POINT COUNTING ON SINGULAR HYPERSURFACES

REMKE KLOOSTERMAN

ABSTRACT. We discuss how one can extend to hypersurfaces with isolated singularities the methods of Gerkmann, Abbott-Kedlaya-Roe and Lauder for counting points on smooth hypersurfaces.

1. Introduction

Let $q = p^r$ be a prime power. Let $\overline{F} \in \mathbf{F}_q[X_0, \dots, X_{n+1}]$ be a homogeneous polynomial of degree d. Let $\overline{V} \subset \mathbf{P}^{n+1}$ be the hypersurface defined by $\overline{F} = 0$. A natural question to ask is how to determine $\#\overline{V}(\mathbf{F}_q)$.

Recently, several algorithms were presented that calculate $\#\overline{V}(\mathbf{F}_q)$ if $\overline{V}$ is a smooth hypersurface. We would like to investigate whether these algorithms extend to singular hypersurfaces.

In the case $n = 1$ (curves) there are many special algorithms to determine $\#\overline{V}(\mathbf{F}_q)$. For the sake of simplicity we leave these out of consideration, and we focus on the case $n > 1$. To our knowledge, there exist the following types of algorithms to determine $\#\overline{V}(\mathbf{F}_q)$ for a smooth hypersurface of degree d:

- A direct method by Abbott, Kedlaya and Roe [1].
- A deformation method by Lauder [8] and a slightly different one by Gerkmann [3].
- A recursive method by Lauder [9].

In this paper we identify an obstruction to extend the deformation method to singular varieties, i.e., for singular $\overline{V}$ the deformation method might give an output different from $\#\overline{V}(\mathbf{F}_q)$. Since the recursive method is based on the deformation method we expect that a similar obstruction plays a role there, therefore we leave this method out of consideration.

Theorem 1. *There exist hypersurfaces $\overline{V} \subset \mathbf{P}^{n+1}_{\mathbf{F}_q}$ such that*

1. $H^i_{\text{rig}}(\overline{V}, \mathbf{Q}_q) \cong H^i_{\text{rig}}(\mathbf{P}^{n+1}, \mathbf{Q}_q)$ *for $i \neq n, 2n+2$.*
2. *Lauder's deformation algorithm and Gerkmann's deformation algorithm terminate, but the output of the algorithm might differ from $\#V(\mathbf{F}_q)$.*
3. *a modification of Abbott-Kedlaya-Roe's algorithm gives $\#V(\mathbf{F}_q)$.*

The exact modification of Abbott-Kedlaya-Roe is described in 2.4. We illustrate this theorem by giving two explicit examples of hypersurfaces satisfying (1)-(3) of Theorem 1. Due to space restrictions we will not describe the precise class of hypersurfaces for which Theorem 1 holds. We intend to come back to this issue in [7].

Unfortunately, in the smooth case the algorithm of Abbott-Kedlaya-Roe is expected to have worse complexity than the Lauder-Gerkmann type of algorithm. This latter algorithm requires $(pd^n \log(q))^{O(1)}$ bit operations (for a discussion see

[8]). Abbott, Kedlaya and Roe did not include an analysis of the complexity of their algorithm.

For singular hypersurfaces we will use a variant of Abbott-Kedlaya-Roe where we replace the Frobenius operator Frob_q^* with the so-called Dwork ψ-operator. This ψ-operator is a left inverse to Frob_q^*.

In the case of a singular hypersurface the choice for ψ is essential, since the original version of Abbott-Kedlaya-Roe will encounter the problem of 'exploding coefficients' if applied to a singular hypersurface: Abbott-Kedlaya-Roe relate the trace of Frob_q^* on a certain $\mathbf{Q}_q$-vector space W with $\#\overline{V}(\mathbf{F}_q)$. If $\overline{V}$ is singular, ψ might have eigenvalues on this vector space W with small p-adic absolute value, hence Frob_q^* might have eigenvalues with very large p-adic absolute value. The eigenvalues of ψ with small p-adic absolute value should be ignored if one wants to calculate $\#\overline{V}(\mathbf{F}_q)$.

If $H^i_{\mathrm{rig}}(\overline{V},\mathbf{Q}_q) \not\cong H^i_{\mathrm{rig}}(\mathbf{P}^{n+1},\mathbf{Q}_q)$ for some i with $n+1 \leq i \leq 2n$ then it is easy to see that none of [1, 3, 8] can work. This follows from Obstruction 5 (PD-Failure). An approach to resolve this PD-Failure will be given in the paper [7]. In the sequel we will assume that the hypersurfaces under consideration do not have this obstruction.

The organization of this paper is as follows. In Section 2 we describe the deformation methods of Lauder and of Gerkmann, and the method of Abbott-Kedlaya-Roe. We indicate which results from algebraic geometry are used. Some of these results hold only for smooth varieties, whereas many other results hold also for certain classes of singular varieties.

In the case of the deformation method we describe an obstruction that is hard to resolve. In the case of the direct method we indicate how one can bypass the obstructions for a certain class of varieties. The main difference between our method and that of [1] is that we use Dwork's left-inverse ψ of a lift of Frobenius instead of the lift itself.

In Section 3 we study the surface $X^2+Y^2+Z^2=0$ in $\mathbf{P}^3$. This is a cone over a conic, i.e., a quadric with an A_1 singularity. This is the prototype of an example for which [1] works, but [3, 8] might give wrong answers.

2. A short description of the algorithms under consideration

Notation 2. Let p be a prime number, $q=p^r$ a power of p. Let $\mathbf{F}_q$ be the finite field with q elements. Denote the ring of Witt vectors of $\mathbf{F}_q$ by $\mathbf{Z}_q$, its maximal ideal by π, and its fraction field by $\mathbf{Q}_q$. Equivalently, the field $\mathbf{Q}_q$ is the unique unramified extension of degree r of $\mathbf{Q}_p$.

We proceed by giving a short summary of the ideas used in [1, 3, 8].

