EXERCISES FOR THE COURSE SUPERFICI DI RIEMANN A.A. 2016/17

(1) Let $\mathbf{T}$ be complex torus.
   (a) Fix a point $O \in \mathbf{T}$. Use Riemann-Roch to show that for every two points P, Q on $\mathbf{T}$ there is a unique point R such that $P+Q$ and $R+O$ are linearly equivalent.
   (b) Consider the map $\mathbf{T} \to \mathrm{Pic}^0(\mathbf{T})$ mapping P to $P-O$. Show that this map is bijective.
   (c) Assume now that O is the origin. Let C be the cubic curve isomorphic with $\mathbf{T}$. Recall that $O=(0:1:0)$ and that O is a flex point. Let $\varphi : C \to \mathrm{Pic}^0(C)$ be the map mapping P to $P-O$. Show that if P, Q, R are collinear points on C then $\varphi(P)+\varphi(Q)+\varphi(R)=0$ in $\mathrm{Pic}^0(C)$ and that $\varphi(-P)=-\varphi(P)$. (Here the first minus is with respect to the group law on C, the second minus is with respect to the group law on $\mathrm{Pic}(C)$.)
   Use this to show that φ defines an isomorphism of groups.

(2) Let C be a compact Riemann surface of genus g. Fix a base point O. Let $P_1, \ldots P_{g+1}$ be points on C. Use Riemann-Roch to show that are g points $Q_1, \ldots, Q_g$ on C such that $P_1+P_2+\cdots+P_{g+1} \sim Q_1+\cdots+Q_g+O$. Use this to show that the map

$$C^g \to \mathrm{Pic}^0(C)$$

given by $(P_1, \ldots, P_g) \to P_1+P_2+\ldots P_g - gO$ is surjective.

(3) Let C be the compact Riemann surface associated with the affine curve $y^2 = x^5+1$. Calculate $\mathrm{div}(y)$ and $\mathrm{div}(x)$.

(4) Use Riemann-Roch to show that if a divisor D satisfies $\ell(D)=g$ and $\deg(D)=2g-2$ then $D \sim \mathrm{kdiv}(\omega)$ for some differential form ω.