EXERCISES FOR THE COURSE SUPERFICI DI RIEMANN A.A. 2016/17

(1) Let $U, V \subset \mathbf{C}$ be open subsets. Let $f : U \to V$ be holomorphic and bijective. Show that $f^{-1} : V \to U$ is holomorphic.

(2) Let G and G' be groups. Let $\varphi : G \to G'$ be such that for every $g_1, g_2, g_3 \in G$ with $g_1 \circ g_2 \circ g_3 = e_G$ we have

$$f(g_1) \circ f(g_2) \circ f(g_3) = e_{G'}.$$

Show that f is homomorphism of groups.

(3) Let $f(z) = \frac{z^3}{1-z^2}$. Extend f to a morphism $F : \mathbf{P}^1 \to \mathbf{P}^1$. Determine the degree of F, find all ramification points and verify the Riemann-Hurwitz formula for this particular map.

(4) Let $a, b \in \mathbf{C}$ be such that $(a^2 - 4b)b \neq 0$. Let $C_1 \subset \mathbf{P}^2$ be the curve $Y^2Z = X^3 + aX^2Z + bXZ^2$ and let $C_2 \subset \mathbf{P}^2$ be the curve $Y^2Z = X^3 - 2aX^2Z + (a^2 - b)XZ^2$.

Show that the map $C_1 \setminus \{(0 : 0 : 1)\} \to C_2$ given by $(X : Y : Z) \mapsto (Y^2Z : Y(bZ^2 - X^2) : X^2Z)$ can be extended to a ramified covering $C_1 \to C_2$. Find the degree of this map and the ramification points.

(5) Fix integers k, n. Let $f(y)$ be a polynomial of degree $kn-1$ without multiple factors. Let $C_1 \subset \mathbf{C}^2$ be the curve given by $x^n = f(y)$ and let C_2 be the curve given by $w^n = f(1/z)z^{kn}$. Show that C_1 and C_2 can be glued to a Riemann surface C, such that $C \setminus C_1$ consist of one point.

Show that the map $C_1 \to \mathbf{C}$ mapping (x, y) to x can be extended to a ramified covering $f : C \to \mathbf{P}^1$. Determine the degree of f, the ramification of f and the genus of C.

(6) Let $\Lambda = \mathbf{Z} \oplus \mathbf{Z}i$. Find all $\alpha \in \mathbf{C}$ such that $\alpha\Lambda = \Lambda$.

(7) Let $\Lambda = \mathbf{Z} \oplus \mathbf{Z}\omega$, with $\omega = \exp(2\pi i/3)$. Find all $\alpha \in \mathbf{C}$ such that $\alpha\Lambda = \Lambda$.

(8) Let $\Lambda = \mathbf{Z} \oplus \mathbf{Z}\tau$. Suppose there is a $\alpha \in \mathbf{C} \setminus \mathbf{Z}$ such that $\alpha\Lambda \subset \Lambda$. Show that τ is algebraic over $\mathbf{Q}$ and that $[\mathbf{Q}(\tau) : \mathbf{Q}] = 2$. (Hint: use that $\alpha \cdot 1$ and $\alpha \cdot \tau$ are linear combinations of τ and 1 to find a polynomial of degree 2 which has τ as one of its zeros.)