In all three papers the authors prefer to calculate $\#\overline{U}(\mathbf{F}_q)$, where $\overline{U}=\mathbf{P}^{n+1}\setminus\overline{V}$ is the complement of $\overline{V}$, instead of calculating $\#\overline{V}(\mathbf{F}_q)$. The main advantage is that $\overline{U}$ is a smooth *affine* variety.

The idea now is to use cohomology. Denote by $H^i(\overline{U},\mathbf{Q}_q)$ the i-th Monsky-Washnitzer, rigid or Dwork cohomology of $\overline{U}$. (In our case, all these groups are isomorphic as vector spaces with Frobenius action.) We can use the Lefschetz trace formula, which reads as

$$\sum_{i=0}^{n+1} q^i - \#\overline{V}(\mathbf{F}_q) = \#\overline{U}(\mathbf{F}_q) = \sum (-1)^i \operatorname{trace}\left((q^{n+1}\,\mathrm{Frob}_q^{*-1}) \mid H^i(\overline{U},\mathbf{Q}_q)\right).$$

The use of $q^{n+1}\operatorname{Frob}_q^{*-1}$ rather than Frob_q^* is due to the fact that the usual Lefschetz trace formula holds for (rigid) cohomology $H_c^\bullet(\overline{U},\mathbf{Q}_q)$ with compact support, which is Poincaré dual to $H^{2n+2-\bullet}(\overline{U},\mathbf{Q}_q)$.

We can simplify the Lefschetz trace formula by:

Proposition 3 (Lefschetz hyperplane theorem)**.** *Suppose $\overline{V}$ is smooth then*

- $H^i(\overline{U},\mathbf{Q}_q)=0$ *if* $i\neq 0,n+1$ *and*
- $H^0(\overline{U},\mathbf{Q}_q)$ *is one-dimensional and Frobenius acts as the identity.*

From this lemma it follows that it suffices to determine the eigenvalues of Frob_q^* on $H^{n+1}(\overline{U},\mathbf{Q}_q)$. All methods under consideration calculate the action of Frobenius on $H^{n+1}(\overline{U},\mathbf{Q}_q)$.

Remark 4. Actually this Proposition is a combination of Lefschetz hyperplane theorem with Poincaré duality on $H^\bullet(\overline{V},\mathbf{Q}_q)$. If $\overline{V}$ is singular then Poincaré duality might not hold. In that case Lefschetz hyperplane theorem shows that $H^i(\overline{U},\mathbf{Q}_q)=0$ for $i>n+1$ (and $i<0$).

Here is the first obstruction to extending these algorithms to singular varieties that occurs:

Obstruction 5 (PD-Failure)**.** If $\overline{V}$ is a singular hypersurface then Proposition 3 might fail for i such that $n-\dim\overline{V}_{\text{sing}}\leq i\leq n$. If this happens, one needs a separate algorithm to calculate the Frobenius action on $H^i(\overline{U},\mathbf{Q}_q)$ for these i.

For a strategy to resolve PD-Failure in some cases we refer to [7]. We give two examples of varieties which have PD-failure:

Example 6. Suppose $\overline{V}$ is a hypersurface with two irreducible components. Then $H^{2n}(\overline{V},\mathbf{Q}_q)$ is two-dimensional. A standard argument using Gysin long exact sequence and Poincaré duality yields that $H^1(\overline{U},\mathbf{Q}_q)$ is 1-dimensional.

Example 7. Let $V: x_0^5+x_1^5+x_2^5+x_3^5+x_4^5-5x_0x_1x_2x_3x_4=0$ in $\mathbf{P}^4$. Then V is an irreducible surface with 125 ordinary double points. If $p\neq 2,5$ then $H^4(\overline{V},\mathbf{Q}_q)$ is 25-dimensional [11] and using a similar standard argument as in the previous example we obtain that $H^3(\overline{U},\mathbf{Q}_q)$ is 24-dimensional.

At this stage the methods under consideration diverge. We start to consider them separately.

2.1. Deformation method, smooth case.

For technical reasons assume $p\nmid d$. Consider the family

$$\overline{V}_{\overline{\lambda}}:(1-\overline{\lambda})\left(\sum_{i=0}^{n+1}X_i^d\right)+\overline{\lambda}\,\overline{F}=0.$$

Then $\overline{V}_0$ is the diagonal hypersurface of degree d and $\overline{V}_1=\overline{V}$. Let $\overline{U}_\lambda$ denote the corresponding family of complements. Let V_λ be a family of hypersurfaces lifting $\overline{V}_{\overline{\lambda}}$ to $\mathbf{Z}_q$, i.e., a family given by $F_\lambda\in\mathbf{Z}_q[X_0,\dots,X_{n+1}]$ such that $F_\lambda\equiv\overline{F}_{\overline{\lambda}} \bmod \pi$ for all $\lambda\in\mathbf{Z}_q$, where $\overline{\lambda}\equiv\lambda \bmod \pi$.

The deformation method is built around the following diagram (cf. [5]):

$$\begin{array}{ccc} H^{n+1}(U_{\lambda^q},\mathbf{Q}_q) & \xrightarrow{\mathrm{Frob}^*_{q,\lambda}} & H^{n+1}(U_\lambda,\mathbf{Q}_q) \\ {\scriptstyle A(\lambda^q)}\downarrow & & {\scriptstyle A(\lambda)}\downarrow \\ H^{n+1}(U_0,\mathbf{Q}_q) & \xrightarrow{\mathrm{Frob}^*_{q,0}} & H^{n+1}(U_0,\mathbf{Q}_q). \end{array}$$

It is relatively easy to calculate the Frobenius action on $H^{n+1}(\overline{U}_0,\mathbf{Q}_q)$ and we leave this aside. The operator $A(\lambda)$ is the unique solution to the p-adic Picard-Fuchs equation associated with the family V_λ, such that $A(0)$ is the identity. Equivalently, one can express $A(\lambda)$ in terms of the Gauß-Manin connection of the local system $H^{n+1}(V_\lambda,\mathbf{Q}_q)$.

