EXERCISES FOR THE COURSE SUPERFICI DI RIEMANN A.A. 2016/17

(1) Let
$$0 \to V_0 \stackrel{f_0}{\to} V_1 \stackrel{f_i}{\to} \cdots \stackrel{f_{n-1}}{\to} V_n \to 0$$
be an exact sequence of finite dimensional vector spaces, i.e we have that $\ker(f_i) = \mathrm{Im}(f_{i-1})$. Show that $\sum(-1)^i \dim V_i = 0$.

(2) **Finitely generated groups**

(a) Let G and H be finitely generated free abelian groups, i.e. $G \cong \mathbf{Z}^r$ and $H \cong \mathbf{Z}^s$ Let $g_1, \dots, g_r$ be a set of generators of G and $h_1, \dots, h_s$ be a set of generators. Let $\varphi : G \to H$ be a group homomorphism. Let M be the matrix of φ with respect to these basis.
Show that φ is injective if and only if the $\mathbf{Q}$-rank of the matrix M is r and show that if φ is surjective then the $\mathbf{Q}$-rank of M equals s. In particular if G and H are isomorphic then $r = s$.

(b) Let G and H be as above, but suppose now that $r = s$. Show that φ is an isomorphism if and only if the determinant of M is either 1 or -1.

Let G be a finitely generated abelian group let G_{fin} be the elements of finite order. Then G/G_{fin} is a free abelian group, i.e., isomorphic to $\mathbf{Z}^r$ for some r, and by the above exercises this r depends only on G. We call r the rank of G.

(3) Let $0 \to G_0 \to \dots G_n \to 0$ be an exact sequence of finitely generated groups. Show that $\sum(-1)^i \operatorname{rank}(G_i) = 0$.

(4) Let C be a Riemann surfae of genus g. Let P be a point on C. A gap number n is a positive integer such that $\ell(nP) = \ell((n-1)P)$. We know that 1 is a gap number, that the largest gap number is at most $2g - 1$, and that there are precisely g gap numbers. Simlilarly we can define non-gap numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_g = 2g$ to be those numbers such that $\ell(\alpha_i P) = \ell((\alpha_i - 1)P) + 1$. Show that the sum of two non-gap numbers is again a non-gap number. Use this to show $\alpha_j + \alpha_{g-j} \geq 2g$. Moreover, show that if $\alpha_1 = 2$ then $\alpha_k = 2k$ for $k = 1, \dots, g$.