgroup of $G_{\mathbf{m}}$. The characteristic polynomial of Frobenius acting on $V_{\mathbf{k}}$ is then in $\mathbf{Q}[T]$, since the characteristic polynomials on V' and V^{G_k} are.

Consider now the equivalence relation indistinguishable by automorphisms and denote with $[\mathbf{k}]$ the equivalence classes then

$$V = \oplus_{[\mathbf{k}]} V_{\mathbf{k}}$$

Hence we have that the characteristic polynomial is a product

$$\prod_{[\mathbf{k}]} P_{\mathbf{k}}$$

and each $P_{\mathbf{k}} \in \mathbf{Q}[T]$. The statement about the degree is immediate. □

We give now an example where we combine both factorisations to obtain a finer factorisation. This example supplies some details to the claims made in [3, Example 6.11]

Example 4.5. Consider

$$x_0^5 + \cdots + x_4^5 + \lambda x_0 x_1 x_2 x_3 x_4.$$

Up to permutation of the coordinates we have that the admissible monomial types are of the shape (a,a,a,a,a), (a,a,a,b,c) and (a,a,b,b,c).

Two weakly equivalent monomial types have the same shape. The weakly equivalence class of $(1,1,1,1,1)$ consists of (a,a,a,a,a) with $a \in \{1,2,3,4\}$.

The weakly equivalent class of $(1,1,1,3,4)$ consists further of

$$(2,2,2,1,3),(3,3,3,4,2),(4,4,4,2,1),(4,4,4,1,2),(3,3,3,2,4),(1,1,1,4,3).$$

The weakly equivalent class of $(1,1,2,2,4)$ consists further of

$$(2,2,4,4,3),(3,3,1,1,2),(4,4,3,3,1),(3,3,4,4,1),$$
$$(1,1,3,3,2),(4,4,2,2,3),(2,2,1,1,4).$$

Hence we find that the characteristic polynomial of Frobenius acting on $H^4(U_{\overline{\lambda}})$ is the product of a degree 4 polynomial and polynomials of degree 8. Using the S_5 symmetry we find that many of these polynomials coincide, i.e., there exist a polynomial $Q_1 \in \mathbf{Q}[T]$ of degree 4 and two polynomials $Q_2, Q_3 \in \mathbf{Q}[T]$ of degree 8 such that the characteristic polynomial of Frobenius is $Q_1 Q_2^{10} Q_3^{15}$.

We can factor Q_2 and Q_3 further. Take the involution σ swapping x_3 and x_4. Consider the subgroup K of the automorphism group generated by σ and the stabiliser of $\omega_{(1,1,1,3,4)}$. Then $H^n_c(V_\lambda)^K$ is a four-dimensional subspace of

$$H^n(V_\lambda)^{G_{(1,1,1,3,4)}}.$$

Hence the degree eight polynomial Q_2 factors as a product of two degree 4 polynomials.

Moreover if $q \equiv 1 \bmod 5$ then we can also consider the factorization in $\mathbf{Q}_q[T]$ and we find that Q_2 is a square in $\mathbf{Q}_q[T]$ and we may deduce from this that it is also a square in $\mathbf{Q}[T]$.

Similarly, we find that Q_3 factors as two polynomials of degree 4 in $\mathbf{Q}[T]$, which coincide if $q \equiv 1 \bmod 5$. Hence if $q \equiv 1 \bmod 5$ then we find three quartic polynomials R_1, R_2, R_3 such that the characteristic polynomial of Frobenius is

$$R_1 R_2^{20} R_3^{30}.$$

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