To calculate $\mathrm{Frob}^*_{q,1}: H^{n+1}(U_1,\mathbf{Q}_q)\to H^{n+1}(U_1,\mathbf{Q}_q)$ it suffices to calculate the following operator

$$Fr(\lambda)=\lim_{\mu\to\lambda} A(\mu)^{-1}\,\mathrm{Frob}^*_{q,0}\,A(\mu^q).$$

It should be remarked that the operator $A(\mu)$ itself does not converge on the p-adic unit disc.

The differential equation for $A(\mu)$ induces a differential equation for $Fr(\lambda)$. The methods of Gerkmann and Lauder consist of an efficient calculation of the solution of the latter differential equation.

2.2. **Deformation method, singular case.** We describe which of the above ideas differ in the case that V_1 is singular.

We start with some (false) heuristics. One expects that the dimension of H^n drops, i.e.,

$$\dim H^n(\overline{V}_1,\mathbf{Q}_q)<\dim H^n(\overline{V}_0,\mathbf{Q}_q)$$

and

$$\dim H^{n+1}(U_1,\mathbf{Q}_q)<\dim H^{n+1}(U_0,\mathbf{Q}_q).$$

However,

$$Fr(\lambda):=\lim_{\mu\to\lambda} A(\mu)^{-1}\,\mathrm{Frob}^*_{q,0}\,A(\mu^q)$$

defines for λ close to 1 an operator on a vector space W of dimension equal to the dimension of $H^n(U_0,\mathbf{Q}_q)$, which might have poles at $\lambda=1$.

At the same time one expects that the singularities of the Picard-Fuchs equation are related to the singularities in the family V_λ, so Fr might have singularities at $\lambda=1$. This suggests that $Fr^{-1}(1)=\lim_{\lambda\to 1}Fr^{-1}(\lambda)$ has a kernel K, that W has a decomposition $W_1\oplus K$, which is Fr^{-1}-invariant and $\dim W_1=\dim H^{n+1}(U_1,\mathbf{Q}_q)$. When this happens then it would be likely that $W_1\cong H^{n+1}(U_1,\mathbf{Q}_q)$ as vector space with Frobenius action, and the trace of Frob^{*-1} on $H^{n+1}(U_1,\mathbf{Q}_q)$ would equal the trace of Frob^{*-1} on W.

Unfortunately, one can construct examples such that the Picard-Fuchs equation is 'less' singular than the drop in the dimension of H^{n+1} predicts, i.e., $\dim W_1>\dim H^{n+1}(U_1,\mathbf{Q}_q)$. This is due to the fact that the family V_λ over the punctured disc $\{\lambda\colon 0<\mid\lambda-1\mid<1\}$, considered as a family of abstract varieties, can be completed in different ways. Since the Picard-Fuchs equation depends only on the family V_λ considered in a neighborhood of $\lambda=0$ all these families have the same Picard-Fuchs equation and therefore the same operator $A(\lambda)$. However, the dimension of $H^n(V_1,\mathbf{Q}_q)$ depends on how one completes the family V_λ. The number of points $\#\overline{V}_1(\mathbf{F}_q)$ depends also on the way one completes the family $\overline{V}_\lambda$. So the main obstruction to extend the deformation algorithm is:

Obstruction 8. If $\overline{V}$ is singular then the deformation algorithm might calculate $\#\overline{V}'(\mathbf{F}_p)$ for a variety $\overline{V}'$ different from $\overline{V}$.

Remark 9 (Choice of basis)**.** The differential equation used in the deformation method depends on the choice of the basis for $H^{n+1}(U_\lambda, \mathbf{Q}_q)$.

The basis used in [3] is a "constant basis. I.e., a basis of differential forms $\omega_{i,\lambda} := G_i/F_\lambda\Omega$, $i = 1, \dots, h^{n+1}(u_0)$ such that the $\omega_{i,0}$ form a basis for $H^{n+1}(U_0, \mathbf{Q}_q)$ and the $\omega_{i,1}$ form a basis for $H^{n+1}(U_0, \mathbf{Q}_q)$. These forms are constant, in the sense that the G_i are constant as a function in λ.

To test whether a set of forms are a basis for $H^{n+1}(U_0, \mathbf{Q}_q)$ is rather straight forward, using the form of the equation. To test whether a set of forms is a basis for $H^{n+1}(U_1, \mathbf{Q}_q)$ we are aware of two methods:

- In the smooth cases one can relate the basis for H^{n+1} with graded pieces of the Jacobian ring of the defining equation of F_1. To find a basis one can calculate a the Groebner basis of the Jacobian ideal of F_1. This identification with the Jacobian ring relies on the fact that a certain spectral sequence degenerates at E_2 in the smooth case. In the singular case this spectral sequence cannot degenerate at E_2. So this method does not work in the singular case.
- A second method is using the differential equation, i.e., in the smooth case the differential equation has a pole at $\lambda = 1$ if and only if the forms $\omega_{i,1}$ do not form a basis. However in the singular case the differential equation might have singularities due to singular points on X_1.

2.3. **Direct method, smooth case.** The idea used in [1] is easier to explain. Suppose for the moment that $\overline{V}$ is a smooth hypersurface. Then $\#\overline{V}(\mathbf{F}_q)$ can be calculated by determining the action of Frobenius on the rigid cohomology group

$$H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q).$$

Fix a lift V of $\overline{V}$ to $\mathbf{Z}_q$, let U be the complement of V. A theorem of Baldassarri-Chiarellotto [2] states that

$$H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q) \cong H^{n+1}_{\mathrm{dR}}(U, \mathbf{Q}_q).$$

Due to work of Griffiths [4], the latter group $H^{n+1}_{\mathrm{dR}}(U, \mathbf{Q}_q)$ is very well understood:

Let $\Omega := \prod_i X_i \sum_j (-1)^j \frac{dX_0}{X_0} \wedge \dots \wedge \widehat{\frac{dX_j}{X_j}} \wedge \dots \wedge \frac{dX_n}{X_n}$. Let $F = 0$ be an equation defining V, such that $\overline{F} \equiv F \bmod \pi$. Then $H^{n+1}_{\mathrm{dR}}(U, \mathbf{Q}_q)$ consists of

$$\Omega^{n+1}(U) := \left\{ \frac{G}{F^t}\Omega \colon t \in \mathbf{Z}, t > 0, \deg(G) = t\deg(F) - n - 1 \right\}$$

modulo the following relations

$$\frac{(t-1)GF_{X_i}}{F^t}\Omega = \frac{G_{X_i}}{F^{t-1}}\Omega \tag{1}$$

where X_i is a coordinate on $\mathbf{P}$ and the subscript X_i denotes the partial derivative with respect to X_i. In particular, one can show that H^{n+1}_{dR} can be generated by forms with $t \le n+1$. Let $\{\omega_j\}$ be a basis of $H^{n+1}_{\mathrm{dR}}(U, \mathbf{Q}_q)$ (which in turn is a basis for $H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$).

