EXERCISES FOR THE COURSE SUPERFICI DI RIEMANN A.A. 2016/17

(1) Let C be the curve $y^2 - x^r = 0$ in $\mathbf{C}^2$, (with $r \geq 2$). Let $\tilde{C}$ be the strict transform of C under the blow-up of $(0,0)$. Show that the strict transform has at most one singular point. Determine the number blow-ups needed to obtain a smooth strict transform.

(2) Let C be the curve $y^m - x^m = 0$ in $\mathbf{C}^2$, with $m \geq 2$. Let $\tilde{C}$ be the strict transform of C under the blow-up of $(0,0)$. Show that the strict transform is smooth.

(3) Let $z \in \mathbf{C}$ be a complex number $\mathrm{Im}(z) > 0$. Consider the $\mathrm{SL}_2(\mathbf{Z})$ orbit of z. Let $\mathcal{F} = \{z \in \mathbf{C} \colon |\mathrm{Re}(z)| \leq 1/2, |z| \geq 1/2\}$.

In the lecture we showed that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ we have $\mathrm{Im}(\gamma(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$. Moreover we showed that $\mathrm{Im}(\gamma(z))$ is bounded from above for fixed z and that there is $z_0 := \gamma_0(z) \in \mathcal{F}$ such that $\mathrm{Im}(\gamma z)$ attains its maximum at $\gamma = \gamma_0$.

(a) Let $z = x + iy \in \mathbf{C}, x, y \in \mathbf{R}$. Show that if $|z| \geq 1$ and $|x| \leq 1/2$ then $|cz + d|^2 \geq c^2(|z|^2 - \frac{1}{4})$ for all $c, d \in \mathbf{Z}$. Deduce from this that for $|c| \geq 2$ we have $|cz + d| > 1$.

(b) Show that if (c, d) is the lower row of a matrix in $SL_2(\mathbf{Z})$ and $c = 0$ then $d = \pm 1$. Deduce from this that the matrices with $c = 0$ yield translations $z \to z + n$ for some $n \in \mathbf{Z}$. Find all $(z, \gamma) \in \mathcal{F} \times \mathrm{SL}_2(\mathbf{Z})$ such that $c = 0$ and such that both $\gamma(z)$ and z are in $\mathcal{F}$.

(c) Show that if $c = \pm 1$, $|\mathrm{Re}(z)| \leq 1/2$ and $|cz + d| \leq 1$ hold then $d \in \{-1, 0, 1\}$ and $|\mathrm{Re}(z)| = 1/2$. Use this to show that $|cz + d| \geq 1$ for all $(c, d) \in \mathbf{Z}^2$ with $(c, d) \neq (0, 0)$ and $z \in \mathcal{F}$. (And hence for all $z \in \mathcal{F}$ we have either $\gamma(z) \notin \mathcal{F}$ or $\mathrm{Im}(z) = \mathrm{Im}(\gamma(z))$.)

(d) Find now all $x+iy \in \mathcal{F}$, $c, d \in \mathbf{Z}$ such that $(c, d) \neq (0, 0)$ and $|cz+d| = 1$. Use this to find all $z \in \mathcal{F}$ such that there is a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma(z) \neq z$ and $\gamma(z) \in \mathcal{F}$.

(4) Let $h(x) \in \mathbf{C}[x]$ be a polynomial of degree $2g + 1$ with distinct roots. Let C be the affine complex curve $y^2 = h(x)$. Show that there is a number $R > 0$ such that the map $\varphi(x, y) = y/x^{g+1}$ defines a hole chart on $\{(x, y) \in \mathbf{C}^2 \colon |x| > R\}$.

(5) Let $\mathbf{C}/\Lambda$ be a complex torus. Show that this group is divisible, i.e., for every $p \in \mathbf{C}/\Lambda, n \in\in Z$ there exists a $q \in \mathbf{C}/\Lambda$ with $nq = p$. Moreover, show that for fixed p, n the equation $nq = p$ has n^2 solutions.