Let $\overline{A}$ be the coordinate ring of $\overline{U}$. Let A be the coordinate ring of U. Fix a representation

$$A = \mathbf{Q}_q[Y_0, \dots, Y_m]/(G_1, \dots, G_k).$$

Definition 10. Set

$$A^\dagger = \frac{\{H \in \mathbf{Q}_q[[Y_0, \dots, Y_m]] \colon \text{the radius of convergence of } H \text{ is at least } r > 1\}}{(G_1, \dots, G_k)}.$$

Then $A^\dagger$ is called an *overconvergent completion* (or weak completion) of A.

An overconvergent completion depends on the representation of A. However, the results mentioned below are independent of the chosen representation of A. (See, e.g., [10].)

Fix a lift of Frobenius $\mathrm{Frob}_q^* : A^\dagger \to A^\dagger$. To calculate the Frobenius action on $H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$ we need to express

$$\mathrm{Frob}_q^*(\omega_j) = \left(\sum_{i=0}^{\infty} \frac{G_i}{F^i} \right) \Omega \tag{2}$$

in terms of the basis $\{\omega_i\}$.

For our purposes it suffices to know the characteristic polynomial of Frobenius up to a certain p-adic precision. For this reason we can truncate the series (2) after N steps, where N can be computed in terms of p, n and d. This truncated series gives a class in H^{n+1}_{dR}. We can use the expression (1) to reduce the pole order, and hence to write $\mathrm{Frob}_q^*(\omega_j)$ in the form $\sum_i a_{i,j}\omega_i$. This suffices to calculate the characteristic polynomial of Frobenius.

2.4. **Direct method, singular case.** If $\overline{V}$ is singular then several of the above ideas fail to work. It turns out that a combination of these obstructions yields an outline for an algorithm that works for singular varieties.

The following three steps fail in the singular case:

(1) First of all, the comparison theorem of Baldassarri-Chiarellotto does not hold. Instead one only has a natural map
$$H^{n+1}_{\mathrm{dR}}(U, \mathbf{Q}_q) \to H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q).$$
One of the problems here is that the dimension of the left hand side depends on the choice of the lift U, whereas the dimension of the right hand side is independent of the dimension of the lift, so there is no hope that an arbitrary choice of a lift will work.

(2) To reduce expression (2) one needs to be able to write polynomials G of large degree as $\sum H_i F_{X_i}$ for some H_i. This is possible, since the Jacobian ring of F
$$R = \mathbf{Q}_q[X_0, \dots, X_{n-1}]/(F_{X_0}, \dots, F_{X_{n+1}})$$
is a finite-dimensional $\mathbf{Q}_q$-vector space, provided that F is smooth. If F is singular then R is infinite-dimensional.

(3) If one chooses the lift F of $\overline{F}$ such that $F = 0$ is smooth, then the reduction of
$$\lim_{N \to \infty} \left(\sum_{i=0}^{N} \frac{G_i}{F^i} \right) \Omega$$
might diverge.

The following remark gives an algebro-geometric explanation for these phenomena.

Remark 11. The second point is the most fundamental obstruction. One can filter Ω_U^k, the k-form on U, by the order of the pole along V. The filtered complex $\Omega_U^\bullet$ yields a spectral sequence $E_k^{i,j}$ abutting to $H_{\mathrm{dR}}^{i+j}(U,\mathbf{Q}_q)$. The relations (1) on page 5 describe $E_2^{i,n+1-j}$: Let R be the Jacobian ring of F. Since the Jacobian ideal is homogeneous we can grade elements of R by their degree. Then $\oplus_i E_2^{i,n+i-1} = \oplus_p R_{id-n-2}$.

If V is smooth then this spectral sequence degenerates at E_2, hence it suffices to calculate $H_{\mathrm{dR}}^{n+1}(U,\mathbf{Q}_q)$. If V is singular then this spectral sequence *cannot* degenerate at E_2 but degenerates at a higher step. One could try to adjust the algorithm [1] by trying to take an "equisingular" lift, and try to identify the extra relations one needs to obtain $H^{n+1}(U,\mathbf{Q}_q)$ as a quotient of Ω_U^n. Unfortunately, such a lift might not exist and, except for a few cases, it is not clear at all which relations one needs to add,.

We give a procedure to determine the kernel of $H_{\mathrm{dR}}^{n+1}(U,\mathbf{Q}_q) \to H_{\mathrm{rig}}^{n+1}(\overline{U},\mathbf{Q}_q)$ under some restrictions on the singularities of $\overline{V}$. In practice (e.g., the case of a surface with $A-D-E$ singularities in sufficiently large characteristic) it turns out that this kernel has the same size as the difference between $\dim H_{\mathrm{dR}}^{n+1}(U,\mathbf{Q}_q)$ and $\dim H_{\mathrm{rig}}^{n+1}(\overline{U},\mathbf{Q}_q)$.

For simplicity, let us assume we have a sequence $F_k \in \mathbf{Z}_q[X_0,\dots,X_{n+1}]$, such that

- $F_k \equiv F_{k-1} \bmod \pi^{k-1}$,
- the singular locus of $F_k \bmod \pi^k$ coincides with a lift of the singular locus of $\overline{F}$,
- $F_k \bmod p^{k+1}$ is smooth.

I.e., we have a sequences of polynomials F_k, defining smooth hypersurfaces, lifting the singular locus modulo π^k. In general such a sequence of polynomials might not exist.

Since the Jacobian ideal of F_k is finite-dimensional we can try to mimic [1], i.e., we make a power series expansion $\mathrm{Frob}_q(\omega_j)$, truncate this after N steps and try to reduce this form in $H_{\mathrm{dR}}^{n+1}(U_k,\mathbf{Q}_q)$.

It turns out that if $N\to\infty$ or $k\to\infty$ then the p-adic absolute value of some of the coefficients of the reductions tend to increase. This is due to the fact that the Jacobian ideal of $\overline{F}$ is infinite-dimensional:

Example 12. Suppose we have a form

$$\frac{G}{F_k^t}\Omega.$$

After dividing or multiplying by π we may assume that $G\in\mathbf{Z}_q[X_0,\dots,X_n]$ and $G\not\equiv 0 \bmod \pi$.

In order to reduce the pole order we need to write G as $\sum H_i F_{k,X_i}$. Let P be a lift of a point in the singular locus. Suppose G is general, i.e., $G(P)\not\equiv 0 \bmod \pi$. Now,

$$G(P) \equiv \sum H_i(P) F_{k,X_i}(P) \equiv 0 \bmod \pi^k,$$

hence some of the coefficients in H_i need to have negative p-adic valuation. In practice this means that after each reduction step the p-adic valuation of the coefficient decreases rapidly.

Since $\mathrm{Frob}_p^*(\omega_j) = \sum \frac{G_t}{F_k^t}\Omega$ is an overconvergent power series one has that the p-adic valuation of the coefficient of G_t increases when t increases. However, the minimum of the valuation of the coefficients of G_t is around t/p. This turns out to be insufficient to compensate for the high power of p in the denominator obtained by reducing the pole order. In the next section we give an example where the inverse of Frobenius has an eigenvalue with very small p-adic absolute value, hence Frobenius has an eigenvalue with large absolute value.

Next, the main idea is to consider the action of Frob_q^{*-1}. We could do this by considering $\mathrm{Frob}_q^*(\omega)$ and truncating at pole order N and then invert the obtained operator. This operator has several eigenvalues with small q-adic absolute value, i.e., very positive q-adic valuation. At the same time we know that the eigenvalues of $q^{n+1}\,\mathrm{Frob}_q^{*-1}$ on $H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$ are algebraic integers with complex absolute value at most q^{n+1}. In particular, the q-adic valuation of such an eigenvalue is between 0 and $n+1$. Therefore, all eigenvalues that have q-adic valuation bigger than $n+1$ cannot be eigenvalues of Frobenius on $H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$, hence the corresponding eigenvectors lie in the kernel of $H^{n+1}_{\mathrm{dR}}(U_k, \mathbf{Q}_q) \to H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$. This idea seems to be very hard to use in practice, since by inverting the approximation of the operator Frob_q^* one encounters severe problems in obtaining the necessary p-adic precision.

Instead we study a left-inverse of Frob_q^*:

Notation 13. Let $\psi : A^\dagger \to A^\dagger$, be the $\mathbf{Q}_q$-linear operator defined by

$$\psi\left(\prod X_i^{a_i}\right) = \begin{cases} \prod X_i^{a_i/q} & \text{if } a_i \equiv 0 \bmod q \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

and $\psi(\Omega/\prod X_i) = \Omega/(p^{n+1}\prod X_i)$.

Since Frob_q^* on $H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$ is invertible and $\psi \circ \mathrm{Frob}_q^*$ is the identity, one has that ψ on $H^{n+1}_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$ is the inverse of Frob_q^*.

Remark 14. This operator ψ behaves much better than Frob_q^*. Assume for simplicity that $n < q$. We need only consider forms with pole order $t \leq q$

$$\begin{aligned} (3)\qquad \psi\left(\frac{G}{F^t}\Omega\right) &= \psi\left(\frac{F^{q-t}G\prod X_k}{F^q}\frac{\Omega}{\prod X_k}\right) \\ (4)\qquad &= \psi\left(\sum_i \frac{F^{q-t}G\prod X_k \Delta^i}{F(X_0^q,\dots,X_{n+1}^q)^{i+1}}\frac{\Omega}{\prod X_k}\right) \\ (5)\qquad &= \left(\sum_i \frac{\psi(F^{q-t}G\prod X_k\Delta^i)}{F(X_0,\dots,X_{n+1})^{i+1}}\right)\frac{\Omega}{p^{n+1}\prod X_k} \end{aligned}$$

with $\Delta = F(X_0^q,\dots,X_{n+1}^q) - F(X_0,\dots,X_{n+1})^q$.

Abbott-Kedlaya-Roe reduce the form

$$(6)\qquad \mathrm{Frob}_q^*\left(\frac{G}{F^t}\Omega\right) = \sum_i \binom{t+i-1}{i}(-\Delta)^i\frac{\mathrm{Frob}^*(G\prod X_k)}{F^{qi+t}}p^{n+1}\frac{\Omega}{\prod X_k}.$$

Very roughly the convergence of power series in (5) is q times faster than in (6). If we reduce the pole order in (5) then the valuation of Δ^i is sufficiently high to compensate for the high power of π one gets in the denominator by reducing.

We would like to remark that in the case of a *smooth* hypersurface one can also use ψ rather than Frob_q^*. By using ψ one can lower the necessary pole order roughly by a factor q.

3. Examples

We apply the above observations to one particular example.

Let q be an odd prime power. In this section we consider the surface S_1 : $X^2+Y^2+Z^2=0$ in $\mathbf{P}^3_{\mathbf{F}_q}$. The surface S_1 is a cone over a conic in $\mathbf{P}^2$. This implies that S_1 has an A_1-singularity at $P:=[1:0:0:0]$. Let $\tilde{S}_1$ be the blow-up of S_1 at P. Then $\tilde{S}_1$ is a ruled surface over $\mathbf{P}^1$. In particular it has the following Betti numbers:

$$h^0(\tilde{S}_1)=h^4(\tilde{S}_1)=1, h^2(\tilde{S}_1)=2$$

and all other Betti numbers vanish. From this it follows that $h^2(S_1)=1$ and $h^i(S_1)=h^i(\tilde{S}_1)$ for $i\neq 2$.

One can easily see that $\#V(\mathbf{F}_q)=q^2+q+1$. We will show that a slight modification of Lauder's (or Gerkmann's) method yield the output q^2+2q+1 or q^2+q+1 (depending on some choices that seem quite arbitrary), whereas a slight modification of Abbott-Kedlaya-Roe gives the correct answer $\#V(\mathbf{F}_q)=q^2+q+1$.

3.1. **Deformation method.** Consider the family

$$\overline{V}_\lambda : (1-\lambda)W^2+X^2+Y^2+Z^2=0.$$

Let $\overline{U}_\lambda$ be the complement of $\overline{V}_\lambda$. The methods of Lauder and Gerkmann require to calculate the Frobenius action on $H^3(U_0,\mathbf{Q}_q)$. It is easy to see that Frobenius acts as multiplication by q^2 on this one-dimensional vector space.

Secondly, one defines an operator $A(\lambda): H^3(U_\lambda)\to H^3(U_0)$.

For this one can proceed as follows: following [3] we need first to calculate the reduction of

$$\frac{-2\frac{\partial F_\lambda}{\partial\lambda}}{F_\lambda^3}\Omega=\frac{2W^2}{F_\lambda^3}\Omega$$

Since $\partial F/\partial w=2(1-\lambda)W$, we obtain that this reduction equals

$$\frac{1}{2(1-\lambda)}\frac{1}{F_\lambda^2}\Omega$$

The associated differential equation

$$A'(\lambda)=\frac{-1}{2(\lambda-1)}A(\lambda)$$

has solution $A(\lambda)=(\lambda-1)^{-1/2}$. This implies that $Fr(\lambda)$ is the analytic continuation of $q^2\frac{(1-\lambda)^{1/2}}{(1-\lambda^q)^{1/2}}=q^2(\sum_{i=0}^{q-1}\lambda^i)^{-1/2}$. Squaring yields $Fr(\lambda)^2=q^4$ if $\lambda\neq 1$ and $Fr(1)^2=q^3$.

One can proceed as follows: $q^3\,\mathrm{Frob}^{-1}(1)$ acting on the rigid cohomology group $H^2(\overline{V}_1,\mathbf{Q}_q)$ has eigenvalues with complex absolute value at most q (since $\overline{V}_1$ is singular, we have this weak form of the Riemann hypothesis, i.e., Frob might have eigenvalues with complex absolute value smaller than q).

From this observation we get that the eigenvalue $q^{3/2}$ of Frob(1) has too big complex absolute value, hence we should ignore this. By doing so we get the correct answer, i.e., this modification of the deformation method yields $\#\overline{V}(\mathbf{F}_q) = q^2+q+1$.

Remark 15. Note that in the case of the direct method we will again argue that certain eigenvalues can not come from Frobenius, since its absolute value does not satisfy the Riemann Hypothesis. However, in that case the complex absolute value of the eigenvalues that we need to ignore is much bigger, i.e., by adjusting the lifting the complex absolute value of this eigenvalue becomes arbitrarily large.

If we consider more complicated singularities it might happen that the eigenvalues corresponding to eigenvectors that should be ignored, are much closer to q. Therefore, it seems harder to find good criteria to filter out these eigenvalues.

Remark 16. The differential equation associated to above family of varieties had a singularity at $\lambda = 1$. The singularity of this differential equation can be explained by the fact that S_1 is singular. As argued before one can also obtain singularities by taking a bad choice of basis.

One can show that the residue at a singularity due to bad choice of basis is integral. In the above example we had residue $-1/2$, hence a non-integral residue. This singularity has to come from the fact that our surface is singular. We will now give a different example, where the residue is integral, but is caused by the singularity rather than a bad choice of basis.

Example 17. Gerkmann's version of the deformation method allows us to consider the following family:

$$\overline{V}'_\lambda : (1-\lambda)^2 W^2 + X^2 + Y^2 + Z^2 = 0.$$

In this case the associated differential equation is

$$A'(\lambda) = \frac{-1}{(\lambda-1)} A(\lambda).$$

The solution with $A(0) = 1$ is $A(\lambda) = (\lambda - 1)^{-1}$. This implies that $\mathrm{Frob}(\lambda) = q^2 (\sum_{i=0}^{q-1} \lambda^i)^{-1}$. Hence $\mathrm{Frob}(\lambda) = q^2$ for $\lambda \neq 1$ and $\mathrm{Frob}(\lambda) = q$ for $\lambda = 1$.

However, we can take a different basis for $H^3(U_\lambda)$, namely $\frac{1}{\lambda-1}\frac{1}{F_\lambda^2}\Omega$. If we take this basis our differential equation becomes

$$A'(\lambda) = 0,$$

hence $\mathrm{Frob}(\lambda) = 1$ for all λ. The output of the deformation method would be in this case that $\#\overline{V}'_1 = q^2 + 2q + 1$, which is wrong, but is the number of point of $\tilde{S}_1$.

Remark 18. The question here is if there is an algorithmic method to determine the correct basis. This seems rather hard for more complicated varieties (such as complete intersections). In the non-singular case, the method used to check that a certain set of generators is a basis relies on the fact that a certain spectral sequence degenerates at E_2. It is well-known that this spectral does not degenerate at E_2 in the singular case, and it is hard to determine at which step this spectral sequence degenerates.

As argued in the previous section, the singularities of the differential equation give information on whether a certain set is a basis or not. Unfortunately, this information is not conclusive.

Remark 19. By changing the basis for $H^{n+1}(U, \mathbf{Q}_q)$ one can introduce extra singularities in the differential equation (and also remove some singularities). Suppose for simplicity that we have a differential equation of the form $A'(\lambda) = (\frac{a}{\lambda-1} + h.o.t.)A(\lambda)$. Then by multiplying a basis vector with $(\lambda-1)^m$ forces a shift of a by m, hence by changing the basis on the cohomology we can change a residue by an integer. So, if the residue were an integer then by changing our basis we would resolve one of the singularities of the differential equation.

Of course, in such an easy example one can control these issues, but in slightly more complicated families of varieties, such as degree 4 surfaces (not to speak of e.g. complete intersections), one can be in the situation that some of the singularities of the differential equation with integral residue are due to a bad choice of basis and should be resolved, whereas other singularities with integral residue are due to the fact that we have a singular hypersurface and should not be resolved.

Remark 20. We can explain this problem of choice of basis in a different way: We construct now a family Y_λ with $V_\lambda = Y_\lambda$ for $\lambda \neq 1$ and such that the deformation method calculates the zeta function of Y_1.

It is known that over an algebraically closed field one can construct a family of vector bundles $\mathcal{V}_\lambda$ on $\mathbf{P}^1$ with

$$\mathcal{V}_\lambda = \begin{cases} \mathcal{O} \oplus \mathcal{O} & \text{if } \lambda \neq 1 \\ \mathcal{O}(-1) \oplus \mathcal{O}(1) & \text{if } \lambda = 1. \end{cases}$$

This yields a family of projective bundles $Y_\lambda := \mathbf{P}(\mathcal{V}_\lambda)$. For $\lambda \neq 1$ we have that $\mathbf{P}(\mathcal{V}_\lambda) \cong \mathbf{P}^1 \times \mathbf{P}^1$, whereas for $\lambda = 1$ we have that $\mathbf{P}(\mathcal{V}_\lambda)$ is isomorphic to the Hirzebruch surface F_2.

We can map this family in to $\mathbf{P}^2$ by fixing a degree 2 line bundle $\mathcal{L}_\lambda$ on Y_λ. On $\mathbf{P}^1 \times \mathbf{P}^1$, let f_1 be a fiber of the first projection, f_2 be a fiber of the second projection, then $\mathcal{L}_\lambda := \mathcal{O}(f_1 + f_2)$ has degree 2 and $\mathcal{L}_\lambda$ is ample. Actually, the family of line bundles $\mathcal{L}_\lambda$ for $\lambda \neq 1$ is a line bundle on the 3-dimensional variety $\cup_{\lambda,\lambda\neq 1} Y_\lambda$. We can extend $\mathcal{L}$ to all of $\cup_\lambda Y_\lambda$: On $Y_1 \cong F_2$ there is only one ruling, let f be a fiber of this ruling, let z be the exceptional section, i.e., the self-intersection (z, z) equals -2 and $(z, f) = 1$. Then $\mathcal{L} \mid_{Y_1} = \mathcal{O}(2f + z)$. This line bundle is of degree 2, but not ample, since $(2f + z, z) = 0$. If we use $\mathcal{L}$ to map the family Y_λ in $\mathbf{P}^3$ then we obtain a family of surfaces V_λ in $\mathbf{P}^3$ such that $V_\lambda \cong Y_\lambda$ for $\lambda \neq 1$ and Y_1 is a resolution of singularities of V_1. In other words, the map $Y_1 \to V_1$ contracts z.

The deformation method (second family, second choice of basis) calculates $\#Y_1(\mathbf{F}_q)$ rather than $\#V_1(\mathbf{F}_q)$.

Remark 21. It would be interesting to have examples with integral residues where F_λ is linear in λ. We are not aware of such examples, but we have not identified obstructions for their existence.

3.2. **Direct method.** For the direct method we only need to consider $F = X^2 + Y^2 + Z^2 = 0$. To simplify the exposition, assume that $q = p$ a prime number.

Let $F_k := X^2 + Y^2 + Z^2 + p^k W^2$. Then $F_k = 0$ defines a smooth hypersurface, such that its reduction modulo p^k is singular. The cohomology group $H^{n+1}_{\mathrm{dR}}(U_k, \mathbf{Q}_p)$ is one-dimensional and it is generated by

$$\frac{1}{F_k^2}\Omega.$$

From (5) it follows that

$$\psi\left(\frac{1}{F_k^2}\Omega\right) \quad = \quad \sum_i \frac{\psi(XYZWF_k^{p-2}\Delta^i)}{F_k^{i+1}} \frac{\Omega}{p^3XYZW}$$

If we truncate this expression at pole order N we get

$$\sum_{j=0}^{N-1}(-1)^j\left(\sum_{i=j}^{N-1}\binom{i}{j}\right)\frac{\psi(XYZWF_k^{(j+1)p-2})}{F_k^{j+1}}\frac{\Omega}{p^3XYZW}.$$

From the definition of ψ it follows that we only have to consider monomials in $XYZWF_k^{(j+1)p-2}$ such that all the exponents are divisible by p. This observation combined with expanding $XYZWF_k^{(j+1)p-2}$ yields:

Lemma 22. *Set $T_j = \{(t_1,t_2,t_3,t_4)\colon t_1,t_2,t_3,t_4 \geq 0, \sum t_i = j-1\}$. For t_1, t_2, t_3, t_4 in T_j set*

$$B(t_1,t_2,t_3,t_4) := \binom{(j+1)p-2}{\frac{p-1}{2}+t_1p \quad \frac{p-1}{2}+t_2p \quad \frac{p-1}{2}+t_3p \quad \frac{p-1}{2}+t_4p}.$$

Then $\psi(XYZWF_k^{(j+1)p-2})$ equals

$$\sum_{(t_1,t_2,t_3,t_4)\in T_j} B(t_1,t_2,t_3,t_4)p^{k(\frac{p-1}{2}+t_4p)}X^{1+2t_1}Y^{1+2t_2}Z^{1+2t_3}W^{1+2t_4}.$$

Denote by $(a)_m$ the Pochhammer symbol $a(a+1)\dots(a+m-1)$. Successively applying (1) yields the following result:

Lemma 23. *The reduction of*

$$\frac{X^{2t_1}Y^{2t_2}Z^{2t_3}W^{2t_4}}{F_k^{t_1+t_2+t_3+t_4+2}}\Omega$$

in $H^{n+1}_{\mathrm{dR}}(U,\mathbf{Q}_q)$ equals

$$\frac{(1/2)_{t_1}(1/2)_{t_2}(1/2)_{t_3}(1/2)_{t_4}}{(t_1+t_2+t_3+t_4+1)!p^{kt_4}}\frac{1}{F_k^2}\Omega.$$

Combing the above Lemmas yields:

Lemma 24. *For $j>0$ the reduction of*

$$\frac{\psi(XYZWF^{(j+1)p-2})}{F^{j+1}}\frac{\Omega}{p^3XYZW}$$

in $H^{n+1}_{\mathrm{dR}}(U,\mathbf{Q}_q)$ equals

$$\left(\sum_{(t_1,t_2,t_3,t_4)\in T_j} B(t_1,t_2,t_3,t_4)p^{\frac{1+2t_4}{2}k(p-1)}\frac{(1/2)_{t_1}(1/2)_{t_2}(1/2)_{t_3}(1/2)_{t_4}}{(t_1+t_2+t_3+t_4+1)!}\right)\frac{1}{F_k^2}\frac{\Omega}{p^3}.$$

Lemma 25. *The quantity*

$$\gamma = B(t_1,t_2,t_3,t_4)\frac{(1/2)_{t_1}(1/2)_{t_2}(1/2)_{t_3}(1/2)_{t_4}}{(t_1+t_2+t_3+t_4+1)!}$$

is a p-adic integer.

Proof. Let $\alpha = (p-1)/2$. Note that $B_0 := B(t_1, t_2, t_3, t_4)$ equals

$$\binom{(t_1+t_2+t_3+t_4)p+4\alpha}{t_1p+\alpha}\binom{(t_2+t_3+t_4)p+3\alpha}{t_2p+\alpha}\binom{(t_3+t_4)p+2\alpha}{t_3+\alpha}.$$

It is well-known that the p-adic valuation of $\binom{m}{i}$ equals the number of carries $c(i, m-i)$ (in base p) if one sums i and $m-i$. Hence $v(B_0)$ equals

$$c\left(t_1p+\alpha, (t_2+t_3+t_4)p+3\alpha\right) + c\left(t_2p+\alpha, (t_3+t_4)p+2\alpha\right) + c\left(t_3p+\alpha, t_4p+\alpha\right).$$

We want to compare the valuation of B_0 with the valuation of $\frac{(t_1+t_2+t_3+t_4+1)!}{t_1!t_2!t_3!t_4!}$. Let B_1 denote the latter quantity. One has that B_1 equals

$$\binom{t_1+t_2+t_3+t_4+1}{t_1}\binom{t_2+t_3+t_4+1}{t_2}\binom{t_3+t_4}{t_3}(t_3+t_4+1),$$

hence its valuation $v(B_1)$ equals

$$c(t_1, t_2+t_3+t_4+1) + c(t_2, t_3+t_4+1) + c(t_3, t_4) + v(t_3+t_4+1).$$

It is easy to see that

$$c(t_1p+\alpha, (t_2+t_3+t_4)+3\alpha) = c(t_1, t_2+t_3+t_4+1) \text{ and } c(t_3p+\alpha, t_4p+\alpha) = c(t_3, t_4).$$

Let $m := v(t_3+t_4+1)$. Since $t_3+t_4 \equiv 1 \bmod p$ we can write

$$t_3+t_4 = (p-1)+(p-1)p+\cdots+(p-1)p^{m-1}+\beta_m,$$

with $\beta_m \equiv 0 \bmod p^{m-1}$. Since $\alpha \not\equiv 0 \bmod p$ we get

$$c(t_2p+\alpha, (t_3+t_4+1)p-1) = m + c(t_2p, \beta_m+p^m) = m + c(t_2, t_3+t_4+1),$$

hence $v(B_0) = v(B_1)$.

Since $(1/2)_{t_j}$ is the product of the first t_j odd numbers divided by 2^{t_j}, we get $v((1/2)_{t_j}) \geq v(t_j!)$ and

$$v(\gamma) \geq v\left(\frac{B_0}{B_1}\right) = 0,$$

which shows that γ is a p-adic integer. □

Combining these lemmas shows that the reduction ω_N of $p^3\psi\left(\frac{1}{F_k^2}\Omega\right)$ truncated after N steps satisfies $\omega_N \equiv 0 \bmod p^{k(p-1)/2}$, provided $N > 1$. The eigenvalues of ψp^3 on $H^3_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$ are algebraic integers with complex absolute value at most p^3. If k is such that $k(p-1) \geq 8$ then $\frac{1}{F_k}\Omega$ lies in the kernel of $H^3_{\mathrm{dR}}(U, \mathbf{Q}_q) \to H^3_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$, and the latter group vanishes.

If k is chosen large enough, then (modified) Abbott-Kedlaya-Roe does not see the eigenvalue corresponding to ω_N. Therefore its output is p^2+p+1, which is the correct number of points.

3.3. **Another example.** We did some computer experiments with the cubic surface S defined by $W^3+X^3+Y^3+Z^3+\overline{3}WX^2$ in $\mathbf{F}_5$. This cubic surface has a D_4 singularity.

We applied the modified algorithm of Abbott-Kedlaya-Roe (with ψ rather than Frob_q^*), where we took the naive lift $W^3+X^3+Y^3+Z^3+3WX^2$. Truncating at $N = 3$ revealed that $p^3\psi$ has eigenvalues $p, -p$ and four eigenvalues with valuation at least 2, two of which are only defined over a degree 2 extension of $\mathbf{Q}_p$. One can show that for a surface with $A-D-E$-singularities in 'large' characteristic (where large depends on the type of singularity) the eigenvalues of Frobenius on $H^3_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$ have complex absolute value p, (i.e., the Riemann hypothesis holds for such surfaces). Hence the eigenvectors corresponding to eigenvalues with p-adic valuation at least 2 generate the kernel of $H^3_{\mathrm{dR}}(U, \mathbf{Q}_q) \to H^3_{\mathrm{rig}}(\overline{U}, \mathbf{Q}_q)$. This yields that the zeta function $Z(S,t)$ equals $\big((1-t)(1-5t)(1+5t)(1-5^2t)\big)^{-1}$ and $\#S(\mathbf{F}_5) = 5^2 + 1 = 26$, which are correct.

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Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany