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Arithmetic and Moduli of Elliptic Surfaces

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Introduction

This thesis consists of a study of elliptic curves, and of elliptic surfaces.

An elliptic curve (considered over the complex numbers) is a pair (E, O) where E is a curve, which topologically is a torus, and O is a point on E. One can prove that $E \setminus \{O\}$ can be mapped isomorphically to a curve in $\mathbf{C}^2$ of the form

$$y^2 = x^3 + Ax + B,$$

for some $A, B \in \mathbf{C}$, such that $4A^3 + 27B^2 \neq 0$. Fix a curve C then an elliptic surface with base curve C can be represented by an equation of the above form, replacing A, B by functions C to $\mathbf{P}^1$.

1. Elliptic surfaces

The first five chapters deal with the study of elliptic surfaces. We study only projective surfaces, and this forces the base curve to be a projective curve. In this case either all members of the family are isomorphic or we have to allow certain other (singular) curves in our family. Kodaira gave a list of all possible singular fibers in a "minimal" family. The Kodaira types of the singular fibers give a lot of information about the elliptic surface.

To give an overview of the first five chapters we need to introduce some notions from algebraic geometry.

Let X be an elliptic surface. One introduces a group $NS(X)$, called the Néron-Severi group of X.A *divisor* D of X is a finite formal sum $n_1D_1 + \cdots + n_kD_k$, where the D_i are irreducible curves and the n_i are integers. On $\text{Div}(X)$ one defines the intersection pairing, denoted by $\cdot$. If D and D' are irreducible curves in X intersecting transversely, then $D \cdot D' = \#D \cap D'$. Using linearity one defines this pairing on a large part on $\text{Div}(X) \times \text{Div}(X)$. One extends the intersection pairing to $\text{Div}(X) \times \text{Div}(X)$ as explained in, e.g., [**28**, Section V.1]. One defines an equivalence relation $\equiv$ on $\text{Div}(X)$, called numerical equivalence, by $D \equiv D'$ if and only if $D \cdot C = D' \cdot C$ for all $C \in \text{Div}(X)$. The quotient of $\text{Div}(X)$ by numerical equivalence is the group denoted by $NS(X)$. In fact, in a more general situation one uses a different equivalence relation, called algebraic equivalence, see [**7**, p. 21]. For the surfaces we study in this thesis, algebraic equivalence and numerical equivalence coincide. We have that $NS(X)$ is a finitely generated group, i.e., $NS(X)/NS(X)_{\text{tor}} \cong \mathbf{Z}^{\rho(X)}$ as groups, for some integer $\rho(X)$, called the Picard number of X.

We need to introduce two other groups, called $H^0(X, \Omega_X^2)$ and $H^1(X, \Omega_X^1)$, which are both finitely dimensional $\mathbf{C}$-vector spaces. There exists an injection $NS(X) \otimes \mathbf{C} \to H^1(X, \Omega_X^1)$, which shows that $\rho(X) \leq h^{1,1} := \dim H^1(X, \Omega_X^1)$.

If X is an elliptic surface with base curve $\mathbf{P}^1$, then the dimension of these spaces are related:

$$h^{1,1} = 10(1 + \dim H^0(X, \Omega_X^2)) =: 10n.$$

One splits $NS(X)$ into two parts: let $\pi : X \to \mathbf{P}^1$ be the elliptic family. Then there is the so-called trivial subgroup $T(\pi)$ of $NS(X)$ generated by the classes of a smooth fiber of π, the image of a fixed section σ_0 of π and all the components of fibers not intersecting the image of σ_0. The quotient $NS(X)/T(\pi)$ is generated by the classes of images of sections other then σ_0. This quotient is denoted by $MW(\pi)$ and called the Mordell-Weil group.

The first two chapters are an attempt to understand the following class of elliptic surfaces and to embed them in a broader theory: take $\alpha, \beta, \gamma \in \mathbf{C} - \{0, 1\}$, pairwise distinct, and consider

$$y^2 = x^3 + t^5(t-1)^5(t-\alpha)^5(t-\beta)^5(t-\gamma^5).$$

Then the associated (smooth compact) surface has very remarkable properties. For example one can show that $h^{1,1} = \rho(X)$ for almost all members of this family, and for two general choices of the parameters α, β, γ the associated surfaces are not isomorphic.

The objects of study in Chapter 1 are the elliptic surfaces such that $MW(\pi)$ is finite and $h^{1,1} = \rho(X)$. We call such surfaces extremal elliptic surfaces. The main result is a classification of extremal elliptic surfaces: there are a few families of extremal elliptic surfaces (see the example above) and there are infinitely many isolated examples. We also prove an technical result concerning the so-called infinitesimal Torelli property of elliptic surfaces. For details see Corollary 1.4.11 and the Introduction of Chapter 1.

In Chapter 2 we consider the following problem. Fix an integer $n \geq 2$. Let $\mathcal{M}_n$ be the space parameterizing all elliptic surfaces $\pi : X \to \mathbf{P}^1$ with $\dim H^0(X, \Omega^2_X) + 1 = n$. Then we define the Noether-Lefschetz locus $NL_r \subset \mathcal{M}_n$ as the subspace

$$NL_r := \{\pi : X \to \mathbf{P}^1 \in \mathcal{M}_n \mid \rho(X) \geq r\}.$$

It is known for a long time that $NL_2 = \mathcal{M}_n$ and $\dim \mathcal{M}_n = 10n - 2$. Cox [**18**] proved $\dim NL_3 = 10n - 3$. We prove that

$$\dim NL_r \geq 10n - r$$

for $2 \leq r \leq 10n$, and that equality holds for $2 \leq r \leq 4n + 3$. For several values of $r \geq 8n$ we know that the bound $\dim NL_r \geq 10n - r$ is *not* sharp.

In Chapters 3 and 4 we focus $MW(\pi)$. In Chapter 3 we study six families of elliptic surfaces. Kuwata [**41**] determined for all members of this family the rank of the groups $MW(\pi)$.

For several of the six families we manage to write down explicit sections which generate a subgroup of finite index of $MW(\pi)$. We use this explicit knowledge to write down elliptic surfaces with Mordell-Weil rank 8 over $\mathbf{Q}$, and we believe that a more thorough study of these surfaces may produce examples with higher rank.

In Chapter 4 we present a three-dimensional family of elliptic $K3$ surfaces such that a general member of this family has Mordell-Weil rank 15. An elliptic $K3$ surface is an elliptic surface for which $n = 2$, i.e., $\dim H^0(X, \Omega^2_X) = 1$ and $C \cong \mathbf{P}^1$. Such a surface satisfies $0 \leq \operatorname{rank} MW(\pi) \leq 18$. Kuwata [**41**] gave examples of elliptic $K3$ surfaces of Mordell-Weil rank $0, \dots, 14$ and $16, 17, 18$. Hence this chapter completes the program of explicitly writing down examples of $K3$ surfaces for every $0 \leq r \leq 18$.

In most cases the underlying surface X of an elliptic surface $\pi : X \to \mathbf{P}^1$ has a unique structure (up to isomorphism) as an elliptic surface, i.e., there is only one morphism $\pi : X \to \mathbf{P}^1$, such that the fibers of π are elliptic curves. If X is a surface admitting more structures as a family of elliptic curves, then one can show that X is a $K3$ surface. In

Chapter 5 we classify all structures as families of elliptic curves on a class of $K3$ surfaces. This class can be described as follows. Fix six lines in $\mathbf{P}^2$ in general position. Then one can make a surface X, such that $\psi : X \to \mathbf{P}^2$ is two-to-one, except for points in the inverse image of the six lines, there the map ψ is one-to-one. The surface X is singular. The class of surfaces in Chapter 5 are the desingularizations of such surfaces X.

2. Tate-Shafarevich groups

In Chapter 6 a topic from the arithmetic of elliptic curves is discussed. To explain this, we first recall some general results on elliptic curves.

Let K be a number field and E/K an elliptic curve. Then the set of K-valued points $E(K)$ is a finitely generated abelian group (see [**72**, Theorem VIII.4.1].) Hence it makes sense to consider the following problem: given E and K, describe the torsion part $E(K)_{\text{tor}}$ and determine the rank of $E(K)$. Determining the torsion part is relatively easy (compare [**72**, Section VII.3]). A strategy to obtain information on the rank is the following. Fix a prime number p. From the fact that $E(K)$ is a finitely generated group, it follows that the dimension of the $\mathbf{F}_p$-vector space $E(K)/pE(K)$ has dimension equal to the dimension of the p-torsion $E(K)[p]$ plus the rank of $E(K)$. In order to determine the rank of $E(K)$ it suffices to compute $\dim E(K)/pE(K)$.

Consider all genus 1 curves C/K, such that over a finite extension of K, the curves C and E are isomorphic. Consider the set WC (the so-called Weil-Chatêlet group) of pairs of isomorphism classes (C, μ) such that μ defines a simply transitive group action of E on C. (This last point is merely technical.)

One can define an injective map $E(K)/pE(K)$ into a certain group $H \subset WC$ (for the specialist $H = H^1(K, E[p])$) and to each element $\xi \in H$, we can associate a genus 1 curve C_ξ, which is isomorphic to E over some finite extension of K. The image of $E(K)/pE(K)$ consist of all (C_ξ, μ_ξ) such that $C_\xi(K) \neq \emptyset$. In general, it is very hard to determine whether $C_\xi(K)$ is empty or not. For a place (prime) $\mathfrak{p}$ of K denote $K_\mathfrak{p}$ the completion with respect to $\mathfrak{p}$. Define

$$S^p(E/K) := \{\xi \in H \colon C_\xi(K_\mathfrak{p}) \neq \emptyset \text{ for all places } \mathfrak{p} \text{ of } K\}.$$

Obviously, the vector space $E(K)/pE(K)$ can be injected into $S^p(E/K)$, yielding

$$E(K)/pE(K) \hookrightarrow S^p(E/K) \subset H \subset WC.$$

(A technical correct definition involves Galois cohomology, which we do not discuss in this Introduction. For more details see Section 6.3.)

Given E and p, one can easily determine a finite subset H' of H with the property $S^p(E/K)$ is contained in H'. One can show that for all but finitely many primes $\mathfrak{p}$ the set $C_\xi(K_\mathfrak{p})$ is not empty for all $\xi \in H'$. To conclude that $C_\xi(K_\mathfrak{p})$ is not empty for all $\mathfrak{p}$, one needs to check only the primes $\mathfrak{p}$ such that the reduction $C_\xi \bmod \mathfrak{p}$ is singular, and the primes $\mathfrak{p}$ such that $\mathfrak{p}$ divides p. Checking whether $C_\xi(K_\mathfrak{p})$ is empty or not in these finitely many cases, can be done using Hensel's Lemma in a finite amount of time.

The above considerations imply that one can find in a finite amount of time the set $S^p(E/K)$. (We cheated a bit, if $p > 5$ then the equations for the C_ξ are not known, but for small degrees one can find them for example in [**72**, Chapter X].)

Consider the quotient of $S^p(E/K)$ by the image of $E(K)/pE(K)$. We call this the p-part of the Tate-Shafarevich group of E/K, denoted by $Ш(E/K)[p]$. This group measures how large the defect of $S^p(E/K)$ is, if one wants to compute $E(K)/pE(K)$, i.e., we have a short exact sequence

$$0 \to E(K)/pE(K) \to S^p(E/K) \to Ш(E/K)[p] \to 0.$$

It is conjectured that the rank of $E(\mathbf{Q})$ is arbitrarily large if one considers all elliptic curves $E/\mathbf{Q}$. If this were true then for all p the number of elements of $S^p(E/\mathbf{Q})$ is arbitrarily large. This is actually proven for $p = 2, 3$ (see [**11**], [**13**], [**33**] and [**40**]), $p = 5, 7$ (see [**23**], [**33**]) and $p = 13$ (see [**33**]) Hence for these prime numbers at least one of $\operatorname{rank} E(\mathbf{Q}), \#Ш(E/\mathbf{Q})[p]$ is arbitrarily large. For $p = 2, 3$ and 5, some of the above mentioned papers actually prove that $Ш(E/\mathbf{Q})[p]$ is arbitrarily large.

In the paper [**38**] it is proven that $\#S^p(E/K)$ is unbounded if one considers pairs (E, K) with K a number field of degree at most $(p + 13)/12$, and E/K an elliptic curve.

The aim of Chapter 6 is to produce for every prime number p examples of pairs (E, K) with E an elliptic curve and K a number field of degree bounded polynomially in p such that $Ш(E/K)[p]$ is unbounded. (Theorem 6.1.1)

In a certain sense, this is a negative result, it shows that the above mentioned strategy to determine $\operatorname{rank} E(K)$ might fail heavily. One can interpret this result much more positively. First of all it takes a lot of effort to produce examples of elliptic curves E/K such that one *can prove* that $Ш(E/K)$ is not trivial (here we consider *all* prime numbers p). Secondly, it gives a little information in which cases one might except large Tate-Shafarevich groups.

The strategy of our prove is a bit involved. We start in Section 6.3 by considering how one proves that $S^p(E/K)$ is very large. Here we assume that there exists a morphism $\varphi : E \to E'$ of degree p defined over K. Under this assumption one can give an upper bound on $\#S^p(E'/K)$ in terms of the geometry of the reduction $E \bmod \mathfrak{p}$, for primes $\mathfrak{p}$ such that $E \bmod \mathfrak{p}$ is singular. Suppose E admits a second morphism $\psi : E \to E''$ with the property that exists a point $Q \in E(K)[p]$ with $\varphi(Q) \neq O$, with O the neutral element of $E''(K)$. (This condition implies that ψ and φ are different morphisms.) One can use this morphism ψ to bound the number of elements of $S^p(E''/K)$ (and hence bound the rank of $E''(K)$). Since the rank of $E(K), E'(K)$ and $E''(K)$ are the same, we obtain a lower bound for the number of elements in $Ш(E''/K)[p]$. (Contrary to the rank, the number of elements in $Ш(E''/K)[p]$ and $Ш(E/K)[p]$ might differ a lot.)

To produce examples with large Tate-Shafarevich groups we will combine both assumptions. It turns out that this is possible in the case that E/K comes from a K-rational point of the so-called modular curve $X(p)$. One now translates both assumptions in terms of the reductions of the point $P \in X(p)(K)$. Then it remains to show that there exist points with the required reduction properties. In Section 6.4 one can find this translation and the actual construction of these points with the required properties.

CHAPTER 1

Extremal elliptic surfaces & Infinitesimal Torelli

The paper [**34**] is based on this chapter and appeared in the Michigan Mathematical Journal.

1. Introduction

An extremal elliptic surface over $\mathbf{C}$ is an elliptic surface such that the rank of the Néron-Severi group equals $h^{1,1}$ and its associated Jacobian surface admits at most finitely many sections.

These surfaces are very useful if one wants to classify all configurations of singular fibers on certain families of elliptic surfaces: given a configuration of singular fibers on an extremal elliptic surface, using so-called deformations of the j-map and twisting one can construct other elliptic surfaces with different configurations of singular fibers (terminology from [**47**, VIII.2] and [**48**]). For these surfaces, the genus of the base curve and the geometric and arithmetic genus of the surface are the same as for the original surface. In [**66**] this is done for the case of $K3$ surfaces. In [**47**, VIII.2] this is done for any elliptic surface.

The classification of singular fibers on a rational elliptic surface has been given more than 10 years ago (see [**48**], [**53**], [**54**]). Recently there has been given a classification of all singular fibers of elliptic $K3$ surfaces with a section (see [**66**]). From the classification of configurations of singular fibers on rational (see [**48**]) and on elliptic $K3$ surfaces with a section (see [**66**]) we know that any configuration can be obtained from an extremal configuration using deformations of the j-map and twisting. Whether this is true for arbitrary elliptic surfaces seems to be unknown.

In this chapter we give a complete classification of extremal elliptic surfaces with constant j-invariant (Theorem 3.7). From this we deduce that if $\pi : X \to \mathbf{P}^1$ is extremal then either $p_g(X)$ is at most 1 or the j-invariant is 0 or 1728. There are some examples of non-trivial families of extremal elliptic surfaces. For example the family of elliptic surfaces associated to $y^2 = x^3 + t^5(t-1)^5(t-\alpha)^5(t-\beta)^5(t-\gamma)^5$ is such a family. We use this fact to prove the following:

THEOREM 1.1. *Let $\pi : X \to \mathbf{P}^1$ be an elliptic surface without multiple fibers. Assume that $p_g(X) > 1$. Then X satisfies infinitesimal Torelli (cf. Definition 4.1) if and only if $j(\pi)$ is non-constant or π is not extremal.*

Kiĭ ([**31**, Theorem 2]) proved infinitesimal Torelli for elliptic surfaces without multiple fibers and non-constant j-invariant. Saitō ([**58**]) proved in a different way infinitesimal Torelli for elliptic surfaces over $\mathbf{P}^1$ without multiple fibers and j-invariant not identical 0 or 1728 (and also for large classes of elliptic surfaces over other base curves). The case $p_g(X) = 0$ is trivial, since $H^2(X, \mathbf{C}) = H^{1,1}(X, \mathbf{C})$, the case $p_g(X) = 1$ follows from [**55**].

For elliptic surfaces with non-constant j-invariant we will give the following structure theorem:

THEOREM 1.2. *Suppose $\pi : X \to C$ is an elliptic surface without multiple fibers and non-constant j-invariant, then the following three statements are equivalent:*

(1) *π is extremal*
(2)
- *$j(\pi) : C \to \mathbf{P}^1$ is unramified outside* $0, 1728, \infty$*;*
- *the only possible ramification indices above* 0 *are* $1, 2, 3$ *and above* 1728 *are* $1, 2$*;*
- *and π has no fibers of type II, III, IV or I_0^*.*

(3) *There exists an elliptic surface $\pi' : X' \to C$, such that $j(\pi') = j(\pi)$, the fibration π' has no fibers of type II^*, III^* or IV^*, at most one fiber of type I_0^*, and π' has precisely $2p_g(X) + 4 - 4g(C)$ singular fibers.*

We will now present a more precise theorem which moreover gives the possible Mordell-Weil groups for an extremal elliptic surface. Let $m, n \in \mathbf{Z}_{\geq 1}$ be such that $m|n$ and $n > 1$. Let $X(m, n)$ be the modular curve parameterizing triples $((E, O), P, Q)$, such that (E, O) is an elliptic curve, $P \in E$ is a point of order m, the point $Q \in E$ a point of order n and the group generated by P and Q has mn elements.

If

$$(m, n) \notin \{(1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\}$$

then there exists a universal family for $X(m, n)$, which we denote by $E(m, n)$. Denote by $j_{m,n} : X(m, n) \to \mathbf{P}^1$ the map usually called j.

From the results of [**69**, Sections 4 and 5] it follows that $E(m, n)$ is an extremal elliptic surface. The following theorem explains how to construct many examples of extremal elliptic surfaces with a given torsion group.

THEOREM 1.3. *Fix $m, n \in \mathbf{Z}_{\geq 1}$ such that $m|n$ and (m, n) is not one of the pairs* $(1, 1)$, $(1, 2)$, $(2, 2)$, $(1, 3)$, $(1, 4)$, $(2, 4)$. *Let C be a (projective smooth irreducible) curve, let $j \in \mathbf{C}(C)$ be a non-constant function. Then the following are equivalent*

- *there exists a unique extremal elliptic surface $\pi : X \to C$, with $j(\pi) = j$ and the group of sections has $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ as a subgroup*
- *j is unramified outside* $0, 1728, \infty$ *and $j = j_{n,m} \circ h$ for some $h : C \to X_m(n)$.*

An extremal elliptic surface $\pi : X \to C$ with $j(\pi) = j$ and $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ is a subgroup of the group of section, is one of the $2^{2g(C)}$ surfaces with $j(\pi) = j$ and all singular fibers are of type I_ν.

If $\pi : X \to \mathbf{P}^1$ is an extremal semi-stable rational elliptic surface then X is determined by the configuration of singular fibers (see [**50**, Theorem 5.4]). It seems that this quite special for rational elliptic surfaces. If X is a $K3$ surface a similar statement does not hold:

THEOREM 1.4. *There exists pairs of* extremal semi-stable elliptic $K3$ surfaces, $\pi_i : X_i \to \mathbf{P}^1$, $(i = 1, 2)$ *such that the groups of section of π_1 and π_2 are isomorphic, the configuration of singular fibers of the π_i coincide and X_1 and X_2 are non-isomorphic.*

This gives a negative answer to [**5**, Question 0.2] (for a precise formulation of this question see Section 8). The essential ingredient for the proof of Theorem 1.4 comes from [**67**, Table 2].

This chapter is organized as follows:

Section 2 contains some definitions and several standard facts. In Section 3 we give a list of extremal elliptic surfaces with constant j-invariant. They behave differently from the non-constant ones. There are exactly 5 infinite families of extremal elliptic surfaces with constant j-invariant (3 of dimension 1, 1 of dimension 2 and 1 of dimension 3). In Section 4 we explain this different behavior by proving Theorem 1.1. In Section 5 we explain how twisting can reduce the problem of classification. In Section 6 we link the ramification of the j-map and the number of singular fibers of a certain elliptic surface. This combined with the results of Section 5 gives a proof of Theorem 1.2. Section 7 contains a proof of the version with the description of the group of sections (Theorem 1.3). In Section 8 we prove Theorem 1.4. In Section 9 we give a classification of extremal elliptic surfaces with $g(C) = p_g(X) = q(X) = 1$. Section 10 contains a proof of the fact that there exist elliptic surfaces with exactly one singular fiber. It is easy to see that the singular fiber is of type I_{12k} or I^*_{12k-6}, for some $k > 0$. In this section we prove that for every positive k, both I_{12k} and I^*_{12k-6} occur.

2. Definitions and Notation

ASSUMPTION 2.1. *By a curve we mean a non-singular projective complex connected curve.*

By a surface we mean a non-singular projective complex surface.

DEFINITION 2.2. An *elliptic surface* is a triple (π, X, C) with X a surface, C a curve, π is a morphism $X \to C$, such that almost all fibers are irreducible genus 1 curves and X is relatively minimal, i.e., no fiber of π contains an irreducible rational curve D with $D^2 = -1$.

We denote by $j(\pi) : C \to \mathbf{P}^1$ the rational function such that $j(\pi)(P)$ equals the j-invariant of $\pi^{-1}(P)$, whenever $\pi^{-1}(P)$ is non-singular.

A *Jacobian elliptic surface* is an elliptic surface together with a section $\sigma_0 : C \to X$ to π. The set of sections of π is an abelian group, with σ_0 as the identity element. Denote this group by $MW(\pi)$.

By an *elliptic fibration* on X we mean that we give a surface X a structure of an elliptic surface.

Let $\mathbf{L}$ be the line bundle $[R^1\pi_*\mathcal{O}_X]^{-1}$. We call $\mathbf{L}$ the *fundamental line bundle* (terminology from [**47**]). Let $\rho(X)$ denote the rank of the Néron-Severi group of X. We call $\rho(X)$ the *Picard number*.

We use the line bundle $\mathbf{L}$ only to keep track of some numerical data. Note that $\deg(\mathbf{L}) = p_g(X) + 1 - g(C) = p_a(X) + 1$. (See [**47**, Lemma IV.1.1].)

ASSUMPTION 2.3. *All elliptic surfaces in this thesis are without multiple fibers.*

REMARK 2.4. To an elliptic surface $\pi : X \to C$ we can associate its Jacobian elliptic surface $\mathrm{Jac}(\pi) : \mathrm{Jac}(X) \to C$. The Hodge numbers $h^{p,q}$, the Picard number $\rho(X)$, the type of singular fibers of π and $\deg(\mathbf{L})$ are the same for π and its associated Jacobian surface. We have that $\mathrm{Jac}(\pi) \cong \pi$ if and only if π admits a section.

DEFINITION 2.5. An *extremal elliptic surface* is an elliptic surface such that $\rho(X) = h^{1,1}(X)$ and $MW(\mathrm{Jac}(\pi))$ is finite.

Definition 2.6. Let $\pi : X \to C$ be an elliptic surface. Let P be a point of C. Define $v_P(\Delta_P)$ as the valuation at P of the minimal discriminant of the Weierstrass model, which equals the topological Euler characteristic of $\pi^{-1}(P)$.

Proposition 2.7. *Let $\pi : X \to C$ be an elliptic surface. Then*

$$\sum_{P \in C} v_P(\Delta_P) = 12 \deg(\mathbf{L}).$$

In particular, $\deg(\mathbf{L}) \geq 0$.

Proof. This follows from Noether's formula (see [**7**, p. 20]). The precise reasoning can be found in [**47**, Section III.4]. ☐

Remark 2.8. If P is a point on C, such that $\pi^{-1}(P)$ is singular then $j(\pi)(P)$ and $v_p(\Delta_p)$ behave as follows:

Kodaira type of fiber over P	$j(\pi)(P)$	$v_p(\Delta_p)$	number of components
I_0^*	$\neq \infty$	6	1
I_ν $(\nu > 0)$	∞	ν	$\nu + 1$
I_ν^* $(\nu > 0)$	∞	$6 + \nu$	$\nu + 5$
II	0	2	1
IV	0	4	3
IV^*	0	8	7
II^*	0	10	9
III	1728	3	2
III^*	1728	9	8

For proofs of these facts see [**7**, p. 150], [**73**, Theorem IV.8.2] or [**47**, Lecture 1].

Definition 2.9. Let X be a surface, let C and C_1 be curves. Let $\varphi : X \to C$ and $f : C_1 \to C$ be two morphisms. Then we denote by $\widetilde{X \times_C C_1}$ the smooth, relatively minimal model of the ordinary fiber product of X and C_1.

Definition 2.10. Let $\pi : X \to C$ be an elliptic surface. We say that $\pi : X \to C$ is a *semi-stable elliptic surface*, if for all $p \in C$ we have that $\pi^{-1}(p)$ is of type I_ν.

Recall the following theorem.

Theorem 2.11 (Shioda-Tate ([**70**, Theorem 1.3 & Corollary 5.3])). *Let $\pi : X \to C$ be a Jacobian elliptic surface, such that* $\deg(\mathbf{L}) > 0$. *Then the Néron-Severi group of X is generated by the classes of $\sigma_0(C)$, a non-singular fiber, the components of the singular fibers not intersecting $\sigma_0(C)$, and the generators of the Mordell-Weil group. Moreover, let S be the set of points P such that $\pi^{-1}(P)$ is singular. Let $m(P)$ be the number of irreducible components of $\pi^{-1}(P)$, then*

$$\rho(X) = 2 + \sum_{P \in S} (m(P) - 1) + \operatorname{rank}(MW(\pi)).$$

Definition 2.12. Suppose $\pi : X \to C$ is an elliptic surface. Denote by $T(\pi)$ the subgroup of the Néron-Severi group of $\operatorname{Jac}(\pi)$ generated by the classes of the fiber, $\sigma_0(C)$ and the components of the singular fibers not intersecting $\sigma_0(C)$. Let $\rho_{tr}(\pi) := \operatorname{rank} T(\pi)$. We call $T(\pi)$ the *trivial part* of the Néron-Severi group of $\operatorname{Jac}(\pi)$.

REMARK 2.13. In Section 2.7 we give an alternative description for the trivial part of the Néron-Severi group.

REMARK 2.14. Suppose $\pi : X \to \mathbf{P}^1$ has $\deg(\mathbf{L}) = 0$, then there are no singular fibers hence $\rho_{tr} = 2$.

DEFINITION 2.15. Let $\pi : X \to C$ be an elliptic surface, define

- $a(\pi)$ as the number of fibers of type II^*, III^*, IV^*.
- $b(\pi)$ as the number of fibers of type II, III, IV.
- $c(\pi)$ as the number of fibers of type I_0^*.
- $d(\pi)$ as the number of fibers of type I_ν^*, with $\nu > 0$.
- $e(\pi)$ as the number of fibers of type I_ν, $\nu > 0$.

PROPOSITION 2.16. *For any elliptic surface $\pi : X \to C$, not a product, we have*

$$h^{1,1}(X) - \rho_{tr}(\pi) = 2(a(\pi) + b(\pi) + c(\pi) + d(\pi)) + e(\pi) - 2\deg(\mathbf{L}) - 2 + 2g(C).$$

PROOF. Recall from [**47**, Lemma IV.1.1] that

$$h^{1,1} = 10\deg(\mathbf{L}) + 2g(C).$$

From Kodaira's classification of singular fibers (see Remark 2.8 and Proposition 2.7) it follows that

$$\rho_{tr}(\pi) = 2 + 12\deg(\mathbf{L}) - 2(a(\pi) + b(\pi) + c(\pi) + d(\pi)) - e(\pi).$$

Combining these yields the proof. □

COROLLARY 2.17. *Let $\pi : X \to C$ be an elliptic surface with constant j-invariant, not a product. Then π is extremal if and only if π has* $\deg(\mathbf{L}) + 1 - g(C)$ *singular fibers.*

PROOF. If j is constant then $e(\pi) = d(\pi) = 0$, hence $a(\pi) + b(\pi) + c(\pi)$ equals the number of singular fibers of π. Then apply Proposition 2.16. □

3. Constant j-invariant

In this section we give a list of all extremal elliptic surfaces with constant j-invariant.

LEMMA 3.1. *Suppose $\pi : X \to C$ is an extremal elliptic surface such that $j(\pi)$ is constant. Then $g(C) \leq 1$.*

PROOF. If $j(\pi)$ is constant then $v_P(\Delta_P) \leq 10$ (see Remark 2.8), for every point P. From this, Proposition 2.7 and Corollary 2.17 it follows that

$$12\deg(\mathbf{L}) = \sum_{P|\pi^{-1}(P) \text{ singular}} v_P(\Delta_p) \leq 10(\deg(\mathbf{L}) + 1 - g(C)).$$

This inequality and the fact that $\deg(\mathbf{L}) \geq 0$ (see Proposition 2.7) imply that $g(C) \leq 1$. □

DEFINITION 3.2. An elliptic surface $\pi : X \to C$, with C a genus 1 curve, is called a *hyperelliptic surface*, if π has no singular fibers, $j(\pi)$ is constant, and X is not isomorphic to $C \times E$.

Remark 3.3. In terms of Section 5, an elliptic surface $\pi : X \to C$, with $j(\pi) \neq 0, 1728$ is called a hyperelliptic surface if and only if the associated elliptic curve $E/\mathbf{C}(C)$ is isomorphic to $E_1^{(f)}$, with $E_1/\mathbf{C}$ an elliptic curve and f a function such that the valuation of f at every place of C is even, but there is no function $g \in \mathbf{C}(C)$ such that $f \neq g^2$. (If $j(\pi)$ equals 0 or 1728, then one can give a similar description.)

Remark 3.4. The usual definition of a hyperelliptic surface (e.g. [**7**, page 148]) is different, but equivalent (see [**47**, Lemma III.4.6.b]), to this one.

Remark 3.5. An hyperelliptic surface does *not* admit a fibration in hyperelliptic curves. For historical reasons (see [**7**, page 148]) these surfaces are called hyperelliptic. Beauville [**8**] calls these surfaces *bi-elliptic*, because they admit two elliptic fibrations.

Lemma 3.6. *Suppose $\pi : X \to C$ is an extremal elliptic surface such that $j(\pi)$ is constant. Then one of the following occurs*

(1) *$g(C) = 1$; $\deg(\mathbf{L}) = 0$ and $\pi : X \to C$ is a hyperelliptic (or bi-elliptic) surface.*
(2) *$g(C) = 0$; $j(\pi) \neq 0, 1728$; $\deg(\mathbf{L}) = 1$.*
(3) *$g(C) = 0$; $j(\pi) = 0$; $1 \leq \deg(\mathbf{L}) \leq 5$.*
(4) *$g(C) = 0$; $j(\pi) = 1728$; $1 \leq \deg(\mathbf{L}) \leq 3$.*

Proof. Suppose $g(C) = 1$. Then Corollary 2.17 implies that π has $\deg(\mathbf{L})$ singular fibers hence

$$12\deg(\mathbf{L}) = \sum v_P(\Delta_P) \leq 10\deg(\mathbf{L})$$

from which it follows that $\deg(\mathbf{L}) = 0$. Hence π has no singular fibers. Since π has finitely many sections, it follows from the definition that $\pi : X \to C$ is a so-called hyperelliptic surface.

Suppose $g(C) = 0$. If $\deg(\mathbf{L}) = 0$ then π is a projection from a product, hence there are infinitely many sections. By definition, π is not extremal.

Suppose $\deg(\mathbf{L}) > 0$. Assume $j(\pi) \neq 0, 1728$. Then all singular fibers are of type I_0^* (see Remark 2.8). Since the Euler characteristic of such a fiber is 6, Proposition 2.7 implies that there are exactly $2\deg(\mathbf{L})$ singular fibers. Applying Corollary 2.17 gives

$$2\deg(\mathbf{L}) = \deg(\mathbf{L}) + 1.$$

From this we know $\deg(\mathbf{L}) = 1$.

Suppose $j(\pi) = 1728$. In this case all singular fibers are of type III, I_0^*, III^* (see Remark 2.8). From this it follows that $v_P(\Delta_P) \leq 9$. By Proposition 2.7 and Proposition 2.16 we obtain

$$12\deg(\mathbf{L}) \leq 9(a(\pi) + b(\pi) + c(\pi)) = 9\deg(\mathbf{L}) + 9,$$

so $1 \leq \deg(\mathbf{L}) \leq 3$.

Suppose $j(\pi) = 0$. In this case we obtain in a similar way $1 \leq \deg(\mathbf{L}) \leq 5$. □

Theorem 3.7. *Suppose $\pi : X \to C$ is an elliptic surface with $j(\pi)$ constant.*

Then π is extremal if and only if either C is a curve of genus 1 and Jac(X) *is a hyperelliptic surface or $C \cong \mathbf{P}^1$ and* Jac(π) *has a model isomorphic to one of the following:*

- $(j(\pi) = 0)$ $y^2 = x^3 + f(t)$ *where* $f(t)$ *comes from the following table (the left hand side indicates the positions of the singular fibers)*

II	IV	I_0^*	IV^*	II^*	p_g	$f(t)$
0				∞	0	t
	0		∞		0	t^2
		$0, \infty$			0	t^3
	1			$0, \infty$	1	$t^5(t-1)^2$
		1	0	∞	1	$t^4(t-1)^3$
			$0, 1, \infty$		1	$t^4(t-1)^4$
		α		$0, 1, \infty$	2	$t^5(t-1)^5(t-\alpha)^3$
			$\alpha, 1$	$0, \infty$	2	$t^5(t-1)^4(t-\alpha)^4$
			β	$\alpha, 0, 1, \infty$	3	$t^5(t-1)^5(t-\alpha)^5(t-\beta)^4$
				$0, 1, \infty, \alpha, \beta, \gamma$	4	$t^5(t-1)^5(t-\alpha)^5(t-\beta)^5(t-\gamma)^5$

where $\alpha, \beta, \gamma \in \mathbf{C} - \{0, 1\}$*, pairwise distinct.*
- $(j(\pi) = 1728)$ $y^2 = x^3 + g(t)x$ *where* $g(t)$ *comes from the following table*

III	I_0^*	III^*	p_g	$g(t)$
0		∞	0	t
	$0, \infty$		0	t^2
	1	$0, \infty$	1	$t^3(t-1)^2$
		$0, 1, \infty, \alpha$	2	$t^3(t-1)^3(t-\alpha)^3$

where $\alpha \neq 0, 1$.
- $(j(\pi) \neq 0, 1728)$ $y^2 = x^3 + at^2x + t^3$*, with singular fibers of type* I_0^* *at* $t = 0$ *and* $t = \infty$

PROOF. The above list follows directly from Corollary 2.17 and Lemma 3.6. Since all cases are very similar, we discuss only the case $j(\pi) = 0$ and $p_g = 2$. In this case $\mathrm{Jac}(\pi)$ has a Weierstrass model isomorphic to

$$y^2 = x^3 + f(t)$$

with f a polynomial such that $13 \leq \deg(f) \leq 18$, and $v_P(f) \leq 5$ for all finite P. At all zeros of f there is a singular fiber. If $\deg(f) < 18$ then the fiber over $t = \infty$ is also singular.

If π is extremal then from Corollary 2.17 it follows that π has exactly 4 singular fibers. Assume that the fibers with the highest Euler characteristic are over $t = \infty, 0, 1$. Since $5+5+5+3 = 5+5+4+4$ are the only two ways of writing 18 as a sum of four positive integers smaller then 6, we obtain that after applying an isomorphism, if necessary, f equals either

$$t^5(t-1)^5(t-\alpha)^3 \text{ or } t^5(t-1)^4(t-\alpha)^4.$$

□

REMARK 3.8. Note that all extremal elliptic surfaces with constant j-invariant and $p_g(X) > 1$ have moduli.

4. Infinitesimal Torelli

In the previous section we gave examples of families of elliptic surfaces with maximal Picard number. In this section we prove that these surfaces are counterexamples to infinitesimal Torelli. Moreover we give a complete solution for infinitesimal Torelli for Jacobian elliptic surfaces over $\mathbf{P}^1$.

Suppose that X is a smooth complex algebraic variety. Then the first order deformations of X are parameterized by $H^1(X,\Theta_X)$, with Θ_X the tangent bundle of X. The isomorphism $H^{p,q}(X,\mathbf{C})=H^q(X,\Omega^p)$ and the contraction map $\Theta_X\otimes_{\mathcal{O}_X}\Omega^p_X\to\Omega^{p-1}_X$ give a cup product map:

$$H^1(X,\Theta_X)\otimes H^{p,q}(X,\mathbf{C})\to H^{p-1,q+1}(X,\mathbf{C}).$$

From this one obtains the infinitesimal period map

$$\delta_k : H^1(X,\Theta_X)\to\oplus_{p+q=k}\operatorname{Hom}(H^{p,q}(X,\mathbf{C}),H^{p-1,q+1}(X,\mathbf{C})).$$

The (holomorphic) map δ_k is closely related to the period map. Assume that $\varphi:\mathcal{X}\to\mathcal{B}$ is a proper, smooth, surjective holomorphic map between complex manifolds having connected fibers, and that for all $t\in\mathcal{B}$ the vector space $H^k(X_t,\mathbf{C})$ carries a Hodge structure of weight k, with $X_t:=\varphi^{-1}(t)$. Fix a point $0\in\mathcal{B}$. Let U be a small simply connected open neighborhood of 0.

Define

$$\begin{array}{rcl}\mathcal{P}^{p,k}:U & \to & \operatorname{Grass}\left(\sum_{i\geq p}h^{i,k-i},H^k(X_0,\mathbf{C})\right)\\ t & \mapsto & (\oplus_{i\geq p}H^{i,k-i}(X_t,\mathbf{C}))\subset H^k(X_0,\mathbf{C}),\end{array}$$

via the identification $H^k(X_0,\mathbf{C})\cong H^k(X_t,\mathbf{C})$. (Note that U is simply connected.)

If the Kodaira-Spencer map $\rho_{U,0}:T_{U,0}\to H^1(X,\Theta_X)$ is injective then the differential at 0 of the period-map $\oplus_p\mathcal{P}^{p,k}$ is injective if and only if δ_k is injective. (See [**58**, Section 2] or [**82**, Chapter 10].)

Note that if X is an Jacobian elliptic surface with base $\mathbf{P}^1$ then δ_k, for $k\neq 2=\dim X$ is the zero-map.

Definition 4.1. We say that X satisfies *infinitesimal Torelli* if and only if $\delta_{\dim(X)}$ is injective.

If X is a rational surface, then $H^{1,1}(X,\mathbf{C})=H^2(X,\mathbf{C})$ hence the image of $\oplus_p\mathcal{P}^{p,2}$ is a point, while the moduli space of rational elliptic surfaces has positive dimension, so infinitesimal Torelli does not hold for rational elliptic surfaces. If X is a $K3$ surface, then infinitesimal Torelli follows from [**55**]. This means that we handled the case $p_g(X)\leq 1$. This section will focus on the case $p_g(X)>1$.

There is an easy sufficient condition of Kiĭ([**31**]), Lieberman, Peters and Wilsker ([**42**]) for checking infinitesimal Torelli for manifolds with divisible canonical bundle. The following result is a direct consequence of [**42**, Theorem 1'].

Theorem 4.2. *Let X be a compact Kähler n-manifold, with $p_g(X)>1$. Let $\mathcal{L}$ be a line bundle such that*

(1) *$\mathcal{L}^{\otimes k}=\Omega^n_X$ for some $k>0$.*
(2) *the linear system corresponding to $\mathcal{L}$ has no fixed components of codimension 1.*

(3) $H^0(X, \Omega_X^{n-1} \otimes \mathcal{L}) = 0$.

Then δ_n is injective.

We want to apply the above theorem, when X is an elliptic surface. We take $\mathcal{L}$ to be the line bundle $\mathcal{O}_X(F)$, where F is the class of a smooth fiber.

LEMMA 4.3. *Let $\pi : X \to \mathbf{P}^1$ be an elliptic surface. Assume that X is not birational to a product $C \times \mathbf{P}^1$. Then for $n > 0$ we have*

$$\dim H^0(X, \Omega^1(nF)) = \begin{cases} n-1 & \text{if } j(\pi) \text{ is not constant} \\ n-1+\max(0, n+d+1) & \text{if } j(\pi) \text{ is constant,} \end{cases}$$

where $d = \deg(\mathbf{L}) - \#\{P \in C(\mathbf{C}) \mid \pi^{-1}(P) \text{ singular }\}$.

PROOF. By [**58**, Prop. 4.4 (I)] we know that $\pi_* \Omega_X^1 \cong \Omega_{\mathbf{P}^1}^1$ if $j(\pi)$ is not constant, which gives the first case.

If $j(\pi)$ is constant then we have the following exact sequence (by [**58**, Prop. 4.4 (II)]):

$$0 \to \Omega_{\mathbf{P}^1}^1 \to \pi_* \Omega_X^1 \to \mathcal{O}_{\mathbf{P}^1}(d) \to 0.$$

Tensoring with $\mathcal{O}_{\mathbf{P}^1}(n)$ gives

$$\begin{aligned} \dim H^0(X, \Omega_X^1(nF)) &= \dim H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(n)) + \dim H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n+d)) \\ &= n-1+\max(0, n+d+1), \end{aligned}$$

using that $\dim H^1(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(n)) = 0$. □

COROLLARY 4.4. *Suppose $\pi : X \to \mathbf{P}^1$ is an elliptic surface (cf. Assumption 2.3), such that $p_g(X) > 1$. Suppose that $j(\pi)$ is non-constant or π is not extremal then X satisfies infinitesimal Torelli.*

PROOF. Let F be a smooth fiber of π. Note that $\mathcal{O}(F)^{\otimes(p_g(X)-1)} = \Omega_X^2$ (see [**70**, Theorem 2.8]) and the linear system $|F|$ is the elliptic fibration, hence without base-points.

We claim that $\dim H^0(X, \Omega^1(F)) = 0$. If this is not the case then by Lemma 4.3 $j(\pi)$ is constant and π has at most $\deg(\mathbf{L}) + 1$ singular fibers. From Lemma 2.17 it follows that π is extremal. Apply now Theorem 4.2 with $\mathcal{L} = \mathcal{O}(F)$. □

The following technical result states, more or less, that if we have a family of surfaces satisfying Infinitesimal Torelli and the generic member has a large Picard number, then the base of this family has small dimension. The main idea in the proof is that a high Picard number gives a severe restriction on the image of "δ_2 restricted to this family". Joseph Steenbrink pointed out to me that the following result is a direct consequence of [**12**, Theorem 1.1].

PROPOSITION 4.5. *Let $\varphi : \mathcal{X} \to \Delta$ be a family of smooth surfaces with Δ a small polydisc. For any $t \in \Delta$ denote by X_t the fiber over t. Assume that for all $t, t' \in \Delta$ such that $t \neq t'$, we have that $X_t \not\cong X_{t'}$. Moreover, assume that the Kodaira-Spencer map $\rho_{\Delta,t} : T_{\Delta,t} \to H^1(X_t, \Theta_{X_t})$ is injective for all $t \in \Delta$.*

Let r be the Picard number of a generic member of the family φ. Suppose that for some t we have $p_g(X_t) > 1$ and

$$\dim \Delta > \frac{1}{2} p_g(X_t)(h^{1,1}(X_t, \mathbf{C}) - r).$$

or $p_g(X_t) = 1$ *and*

$$\dim \Delta > (h^{1,1}(X_t, \mathbf{C}) - r).$$

Then for no t*, the surface* X_t *satisfies infinitesimal Torelli.*

COROLLARY 4.6. *Let* $\varphi : \mathcal{X} \to \Delta$ *be a non-trivial family of surfaces such that* $\rho(X_t) = h^{1,1}(X_t, \mathbf{C})$ *for all* t*. Then for no* t *the surface* X_t *satisfies infinitesimal Torelli.*

COROLLARY 4.7. *Let* $\pi : X \to \mathbf{P}^1$ *be an extremal elliptic surface without multiple fibers, with constant* j*-invariant and* $p_g(X) > 1$*. Then* X *does not satisfy infinitesimal Torelli.*

PROOF. From Remark 3.8 it follows that X is a member of positive dimensional family of surfaces with $\rho(X) = h^{1,1}(X, \mathbf{C})$. The fact that the Kodaira-Spencer map is injective for these families is well-known. □

PROOF OF PROPOSITION 4.5. We start with some reduction steps. Fix a base point $0 \in \Delta$. Denote X the fiber over 0. It suffices to show that

$$H^1(X, \Theta_X) \xrightarrow{\delta_2} \mathrm{Hom}(H^{2,0}(X, \mathbf{C}), H^{1,1}(X, \mathbf{C})) \oplus \mathrm{Hom}(H^{1,1}(X, \mathbf{C}), H^{0,2}(X, \mathbf{C}))$$

is not injective. Using Serre duality one can show that this is equivalent to show that

$$H^1(X, \Theta_X) \xrightarrow{\delta_2'} \mathrm{Hom}(H^{2,0}(X, \mathbf{C}), H^{1,1}(X, \mathbf{C}))$$

is not injective (see [**58**, Section 3]). Since $\rho_{\Delta,0}$ is injective, it suffices to show that the dimension of the image of $\delta_2' \circ \rho_{\Delta,0}$ is less than the dimension of Δ.

Since Δ is simply connected there is a natural isomorphism

$$\psi_t : H^2(X, \mathbf{C}) \cong H^2(X_t, \mathbf{C}),$$

for all $t \in \mathbf{C}$. Let $\Lambda := \cap_{t \in \Delta} \psi_t^{-1}(NS(X_t))$. Then Λ is a rank r lattice.

Let $T(X_t)$ be the orthogonal complement (with respect to the cup-product) of $\psi_t(\Lambda)$ in $H^2(X_t, \mathbf{Z})$. Then $T(X_t, \mathbf{C}) := T(X_t) \otimes \mathbf{C}$ carries a sub-Hodge structure. From

$$\psi_t(\Lambda) \subset NS(X_t) = H^{1,1}(X_t, \mathbf{C}) \cap H^2(X, \mathbf{Z})$$

we obtain that the Hodge structure on $H^2(X_t, \mathbf{C})$ is determined by the Hodge structure on $T(X_t, \mathbf{C})$. We consider the variation of the Hodge structure on $T(X_t)$ (cf. [**24**, Section 6]).

One easily shows that the composed map $\delta_2' \circ \rho_{\Delta,0}$ factors

$$T_{\Delta,0} \to \mathrm{Hom}(T^{2,0}(X, \mathbf{C}), T^{1,1}(X, \mathbf{C})) \to \mathrm{Hom}(H^{2,0}(X, \mathbf{C}), H^{1,1}(X, \mathbf{C})).$$

This follows almost immediately from the fact that Λ has a pure Hodge structure of type $(1, 1)$, hence the Hodge structure does not vary.

The above factorization implies that for any value of $p_g(X) > 0$

$$\begin{aligned} \dim \mathrm{Im}(\delta_2' \circ \rho_{\Delta,0}) &\leq \dim T^{2,0}(X, \mathbf{C}) \cdot \dim T^{1,1}(X, \mathbf{C}) \\ &= p_g(X)(h^{1,1}(X, \mathbf{C}) - r). \end{aligned}$$

If $p_g(X) > 1$ then by Griffiths' transversality we obtain that

$$\begin{aligned} \dim \mathrm{Im}(\delta_2' \circ \rho_{\Delta,0}) &\leq \frac{1}{2} \dim T^{2,0}(X, \mathbf{C}) \cdot \dim T^{1,1}(X, \mathbf{C}) \\ &= \frac{1}{2} p_g(X)(h^{1,1}(X, \mathbf{C}) - r), \end{aligned}$$

(cf. [**24**, Section 6]). □

THEOREM 4.8. *Let $\pi : X \to \mathbf{P}^1$ be an elliptic surface (cf. Assumption 2.3). Assume that $p_g(X) > 1$. Then X satisfies infinitesimal Torelli if and only if $j(\pi)$ is non-constant or π is not extremal.*

PROOF. Combine Corollary 4.4 and Corollary 4.7. □

REMARK 4.9. Note that the hyperelliptic surfaces form a family of elliptic surfaces with $p_g(X) = 0$, so they do not satisfy infinitesimal Torelli.

REMARK 4.10. Chakiris ([**15**, Section 4]) gave different formulae for the dimension of $H^0(X, \Omega^1(nF))$. He used them to deduce a formula for $\dim H^1(X, \Theta_X)$, which he used to prove that generic global Torelli holds. Even with the use of these incorrect formulae his proof of generic global Torelli seems to remain valid, after a small modification. His formulae would imply that infinitesimal Torelli holds for any Jacobian elliptic surface over $\mathbf{P}^1$. The same erroneous formulae leads Beauville ([**10**, p. 13]) to state in a survey paper on Torelli problems that infinitesimal Torelli holds for an *arbitrary* Jacobian elliptic surface. Theorem 4.8 shows instead that this is true only under the condition that $j(\pi)$ is not constant or π is not extremal.

The argument used in Corollary 4.4 to prove that several elliptic surfaces satisfy infinitesimal Torelli, relies heavily on $C = \mathbf{P}^1$. Saitō [**58**] proved, using other techniques, that if C is an *arbitrary* smooth curve and if $\pi : X \to C$ is an elliptic surface with non-constant j-invariant then X satisfies infinitesimal Torelli.

We give now some example of Jacobian elliptic surfaces over curves of positive genus for which infinitesimal Torelli does not hold. Hence the j-invariant is constant.

LEMMA 4.11. *Let $\varphi : \mathcal{X} \to \mathcal{B}$ be a family of elliptic surfaces with $p_g(X_0) > 1$, constant j-invariant and s singular fibers. Let g be the genus of the base curve of a generic member of this family. Suppose that for all t we have that $\{t' \mid X_t \cong X_{t'}\}$ is zero-dimensional and*

$$\dim \mathcal{B} > (s - \deg(\mathbf{L}) + g - 1)p_g.$$

Then there is no t such that X_t satisfies infinitesimal Torelli.

PROOF. Note that $\rho(X_t) \geq \rho_{tr}(\pi)$ for all $t \in \mathcal{B}$ and

$$h^{1,1}(X) - \rho_{tr}(\pi) = 2s - 2\deg(\mathbf{L}) - 2 + 2g.$$

Apply now Proposition 4.5. □

We now give two examples where the conditions of Lemma 4.11 are satisfied. The first example covers the case where the j-invariant is constant, but arbitrary. The second example covers the case where $j = 0$ or $j = 1728$.

EXAMPLE 4.12. Let $\psi : \mathcal{X} \to \mathcal{B}$ be a maximal-dimensional family of Jacobian elliptic surfaces over a base curve of a fixed positive genus g, having $2\deg(\mathbf{L}) > 0$ singular fibers of type I_0^*. Then $\mathcal{B}$ has dimension $3g - 3 + 2\deg(\mathbf{L}) + 1$.

We want to find examples satisfying the conditions of Lemma 4.11. Easy combinatorics show that if $g > 1$ then the conditions $p_g > 1$ and

$$\dim \mathcal{B} > (s - \deg(\mathbf{L}) + g - 1)p_g$$

of Lemma 4.11 hold if and only if (g, p_g) is one of $(1,2)$, $(2,2)$, $(3,2)$, $(4,3)$. In all these cases any member of the family ψ is a counterexample to Infinitesimal Torelli. It is easy to construct examples with these invariants. Fix $j_0 \in \mathbf{C}$. Fix $A, B \in \mathbf{C}$ such that the elliptic curve associated to $y^2 = x^3 + Ax + B$ has j-invariant j_0. Fix a curve C of genus g. Take $f \in K(C)^*$ such that f has an odd valuation at precisely $2\deg(\mathbf{L})$ places of C. Then the elliptic surface

$$y^2 = x^3 + Af^2x + Bf^3$$

has the above mentioned invariants.

EXAMPLE 4.13. Let $\psi : \mathcal{X} \to \mathcal{B}$ be a maximal-dimensional family of Jacobian elliptic surfaces over a base curve of a fixed positive genus g, such that almost all fibers have constant j-invariant 0 or 1728 and s singular fibers. Then $\mathcal{B}$ has dimension $3g - 3 + s$. Using $p_g = \deg(\mathbf{L}) + g - 1$ and some easy combinatorics we obtain that the condition $\dim \mathcal{B} > (s - \deg(\mathbf{L}) + g - 1)p_g$ of Lemma 4.11 holds if and only if

$$s < \frac{\deg(\mathbf{L})^2 - g^2 + 5g - 4}{\deg(\mathbf{L}) + g - 2}. \tag{1}$$

From Noether's condition, the smallest s that is possible is $\lceil \frac{6}{5} \deg \mathbf{L} \rceil$. This implies that

$$\deg(\mathbf{L}) < -3g + 6 + \sqrt{4g^2 - 11g + 16}.$$

All combinations of g and p_g satisfying $\deg(\mathbf{L}) > 0$, $p_g > 1$ and (1) are mentioned in the table below. In this table $s_{\max}$ denotes the number of singular fibers such that an elliptic surface with constant j-invariant and at most $s_{\max}$ singular fibers satisfy (1). In particular, elliptic surfaces with these invariant do not satisfy Infinitesimal Torelli by Lemma 4.11. One can find examples with these invariants in a similar way as above. The columns with $\lceil 6/5 \deg(\mathbf{L}) \rceil$ and $\lceil 4/3 \deg(\mathbf{L}) \rceil$ denote the minimal number of singular fibers for an elliptic surface with j-invariant 0 or 1728, whenever this number is at most $s_{\max}$.

$g(C)$	$p_g(X)$	$\deg(\mathbf{L})$	$s_{\max}$	$\lceil 6/5 \deg(\mathbf{L}) \rceil$	$\lceil 4/3 \deg(\mathbf{L}) \rceil$
1	2	2	3	3	3
1	3	3	4	4	4
1	4	4	5	5	–
1	5	5	6	6	–
2	2	1	2	2	2

5. Twisting

In this section we study the behavior of $h^{1,1}(X) - \rho_{tr}(X)$ under twisting, when $\pi : X \to \mathbf{P}^1$ is a Jacobian elliptic surface. We are mostly interested in the case that $j(\pi)$ is not constant.

Given a Jacobian elliptic surface $\pi : X \to C$, we can associate an elliptic curve in $\mathbf{P}^2_{\mathbf{C}(C)}$ corresponding to the generic fiber of π. This induces a bijection on isomorphism classes of Jacobian elliptic surfaces and elliptic curves over $\mathbf{C}(C)$.

Two elliptic curves E_1 and E_2 are isomorphic over $\mathbf{C}(C)$ if and only if $j(E_1) = j(E_2)$ and the quotients of the minimal discriminants of $E_1/\mathbf{C}(C)$ and $E_2/\mathbf{C}(C)$ is a 12-th power (in $\mathbf{C}(C)^*$).

Assume that E_1, E_2 are elliptic curves over $\mathbf{C}(C)$ with $j(E_1) = j(E_2) \neq 0, 1728$. Then one shows easily that $\Delta(E_1)/\Delta(E_2)$ equals u^6, with $u \in \mathbf{C}(C)^*$. Hence E_1 and E_2 are isomorphic over $\mathbf{C}(C)(\sqrt{u})$. We call E_2 the twist of E_1 by u, denoted by $E_1^{(u)}$. Actually, we are not interested in the function u, but in the places at which the valuation of u is odd.

DEFINITION 5.1. Let $\pi : X \rightarrow C$ be a Jacobian elliptic surface. Fix $2n$ points $P_i \in C(\mathbf{C})$. Let $E/\mathbf{C}(C)$ be the Weierstrass model of the generic fiber of π.

A Jacobian elliptic surface $\pi' : X' \rightarrow C$ is called a *(quadratic) twist* of π by $(P_1, \ldots, P_n)$ if the Weierstrass model of the generic fiber of π' is isomorphic to $E^{(f)}$, where $E^{(f)}$ denotes the quadratic twist of E by f in the above mentioned sense and $f \in \mathbf{C}(C)$ is a function such that $v_{P_i}(f) \equiv 1 \bmod 2$ and $v_Q(f) \equiv 0 \bmod 2$ for all $Q \not\in \{P_i\}$.

The existence of a twist of π by $(P_1, \ldots, P_{2n})$ follows directly from the fact that $\mathrm{Pic}^0(C)$ is 2-divisible.

If we fix $2n$ points $P_1, \ldots P_{2n}$ then there exist precisely $2^{2g(C)}$ twists by $(P_i)_{i=1}^{2n}$.

If P is one of the $2n$ distinguished points, then the fiber of P changes in the following way (see [**47**, V.4]).

$$I_\nu \leftrightarrow I_\nu^* \ (\nu \geq 0) \qquad II \leftrightarrow IV^* \qquad III \leftrightarrow III^* \qquad IV \leftrightarrow II^*$$

The fiber-types with a * have a higher Euler characteristics then the fiber-type one obtains after twisting that fiber (see Remark 2.8).

After fixing a base curve C, the quantities $p_g(X)$ and $p_a(X)$ are increasing functions of the Euler characteristic of X. This motivates us to consider a special class of twists:

DEFINITION 5.2. A **-minimal twist* of π is a twist $\tilde{\pi} : \tilde{X} \rightarrow C$ such that none of the fibers are of type II^*, III^*, IV^* or I_ν^* and at most 1 fiber is of type I_0^*.

Later on we will introduce another notion of minimality: the twist for which $h^{1,1}(X) - \rho_{tr}(\pi)$ is minimal. It turns out that this is a twist which might have several *-fibers.

A *-minimal twist of an elliptic curve need not be unique for two reasons: first of all, if the *-minimal configuration contains a I_0^*-fiber one might move the I_0^*-fiber. Secondly, when fixing the points tot twist with, there are 2^{2g} possibilities for the function to twist with.

One can easily see that the configuration of the singular fibers of any two *-minimal twists of the same surface are equal.

LEMMA 5.3. *Let $\pi : X \rightarrow C$ be a Jacobian elliptic surface. Let $P_i, i = 1 \ldots 2n$ be points of C. Let $\pi' : X' \rightarrow C$ be a twist by (P_i). Then*

$$h^{1,1}(X') - \rho_{tr}(\pi') = h^{1,1}(X) - \rho_{tr}(\pi) + \sum_{i=1}^{2n} c_{P_i}$$

with

$$c_{P_i} = \begin{cases} 1 & \text{if } \pi^{-1}(P_i) \text{ is of type } I_0, IV^*, III^* \text{ or } II^*, \\ 0 & \text{if } \pi^{-1}(P_i) \text{ is of type } I_\nu \text{ or } I_\nu^*, \text{ with } \nu > 0, \\ -1 & \text{if } \pi^{-1}(P_i) \text{ is of type } II, III, IV, \text{ or } I_0^*. \end{cases}$$

PROOF. Suppose $\pi^{-1}(P_i)$ is of type I_0. Then $\pi'^{-1}(P_i)$ is of type I_0^*. The Euler characteristic of this fiber is 6, so this point causes $h^{1,1}$ to increase by 5. An I_0^* fiber has 4 components not intersecting the zero-section. Hence ρ_{tr} increases by 4.

The other fiber types can be done similarly. □

The notation $a(\pi), b(\pi), \dots, e(\pi)$ is introduced in Section 2.

LEMMA 5.4. *Given a Jacobian elliptic surface $\pi : X \to C$ with non-constant j-invariant. There exist finitely many twists π' of π such that the non-negative integer $h^{1,1} - \rho_{tr}$ is minimal under twisting. These twists are characterized by $b(\pi') = c(\pi') = 0$, i.e., there are no fibers of type II, III, IV or I_0^*.*

PROOF. It is easy to see that there are at most finitely many twists with $c(\pi') = 0$. Hence it suffices to show that $b(\pi') = c(\pi') = 0$ if $h^{1,1}(X') - \rho_{tr}(\pi')$ is minimal under twisting. From Lemma 5.3 it follows that it suffices to show that for any elliptic surface there exists a twist with $b(\pi') = c(\pi') = 0$.

Consider a *-minimal twist $\tilde{\pi} : \tilde{X} \to C$. Note that $e(\tilde{\pi}) > 0$ (otherwise the j-invariant would be constant.)

Suppose $b(\tilde{\pi}) + c(\tilde{\pi})$ is even. Twist by all points with a fiber of type II, III, IV or I_0^*. The new elliptic surface has $b = c = 0$.

Suppose $b(\tilde{\pi}) + c(\tilde{\pi})$ is odd. Twist by all points with a fibers of type II, III, IV or I_0^* and one point with a fiber of type I_ν or I_ν^*, with $\nu > 0$ (such a fiber exists because the j-invariant is not constant). The new elliptic surface has $b = c = 0$. □

REMARK 5.5. The classification (in [**67**]) of extremal (Jacobian) elliptic $K3$ surfaces is a classification of the root lattices corresponding to the singular fibers. In general one *cannot* decide which singular fibers correspond to these lattices, since each of the pairs (I_1, II), (I_2, III) and (I_3, IV) give rise to the same lattice (A_0, A_1, A_2). From the Lemma above it follows that this problem does not occur when π is extremal.

PROPOSITION 5.6. *Let $\pi : X \to C$ be a *-minimal twist with non-constant j-invariant and fundamental line bundle $\mathbf{L}$. Let $\tilde{\pi} : \tilde{X} \to C$ be a twist for which $h^{1,1} - \rho_{tr}$ is minimal. Then*

$$h^{1,1}(\tilde{X}) - \rho_{tr}(\tilde{X}) = 2g(C) - 2\deg(\mathbf{L}) - 2 + \#\{\textit{singular fibers for } \pi\}.$$

PROOF. From Proposition 2.16 and the Lemmas 5.4 and 5.3 we have

$$\deg(\tilde{\mathbf{L}}) = \deg(\mathbf{L}) + (d(\tilde{\pi}) + a(\tilde{\pi}) - c(\pi))/2, d(\tilde{\pi}) + e(\tilde{\pi}) = e(\pi), a(\tilde{\pi}) = b(\pi).$$

This yields

$$\begin{aligned} h^{1,1}(\tilde{X}) - \rho_{tr}(\tilde{X}) &= 2g(C) - 2\deg(\tilde{\mathbf{L}}) - 2 + 2(a(\tilde{\pi}) + d(\tilde{\pi})) + e(\tilde{\pi}) \\ &= 2g(C) - 2\deg(\mathbf{L}) - 2 + b(\pi) + c(\pi) + e(\pi). \end{aligned}$$

Finally note that $a(\pi) = d(\pi) = 0$. □

COROLLARY 5.7. *Let $\pi : X \to C$ be an elliptic surface with $j(\pi)$ non-constant, then π is extremal if and only if π has no fibers of type II, III, IV or I_0^* and the *-minimal twist of its Jacobian $\tilde{\pi} : \tilde{X} \to C$ has $2\deg(\tilde{\mathbf{L}}) + 2 - 2g(C)$ singular fibers.*

6. Configurations of singular fibers

In order to apply the results of the previous section, we need to know which elliptic surfaces have a *-minimal twist with $2\deg(\mathbf{L}) + 2 - 2g(C)$ singular fibers.

We need the following definition:

Definition 6.1. A function $f : C \to \mathbf{P}^1$ is called of $(3,2)$-type if the ramification indices of the points in the fiber of 0 are at most 3, and in the fiber of 1728 are at most 2.

Proposition 6.2. *Let $\pi : X \to C$ be a (Jacobian) elliptic surface with $j(\pi)$ non-constant, such that π is a *-minimal twist. Then $j(\pi)$ is of $(3,2)$-type and unramified outside $0, 1728, \infty$ if and only if there are $2\deg(\mathbf{L}) + 2 - 2g(C)$ singular fibers.*

Proof. Denote by

- n_2 the number of fibers of π of type II.
- n_3 the number of fibers of π of type III.
- n_4 the number of fibers of π of type IV.
- n_6 the number of fibers of π of type I_0^*.
- m_ν the number of fibers of π of type I_ν. ($\nu > 0$.)

Let $r = \sum \nu m_\nu$. The ramification of $j(\pi)$ is as follows (using [**47**, Lemma IV.4.1]).

Above 0 we have n_2 points with ramification index 1 modulo 3, we have n_4 point with index 2 modulo 3 and at most $(r - n_2 - 2n_4)/3$ points with index 0 modulo 3. In total, we have at most $n_2 + n_4 + (r - n_2 - 2n_4)/3$ points in $j(\pi)^{-1}(0)$.

Above 1728 we have n_3 points with index 1 modulo 2 and at most $(r - n_3)/2$ points with index 0 modulo 2. So $j(\pi)^{-1}(1728)$ has at most $n_3 + (r - n_3)/2$ points.

Above ∞ we have $\sum m_\nu$ points.

Collecting the above gives

$$\#j(\pi)^{-1}(0) + \#j(\pi)^{-1}(1728) + \#j(\pi)^{-1}(\infty) \leq \frac{2}{3}n_2 + \frac{1}{2}n_3 + \frac{1}{3}n_4 + \frac{5}{6}r + \sum m_\nu$$

with equality if and only if $j(\pi)$ is of $(3,2)$-type.

Hurwitz' formula implies that

$$r + 2 - 2g(C) \leq \#j(\pi)^{-1}(0) + \#j(\pi)^{-1}(1728) + \#j(\pi)^{-1}(\infty)$$

with equality if and only if there is no ramification outside $0, 1728$ and ∞.

So

$$r \leq 12g(C) - 12 + 4n_2 + 3n_3 + 2n_4 + 6\sum m_\nu$$

holds and Proposition 2.7 implies that

$$\sum i n_i + r = 12\deg(\mathbf{L}).$$

Substituting gives

$$2\deg(\mathbf{L}) + 2 - 2g(C) \leq n_2 + n_3 + n_4 + n_6 + \sum m_\nu = \#\{p \in C \mid \pi^{-1}(p) \text{ singular}\}$$

with equality if and only if $j(\pi)$ is unramified outside $0, 1728$ and ∞ and $j(\pi)$ is of $(3,2)$-type. □

This enables us to prove

Theorem 6.3. *Suppose $\pi : X \to C$ is an elliptic surface with non-constant j-invariant, then the following three are equivalent*

1. *π is extremal*
2. *$j(\pi)$ is of $(3,2)$-type, unramified outside 0, 1728, ∞ and π has no fibers of type II, III, IV or I_0^*.*
3. *The minimal twist π' of $\mathrm{Jac}(\pi)$ has $2\deg(\mathbf{L}) + 2 - 2g(C)$ singular fibers.*

PROOF. Apply Proposition 6.2 to Corollary 5.7. □

REMARK 6.4. Frédéric Mangolte pointed out to me that the equivalence of (1) and (2) was already proved in [**51**].

REMARK 6.5. Given a function $f : C \to \mathbf{P}^1$ of $(3,2)$-type, unramified outside 0, 1728, ∞, then there exists a Jacobian elliptic surface $\pi : X_1 \to C$, with $j(\pi) = f$. This can be obtained by taking an elliptic surface $\pi_1 : X_1 \to \mathbf{P}^1$ with $j(\pi_1) = t$.(For example one can take the elliptic surface associated to $y^2 + xy = x^3 - 36/(t-1728)x - 1/(t-1728)$.) After base changing π_1 by f, and then twisting away all the II, III, IV and I_0^* fibers gives the desired surface.

REMARK 6.6. Consider functions $f : C \to \mathbf{P}^1$ up to automorphisms of C. If we fix the ramification indices above $0, 1728, \infty$ and demand that f is unramified at any other point, then there are only finitely many f with that property. Any small deformation of f in $Mor_d(C, \mathbf{P}^1)$, the moduli space of morphisms $C \to \mathbf{P}^1$ of degree d, has more critical values.

So the j-invariants of extremal elliptic surfaces lie form a discrete set in $Mor_d(C, \mathbf{P}^1)$. From Lemma 5.4 it follows that to any j-invariant there correspond only finitely many extremal elliptic surfaces. In particular, extremal elliptic surfaces over $\mathbf{P}^1$ with geometric genus n and non-constant j-invariant, form a discrete set in the moduli space of elliptic surfaces over $\mathbf{P}^1$ with geometric genus n.

Suppose that $\pi : X \to \mathbf{P}^1$ has $2p_g(X) + 4$ singular fibers of type I_ν and no other singular fibers. Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a cyclic morphism ramified at two points P such that $\pi^{-1}(P)$ is singular.

Fastenberg ([**21**, Theorem 2.1]) proved that then the base changed surface has Mordell-Weil rank 0. In fact, she proved that the base-changed surface is extremal. The first surface is also extremal (by Proposition 5.7). A slightly more general variant is the following.

EXAMPLE 6.7. Suppose $\pi : X \to C$ is an extremal elliptic surface. Let $f : C' \to C$ be a finite morphism.

Then the base-changed elliptic surface $\pi' : X' \to C'$ is extremal if f is not ramified outside the set of points P, such that $\pi^{-1}(P)$ is multiplicative or potential multiplicative.

In that case the composition $j' : C' \to C \xrightarrow{j} \mathbf{P}^1$ is not ramified outside $0, 1728$ and ∞ and the ramification indices above 0 and 1728 are at most 3 and 2. Moreover there are no fibers of type II, III, IV or I_0^*.

An easy calculation shows that all elliptic surfaces mentioned in [**21**, Theorem 1] are either extremal elliptic surfaces or have a twist which is extremal. Moreover these surfaces have no fibers of type I_0^*, hence all elliptic surfaces for which her results hold lie discretely in the moduli spaces mentioned above.

7. Mordell-Weil groups of extremal elliptic surfaces

It remains to classify which Mordell-Weil groups can occur. To this we can give only a partial answer to this. Note first of all, that a Mordell-Weil group of an extremal elliptic surface is isomorphic to $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ with $m|n$.

Let $m, n \in \mathbf{Z}_{\geq 1}$ be such that $m|n$ and $n > 1$. Recall from Section 1 that $X(m,n)$ is the modular curve parameterizing triples $((E,O),P,Q)$, such that (E,O) is an elliptic curve, $P \in E$ is a point of order m and $Q \in E$ a point of order n.

If

$$(m,n) \notin \{(1,2),(2,2),(1,3),(1,4),(2,4)\}$$

then there exists a universal family for $X(m,n)$, which we denote by $E(m,n)$. Denote by $j_{m,n} : X(m,n) \to \mathbf{P}^1$ the map usually called j.

From the results of [**69**, Sections 4 and 5] it follows that $E(m,n)$ is an extremal elliptic surface. The following theorem explains how to construct many examples of extremal elliptic surfaces with a given torsion group.

THEOREM 7.1. *Fix $m, n \in \mathbf{Z}_{\geq 1}$ such that $m|n$ and (m,n) is not one of the pairs $(1,1)$, $(1,2)$, $(2,2)$, $(1,3)$, $(1,4)$, $(2,4)$. Let C be a (projective smooth irreducible) curve, let $j \in \mathbf{C}(C)$ be a non-constant function. Then the following are equivalent*

- *there exists a unique extremal elliptic surface $\pi : X \to C$, with $j(\pi) = j$ and the group of sections has $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ as a subgroup*
- *j is unramified outside $0, 1728, \infty$ and $j = j_{n,m} \circ g$ for some $g : C \to X_m(n)$.*

An extremal elliptic surface $\pi : X \to C$ with $j(\pi) = j$ and $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ is a subgroup of the group of section, is one of the 2^{2g} elliptic surfaces with $j(\pi) = j$ and all singular fibers are of type I_ν.

PROOF. Let $\pi : X \to C$ be an elliptic surface, such that $MW(\pi)$ has a subgroup isomorphic to $G := \mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$. Then $j : C \to \mathbf{P}^1$ can be decomposed in $g : C \to X(m,n)$ and $j_{m,n} : X(m,n) \to \mathbf{P}^1$, and X is isomorphic to $\widetilde{E(m,n)} \times_{X(m,n)} C$.

Conversely, for any base change π' of $\varphi_{m,n} : E(m,n) \to X(m,n)$, the group $MW(\pi')$ has G as a subgroup.

Moreover, since $\varphi_{m,n}$ has only singular fibers of type I_ν, the same holds for π'. An application of Theorem 6.3 concludes the proof. $\square$

REMARK 7.2. Let $n = 2$ and $m \leq 2$. Then any elliptic surface such that $j(\pi) = j_{m,n} \circ g$, has $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ as a subgroup of $MW(\pi)$.

8. Uniqueness

Artal Bartolo, Tokunaga and Zhang ([**5**]) expect that an extremal elliptic surface is determined by the configuration of the singular fibers and the Mordell-Weil group. More precise, they raise the following question:

QUESTION 8.1. *Suppose $\pi_1 : X_1 \to \mathbf{P}^1$ and $\pi_2 : X_2 \to \mathbf{P}^1$ are extremal semi-stable elliptic surfaces, such that $MW(\pi_1) \cong MW(\pi_2)$ and the configurations of singular fibers of π_1 and π_2 are the same.*

Are X_1 and X_2 isomorphic, and if so, is there then an isomorphism that respects the fibration and the zero section?

By [**50**, Theorem 5.4] this is true in the case where X_1 and X_2 are rational elliptic surfaces.

In the case where X_1 and X_2 are $K3$ surfaces the answer is the following theorem.

THEOREM 8.2. *There exists precisely 19 pairs* $(\pi_1 : X_1 \to \mathbf{P}^1, \pi_2 : X_2 \to \mathbf{P}^1)$ *of extremal elliptic* $K3$ *surfaces, such that* π_1 *and* π_2 *have the same configuration of singular fibers,* $MW(\pi_1)$ *and* $MW(\pi_2)$ *are trivial, and* X_1 *and* X_2 *are not isomorphic. Of these pairs 13 are semi-stable. There is an unique pair* $(\pi_1 : X_1 \to \mathbf{P}^1, \pi_2 : X_2 \to \mathbf{P}^1)$ *of extremal elliptic* $K3$ *surfaces, such that* π_1 *and* π_2 *have the same configuration of singular fibers,* $MW(\pi_1) = MW(\pi_2) = \mathbf{Z}/2\mathbf{Z}$ *and* X_1 *and* X_2 *are not isomorphic, which is not semi-stable.*

All configurations are listed below in Table 1.

PROOF. From [**67**, Table 2] there exist 19 pairs of surfaces (X_1, X_2) such that the transcendental lattices of X_1 and X_2 lie in distinct $SL_2(\mathbf{Z})$-orbits, they admit elliptic fibrations $\pi_i : X_i \to \mathbf{P}^1$ such that $MW(\pi_1) = MW(\pi_2) = 0$ and the contributions of the singular fibers as sub-lattices of the Néron-Severi lattice coincide. This is not yet enough to conclude that the configuration of singular fibers coincides, but from Remark 5.5 we know that for extremal elliptic surfaces the sub-lattices determine the singular fibers. Since the transcendental lattices are in different $SL_2(\mathbf{Z})$-orbits, the surfaces are not isomorphic.

The rest of the statement follows from the same Table. □

In Table 1 we list 20 (extremal) configurations of singular fibers, such that the configuration of singular fibers plus the Mordell-Weil group does not determine the $K3$ surfaces up to isomorphism. The list is complete with this property.

The column SZ-number indicates the number in the list of Shimada and Zhang, see [**67**, Table 2]. The numbers between square brackets indicate which I_ν fibers occur, all other fiber types are in the usual Kodaira notation.

In Table 2 we list the (extremal) configurations C with the property that there are at least two extremal elliptic $K3$ surfaces $\pi_i : X_i \to \mathbf{P}^1$ such that $C(\pi_i) = C$ and $MW(\pi_i) \not\cong MW(\pi_j)$ for $i \neq j$. For any other configuration of singular fibers on a (Jacobian) extremal elliptic $K3$ surface the configuration of singular fibers determines the Mordell-Weil group. In all cases, it turns out that not more than two different (finite) Mordell-Weil groups occur. The only intersection of both list is the case $[1, 1, 1, 2, 5, 14]$. There are three surfaces admitting a fibration with this configuration of singular fibers. Two of these have a trivial Mordell-Weil group, one has $\mathbf{Z}/2\mathbf{Z}$ as Mordell-Weil group.

REMARK 8.3. From these surfaces one should be able to construct other pairs of extremal elliptic surfaces with isomorphic Mordell Weil groups, and the same configuration of singular fibers, such that the geometric genus is higher then 1.

Start with two non-isomorphic extremal elliptic $K3$ surfaces with the same configuration of singular fibers and the same Mordell Weil group. Then the j-invariant of both surfaces are unequal modulo automorphism of $\mathbf{P}^1$.

We can base-change both surfaces in such a way that the base-changed surfaces remain extremal (cf. Example 6.7), they have the same configuration of singular fibers and their j-invariants are unequal modulo an automorphism of $\mathbf{P}^1$.

The configuration of singular fibers gives restrictions on the possibilities for the torsion part of the Mordell-Weil group. One can hope that this is sufficient to prove that the Mordell-Weil groups are isomorphic.

Note that the base-changed surfaces are not isomorphic, since a surface which is not a $K3$ surface has at most one elliptic fibration.

Number	SZ-number	configuration	Mordell-Weil group
1	18	$[1,2,4,5,5,7]$	0
2	23	$[1,2,3,5,6,7]$	0
3	36	$[1,1,3,5,6,8]$	0
4	40	$[1,1,2,5,7,8]$	0
5	71	$[1,1,3,3,5,11]$	0
6	73	$[1,1,2,4,5,11]$	0
7	74	$[1,1,2,3,6,11]$	0
8	75	$[1,1,1,4,6,11]$	0
9	77	$[1,1,1,3,7,11]$	0
10	78	$[1,1,1,2,8,11]$	0
11	92	$[1,1,2,2,5,13]$	0
12	100	$[1,1,1,2,5,14]$	0
13	109	$[1,1,1,2,2,17]$	0
14	120	$[1,2,6,7]+I_1^*$	0
15	125	$[1,2,5,9]+I_1^*$	0
16	134	$[1,1,2,13]+I_1^*$	0
17	148	$[1,2,3,10]+I_2^*$	$\mathbf{Z}/2\mathbf{Z}$
18	227	$[2,3,4,7]+IV^*$	0
19	267	$[1,2,5,7]+III^*$	0
20	277	$[1,1,2,11]+III^*$	0

TABLE 1. Configurations of singular fibers not determining the surface.

Number	SZ-number	configuration	Mordell-Weil groups	
1	26	$[1,1,3,3,8,8]$	0	$\mathbf{Z}/2\mathbf{Z}$
2	42	$[1,1,2,2,9,9]$	0	$\mathbf{Z}/3\mathbf{Z}$
3	54	$[1,1,1,1,10,10]$	0	$\mathbf{Z}/5\mathbf{Z}$
4	62	$[1,1,1,6,6,10]$	0	$\mathbf{Z}/2\mathbf{Z}$
5	83	$[1,1,3,3,4,12]$	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/6\mathbf{Z}$
6	84	$[1,2,2,3,4,12]$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$
7	87	$[1,1,2,2,6,12]$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/6\mathbf{Z}$
8	96	$[1,1,2,3,3,14]$	0	$\mathbf{Z}/2\mathbf{Z}$
9	100	$[1,1,1,2,5,14]$	0	$\mathbf{Z}/2\mathbf{Z}$
10	103	$[1,1,2,2,3,15]$	0	$\mathbf{Z}/3\mathbf{Z}$
11	107	$[1,1,1,2,3,16]$	0	$\mathbf{Z}/2\mathbf{Z}$
12	111	$[1,1,1,1,2,18]$	0	$\mathbf{Z}/3\mathbf{Z}$
13	241	$[1,1,2,12]+IV^*$	0	$\mathbf{Z}/3\mathbf{Z}$
14	276	$[1,1,3,10]+III^*$	0	$\mathbf{Z}/2\mathbf{Z}$

TABLE 2. Configurations of singular fibers not determining the Mordell-Weil group.

REMARK 8.4. (**"[5]-case 49"**) In [**5**] it is proven that there are two non-isomorphic elliptic surfaces $\pi : X_i \to \mathbf{P}^1$ with $MW(\pi_i) = \mathbf{Z}/5\mathbf{Z}$ and singular fibers $2I_1, I_2, 2I_5, I_{10}$, where isomorphic means that the isomorphism respects the fibration. In [**67**] it is proven that X_1 and X_2 are isomorphic as surfaces.

This case is a bit special: In the same paper it is proven that for any other pair of (semi-stable) extremal elliptic $K3$ surfaces with $\#MW(\pi) > 4$ and the same singular fibers configuration, there exists an isomorphism which respects the fibration. ([**5**, Theorem 0.4])

9. Extremal elliptic surfaces with $p_g = 1, q = 1$

A Jacobian elliptic surface, not a product, with $q = 1$ needs to have a genus 1 base curve. This implies that for an extremal elliptic surface with $p_g = 1, q = 1$ we have $\deg(\mathbf{L}) = 1$.

The minimal twist of an extremal elliptic surface with $p_g = 1, q = 1$ has two singular fibers.

All possible pairs of fiber types such that the sum of the Euler characteristics is 12, are given in the following table.

I_{11} I_1	I_{10} I_2	I_9 I_3	I_8 I_4	I_7 I_5	I_6 I_6	I_{10} II	I_9 III	I_8 IV	I_6 I_0^*
	[**4**]	[**3**]	[**1**]		[**2**]				

Several of these surfaces are already described in the literature. See the above mentioned references. We prove in this section that all these possibilities actually occur except for $I_7\ I_5$.

REMARK 9.1. Note that the 6 configurations with two I_ν fibers are already extremal. The configurations with one additive and one multiplicative fiber are not extremal. In the case $I_6\ I_0^*$ an extremal twist has one singular fiber of type I_6^* and $\deg \mathbf{L} = 1$. All other combinations of one additive and one multiplicative fiber have as extremal twist an elliptic surface, such that the degree of $\mathbf{L}$ is 2.

PROPOSITION 9.2. *All these configurations occur as total configuration of singular fibers on a Jacobian elliptic surface with genus 1 base curve, except $I_7\ I_5$.*

PROOF. This follows from the following lemmas. Note that the existence of elliptic surfaces over $\mathbf{P}^1$ with the below mentioned singular fibers follows from [**54**].

LEMMA 9.3. *The configurations with $I_k\ I_{12-k}$ occur for $k = 2, 4, 6$.*

PROOF. Let $\pi : X \to \mathbf{P}^1$ be an elliptic surface with two III fibers, a fiber of type $I_{k/2}$ and a fiber of type $I_{6-k/2}$, and no other singular fibers. Let $\varphi : C \to \mathbf{P}^1$ be a degree two cover ramified at the four points where the fiber of π is singular. Then $\pi' : \widetilde{X \times_{\mathbf{P}^1} C} \to C$ has two fibers of type I_0^*, a fiber of type I_k and a fiber of type I_{12-k}. Twisting by the two points with I_0^* fibers gives the desired configuration. □

LEMMA 9.4. *The configurations $I_8\ IV$ and $I_6\ I_0^*$ occur.*

PROOF. For the first, let $\pi : X \to \mathbf{P}^1$ be an elliptic surface with two III fibers, a fiber of type II and a fiber of type I_4, and no other singular fibers. Let $\varphi : C \to \mathbf{P}^1$ be a degree two cover ramified at the four points where the fiber of π is singular. Then

$\pi' : \widetilde{X \times_{\mathbf{P}^1} C} \to C$ has two fibers of type I_0^* a fiber of type I_8 and a fiber of type IV. Twisting by the two points with a I_0^* fiber gives the desired configuration.

For the second, let $\pi : X \to \mathbf{P}^1$ be an elliptic surface with three III fibers and a fiber of type I_3, and no other singular fibers. Let $\varphi : C \to \mathbf{P}^1$ be a degree two cover ramified at the four points where the fiber of π is singular. Then $\pi' : \widetilde{X \times_{\mathbf{P}^1} C} \to C$ has three fibers of type I_0^* and a fiber of type I_6. Twisting by two points with a I_0^* fiber gives the desired configuration. □

For the other four configurations we have a different strategy. We simply show that the j-map with the right ramification indices exists. This is equivalent to give the monodromy representation.

Lemma 9.5. *The configurations I_{11} I_1, I_9 I_3, II I_{10}, III I_9 occur.*

Proof. For the two configurations of type I_μ I_ν we need to find curves C and functions $j : C \to \mathbf{P}^1$ of degree 12 such that above ∞ there are two point with ramification indices μ and ν, all points above 0 have ramification index 3 and all points above 1728 have ramification index 2. (see [**47**, Lemma IV.4.1].)

By the Riemann existence theorem it suffices to give two permutations σ_0, σ_1 in S_{12}, such that σ_0 is the product of 6 disjoint 2-cycles, σ_1 the product of 4 disjoint 3-cycles, and $\sigma_0\sigma_1$ is a product of a μ-cycle and a ν cycle, and the subgroup generated by σ_0 and σ_1 is transitive. (See [**49**, Corollary 4.10].)

For I_1 I_{11}, we use

$$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12) * (1\ 3)(2\ 4)(5\ 7)(8\ 10)(9\ 11)(6\ 12) = (3\ 2\ 5\ 8\ 11\ 7\ 6\ 10\ 9\ 12\ 4)$$

For I_3 I_9, we use

$$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12) * (1\ 6)(4\ 9)(3\ 7)(2\ 10)(5\ 12)(8\ 11)(1\ 4\ 7)(2\ 11\ 9\ 5\ 10\ 3\ 8\ 12\ 6)$$

Similarly, the existence of II I_{10}, follows from

$$(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10) * (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9) = (1\ 3\ 5\ 7\ 2\ 6\ 10\ 9\ 4\ 8)$$

and the existence of III I_9, follows from

$$(2\ 3)(4\ 5)(6\ 7)(8\ 9) * (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9) = (1\ 5\ 9\ 2\ 4\ 6\ 8\ 3\ 7).$$

□

Lemma 9.6. *The configuration I_7 I_5 does not occur.*

Proof. A computer search learned us that the permutations needed for the existence of I_7 I_5 do not exist. □

□

Corollary 9.7. *Let k_i be positive integers such that $\sum k_i = 12$, with $i \geq 2$, and if $i = 2$ then $(k_1, k_2) \neq (7, 5)$ or $(5, 7)$. Then there exist a curve C of genus 1, and an elliptic surface $\pi : X \to C$ such that the configuration of singular fibers of π is $\sum I_{k_i}$.*

Proof. Use the monodromy representation as in [**48**, Remark after Corollary 3.5]. □

10. Elliptic surfaces with one singular fiber

In the previous section we proved that there exists an elliptic surface with an I_6 and an I_0^* fiber. Twisting by the points with a singular fiber yields an elliptic surface with one singular fiber, of type I_6^*.

In this section we will prove that for fixed $\deg(\mathbf{L})$ there are exactly two possible configurations of one singular fibers that can be realized as an elliptic surface.

PROPOSITION 10.1. *Suppose $\pi : X \to C$ is an elliptic surface with one singular fiber. The fiber is of type I_{12k-6}^* or I_{12k} for some $k \in \mathbf{Z}_{>0}$, and $g(C) \geq k$ in the first case and $g(C) \geq k+1$ in the second. Conversely, for each $k > 0$ there exists elliptic surface $\pi_k : X_k \to C$ and $\pi'_k : X'_k \to C$, such that π_k and π'_k have precisely one singular fiber. Moreover the singular fiber of π_k is of type I_{12k} and the singular fiber of π'_k is of type I_{12k-6}^*.*

PROOF. Since the Euler characteristic of the singular fiber is $12\deg(\mathbf{L})$, the only possible configurations are $I_{12\deg(\mathbf{L})}$ and $I_{12\deg(\mathbf{L})-6}^*$.

Fix a rational elliptic surface $\pi : X \to \mathbf{P}^1$ with 3 fibers of type III, and one fiber of type I_3. (The existence follows from [**54**].)

Fix k a positive integer. Take a curve C such that $\varphi : C \to \mathbf{P}^1$, has degree $4k-2$, and is ramified at the four points which have the singular fibers, and above such a point there is exactly one point.

The base change $\pi' : \widetilde{X \times_{\mathbf{P}^1} C} \to C$ has 3 fibers of type I_0^* and one fiber of type I_{12k-6}. Twisting by all four points with a singular fiber yields an elliptic surface with one singular fiber and this fiber is of type I_{12k-6}^*.

If we replace $4k-2$ by $4k$, we obtain an elliptic surface with one fiber of type I_{12k}.

For any elliptic surface with only one singular fiber and that fiber is of type I_{12k-6}^*, we have the following: the j-map $C \to \mathbf{P}^1$ has degree $12k-6$, one point above ∞, at most $4k-2$ points above 0, and at most $6k-3$ points above 1728. This implies that the base curve has genus at least k. A similar argument shows that in the case that the only singular fiber is of type I_{12k}, the base curve has genus at least $k+1$. □

CHAPTER 2

Higher Noether-Lefschetz loci of elliptic surfaces

1. Introduction

Let $\mathcal{M}_n$ be the coarse moduli space of Jacobian elliptic surfaces $\pi : X \to \mathbf{P}^1$ over $\mathbf{C}$, such that the geometric genus of X equals $n-1$ and π has at least one singular fiber. It is known that $\dim \mathcal{M}_n = 10n - 2$ (see [**46**]). By $\rho(X)$ we denote the Picard number of X. It is well known that for an elliptic surface with a section we have that $2 \leq \rho(X) \leq h^{1,1} = 10n$.

Fix an integer $r \geq 2$, then in $\mathcal{M}_n$ one can study the loci

$$NL_r := \{[\pi : X \to \mathbf{P}^1] \in \mathcal{M}_n \mid \rho(X) \geq r\}.$$

We call these loci *higher Noether-Lefschetz loci*, in analogy with [**18**]. One can show that NL_r is a countable union of Zariski closed subsets of $\mathcal{M}_n$. This follows from the explicit description of $NS(X)$ for a Jacobian elliptic surface $\pi : X \to \mathbf{P}^1$.

The aim of this chapter is to study the dimension of NL_r.

THEOREM 1.1. *Suppose $n \geq 2$. For $2 \leq r \leq 10n$, we have*

$$\dim NL_r \geq 10n - r = \dim \mathcal{M}_n - (r-2).$$

Moreover, we have equality when we intersect NL_r with the locus of elliptic surfaces with non-constant j-invariant.

The fact that the locus of elliptic surfaces with constant j-invariant has dimension $6n-3$ implies

COROLLARY 1.2. *Suppose $n \geq 2$. For $2 \leq r \leq 4n+3$, we have*

$$\dim NL_r = 10n - r.$$

Since the classes of the image of the zero-section and of a general fiber give rise to two independent classes in $NS(X)$ we have that $NL_2 = \mathcal{M}_n$, proving Theorem 1.1 for the case $r = 2$. For $r = 3$ the result was proven by Cox ([**18**]). If $n = 2$ then we are in the case of $K3$ surfaces, and the above results follow from general results on the period map. In fact, for $K3$ surfaces we have that $\dim NL_r = 20 - r$, for $2 \leq r \leq 20$.

Suppose $\pi : X \to \mathbf{P}^1$ is an elliptic surface with $p_g(X) > 1$, not birational to a product, then we obtain by the Shioda-Tate formula (see Theorem 1.2.11) that the rank of the Mordell-Weil group of π, which we denote by $MW(\pi)$ is at most $\rho(X) - 2$. From this and Theorem 1.1 we obtain

COROLLARY 1.3. *Suppose $n \geq 2$. Let*

$$MW_r := \{[\pi : X \to \mathbf{P}^1] \in \mathcal{M}_n \mid \operatorname{rank} MW(\pi) \geq r\}.$$

Let $U := \{[\pi : X \to \mathbf{P}^1] \mid j(\pi)$ non-constant $\}$. Then for $0 \leq r \leq 10n-2$ we have

$$\dim MW_r \cap U \leq 10n - r - 2.$$

Cox [**18**] proved that $\dim MW_1 = 9n - 1$, which is actually a stronger result than Corollary 1.3 for the special case $r = 1$.

The proof of Theorem 1.1 consists of two parts. In the first part we construct elliptic surfaces with high Picard number. This is done by constructing large families of elliptic surfaces such that the singular fibers have many components. To calculate the dimension of the locus of this type of families, we study the ramification of the j-map, and calculate the dimension of several Hurwitz spaces. This yields $\dim NL_r \geq 10n - r$.

The second part consists of proving that $\dim NL_r \cap U \leq 10n - r$. We choose a strategy similar to what M.L. Green [**25**] uses in order to identify the components of maximal dimension in the Noether-Lefschetz locus in the case of surfaces of degree d in $\mathbf{P}^3$. In order to apply this strategy we consider an elliptic surface over $\mathbf{P}^1$ with a section as a surface Y in the weighted projective space $\mathbf{P}(1, 1, 2n, 3n)$ with $n = p_g(X) + 1$. To obtain Y, we need to contract the zero-section and all fiber components not intersecting the zero-section. Then we use Griffiths' and Steenbrink's identification of the Hodge filtration on $H^2(Y, \mathbf{C})$ with graded pieces of the Jacobi-ring of Y (the coordinate ring of $\mathbf{P}$ modulo the ideal generated by the partials of the defining equation of Y). Using some results from commutative algebra we can calculate an upper bound for the dimension of $NL_r \cap U$.

We would like to point out an interesting detail: the classical Griffiths-Steenbrink identification holds under the assumption that Y is smooth outside the singular locus of the weighted projective space. In our case it might be that Y has finitely many rational double points outside the singular locus of the weighted projective space. Recently, Steenbrink ([**75**]) obtained a satisfactory identification in the case that Y has "mild" singularities.

When we consider elliptic surfaces with constant j-invariant 0 or 1728, the theory becomes a little more complicated: several families of elliptic surfaces over $\mathbf{P}^1$ with a section and constant j-invariant 0 or 1728 and generically Picard number ρ have a codimension one subfamily of surfaces with Picard number $\rho + 2$. It turns out that for certain values of $r \geq 8n$, such families prevent us from proving the equality $\dim NL_r = 10n - r$. These surfaces have other strange properties. For the same reason as above, we can produce examples not satisfying several Torelli type theorems (see Theorem 1.4.8). These surfaces are also the elliptic surfaces with larger Kuranishi families than generic elliptic surfaces. Actually, the difference between the dimension of NL_r and $10n - r$ equals the difference between the dimension of the Kuranishi family of a generic elliptic surface in NL_r and the dimension of the Kuranishi family of a generic elliptic surface X over $\mathbf{P}^1$, with $p_g(X) = n - 1$ and admitting a section.

The organization of this chapter is as follows. In Section 2 we calculate the dimension of a Hurwitz space. In Section 3 we study configurations of singular fibers. In Section 4 we use the results of the previous two sections to identify several components in NL_r of dimension $10n - r$. In Section 5 we study the locus in $\mathcal{M}_n$ of elliptic surfaces with constant j-invariant. In Section 6 we study elliptic surfaces with j-invariant 0 or 1728. We use these surfaces to identify components L of NL_r such that $\dim L + \rho(X) > 10n - r$. In Section 7 we prove that the identified components are of maximum dimension in NL_r. This is done by applying a modified version of the Griffiths-Steenbrink identification of the Hodge structure of hypersurfaces with several graded pieces of the Jacobi-ring. In Section 8 some remarks are made and some open questions are raised.

2. Dimension of Hurwitz Spaces

In this section we calculate the dimension of several Hurwitz spaces.

Definition 2.1. Let C_1 and C_2 be curves. Two morphisms $\varphi_i : C_i \to \mathbf{P}^1$ are called isomorphic, if there exists an isomorphism $\psi : C_1 \to C_2$ such that $\varphi_1 = \varphi_2 \circ \psi$.

Definition 2.2. Let $m > 2$ be an integer. Fix m distinct points $P_i \in \mathbf{P}^1$. Let $\mathcal{H}(\{e_{i,j}\}_{i,j})$ be the Hurwitz space (coarse moduli space) of isomorphism classes of semi-stable morphisms $\mathbf{P}^1 \to \mathbf{P}^1$ of degree d such that $\varphi^* P_i = \sum_j e_{i,j} Q_j$, with $Q_{j'} \neq Q_j$ for $j' \neq j$. (From here on we say that the ramification indices over the P_i are the $e_{i,j}$.)

Remark 2.3. By a semi-stable function we mean the following. All functions of degree d can be parameterized by an open set in $\mathbf{P}^{2d+1}$ by sending a point $[x_0 : x_1 : \cdots : x_{2d+1}]$ to the morphism induced by the function $t \mapsto (x_0 + x_1 t + \cdots + x_d t^d)/(x_{d+1} + x_{d+2} t + \cdots + x_{2d+1} t^d)$. It is not hard to write down a finite set of equations and inequalities, such that every solution corresponds to a function with the required ramification behavior over the P_i. To obtain $\mathcal{H}(\{e_{i,j}\}_{i,j})$ one needs to divide out by the action of the reductive group $\mathrm{Aut}(\mathbf{P}^1) = \mathrm{PGL}_2$. To obtain a good quotient we might need to restrict ourselves to the smaller open subset of so-called 'semi-stable' elements.

Remark 2.4. Note that morphisms corresponding to points of $\mathcal{H}$ might be ramified outside the P_i.

Remark 2.5. If $m > 3$ then $\mathcal{H}(\{e_{i,j}\}_{i,j})$ depends on the points P_i. We will prove in this section that its dimension is independent of the choice of the P_i.

Definition 2.6. Fix m points $P_i \in \mathbf{P}^1$. Let $\varphi : \mathbf{P}^1 \to \mathbf{P}^1$ be a function. Let $n(P) := \deg(\varphi) - \#\varphi^{-1}(P)$. Let $B_{\{P_i\},\varphi}$ be the divisor $\sum_{P \notin \{P_i\}} n(P) P$.

Remark 2.7. One might consider $B_{\{P_i\},\varphi}$ as the push-forward of the ramification divisor of φ outside the pre-images of the P_i. We have almost by definition that $B_{\{P_i\},\varphi} = B_{\{P_i\},\psi}$ if $\psi \cong \varphi$. In Proposition 2.10 we prove that the dimension of $\mathcal{H}(\{e_{i,j}\}_{i,j})$ equals the degree of $B_{\{P_i\},\varphi}$, for any morphism φ corresponding to a point in $\mathcal{H}(\{e_{i,j}\})$.

Recall the following special form of the Riemann existence theorem.

Proposition 2.8. *Fix $m > 2$ points $R_i \in \mathbf{P}^1$. Fix a positive integer d. Fix partitions of d of the form $d = \sum_j^{k_i} e_{i,j}$, for $i = 1, \dots, m$. Let $q = \sum k_i$. Assume that $q = (m-2)d + 2$. Then we have a correspondence (the so-called monodromy representation) between*

- *Isomorphism classes of morphisms $\varphi : \mathbf{P}^1 \to \mathbf{P}^1$ such that the ramification indices over the R_i are the $e_{i,j}$.*
- *Congruence classes of transitive subgroups of S_d (the symmetric group on d letters) generated by $\sigma_i, i = 1, \dots, m$, such that the lengths of the cycles of σ_i are the $e_{i,j}, j = 1, \dots k_j$ and $\prod \sigma_i = 1$.*

Proof. In [**49**, Corollary 4.10] the above equivalence is proven, except that they consider all morphisms $C \to \mathbf{P}^1$ with given ramification indices, and they do not assume $q = (m-2)d + 2$. Hence we need to show that $g(C) = 0$ is equivalent to $q = (m-2)d + 2$. The condition $g(C) = 0$ is equivalent by Hurwitz' formula to

$$-2 = -2d + \sum_{Q \in \mathbf{P}^1} e_Q(\varphi) - 1,$$

where $e_Q(\varphi)$ is the ramification index of φ at Q. The exposed formula is equivalent to $q = (m-2)d + 2$, which yields the claim. □

The following Corollary is immediate.

COROLLARY 2.9. *The dimension of* $\mathcal{H}(\{e_{i,j}\}_{i,j})$ *is independent of the choice of the* P_i.

The above results enable us to calculate the dimension of the Hurwitz scheme.

PROPOSITION 2.10. *Fix a positive integer* d*, and* $m > 2$ *partitions of* d *of the form* $d = \sum_{j=1}^{k_i} e_{i,j}$*,* $i = 1, 2, \ldots, m$*. Let* $q = \sum k_i$*. The dimension of* $\mathcal{H}(\{e_{i,j}\}_{i,j})$ *is* $q-(m-2)d-2$ *provided that* $\mathcal{H}(\{e_{i,j}\}_{i,j})$ *is not empty.*

PROOF. Let $r : \mathcal{H}(\{e_{i,j}\}_{i,j}) \to \mathrm{Div}^{q-(m-2)d-2}\,\mathbf{P}^1$ be the morphism sending an isomorphism class $[f]$ to $B_{\{P_i\},f}$. From Proposition 2.8 it follows that r has finite fibers.

Using the relations

$$(1\ 2\ \ldots\ n) = (1\ 2\ \ldots\ k-1)(k-1\ \ k)(k\ \ldots\ n)$$

and Proposition 2.8 one can show that there is a function φ corresponding to a point of the Hurwitz' space $\mathcal{H}(\{e_{i,j}\}_{i,j})$ such that above every critical value, different from the P_i, there is exactly one ramification point Q, and that $e_Q \leq 2$. This implies that the image of r contains a divisor $Q_1 + Q_2 + \ldots Q_{q-(m-2)d-2}$ with $Q_i \neq Q_j$ if $i \neq j$ and $Q_i \neq P_j$, for all i, j. From Proposition 2.8 it follows that every point $T_1 + T_2 + \cdots + T_{q-(m-2)d-2}$, with $T_i \neq T_j$ if $i \neq j$ and $T_i \neq P_j$ for all i, j, is in the image of r. Hence the dimension of the image of r is $q - (m-2)d - 2$, which yields the proof. □

REMARK 2.11. The inequality $\dim \mathcal{H}(\{e_{i,j}\}_{i,j}) \geq q - (m-2)d - 2$ can also be proven using a parameter-equation count.

If one works over fields of characteristic p then one can define similarly Hurwitz' spaces of separable functions with fixed ramification indices. If one works over algebraically closed fields of positive characteristic the lower bound $\dim \mathcal{H}(\{e_{i,j}\}_{i,j}) \geq q - (m-2)d - 2$ holds by a similar parameter-equation count.

The following corollary tells us that if we know the ramification indices modulo some integers N_i, and for one choice of the ramification indices, the associated Hurwitz space is non-empty, then the same holds for the Hurwitz space associated to the minimal choice of ramification indices, and the Hurwitz space associated to that particular choice is the largest one.

COROLLARY 2.12. *Let* m, d *be positive integers. Fix* m *integers* N_i *such that* $N_i \leq d$*. Let* $a_{i,j}$ *be integers such that* $1 \leq a_{i,j} < N_i$*, and* $r_i N_i + \sum_{j=1}^{s_i} a_{i,j} = d$*, with* r_i *a non-negative integer. Fix* m *points* P_i *on* $\mathbf{P}^1$.

For all $i = 1, \ldots m$*, set*

$$e_{i,j} = \begin{cases} a_{i,j} & 1 \leq j \leq s_i, \\ N_i & s_i + 1 \leq j \leq s_i + r_i. \end{cases}$$

Suppose there exist m *partitions* $d = \sum_j^{s'_i} e'_{i,j}$ *such that* $s'_i \leq s_i$ *and* $e'_{i,j} \equiv e_{i,j} \bmod N_i$ *if* $1 \leq j \leq s'_i$*. Then* $\dim \mathcal{H}(\{e'_{i,j}\}_{i,j}) \leq \dim \mathcal{H}(\{e_{i,j}\}_{i,j})$ *holds.*

PROOF. Using Proposition 2.8 and the relations of permutations mentioned in the proof of Proposition 2.10 we obtain that if $\mathcal{H}(\{e'_{i,j}\}_{i,j})$ is non-empty then $\mathcal{H}(\{e_{i,j}\}_{i,j})$ is non-empty. Now apply Proposition 2.10. □

3. Configuration of singular fibers

Fix some $n \geq 2$. In this section we calculate the dimension of the locus in $\mathcal{M}_n$ corresponding to elliptic surfaces with a fixed configuration of singular fibers, containing a fiber of type I_ν or I_ν^*, with $\nu > 0$. For more on this see also [**47**, Lectures V and X] or Sections 1.5 and 1.6.

Definition 3.1. A *configuration of singular fibers* of an elliptic surface is a formal sum C of Kodaira types of singular fibers, with non-negative integer coefficients.

Let $i_\nu(C)$ denote the coefficient of I_ν in C. Define $ii(C)$, $iii(C)$, $iv(C)$, $iv^*(C)$, $iii^*(C)$, $ii^*(C)$ and $i_\nu^*(C)$ similarly.

A configuration C satisfies *Noether's condition* if

$$\sum_{\nu>0} \nu i_\nu + \sum_{\nu\geq 0} (\nu+6) i_\nu^* + 2ii + 3iii + 4iv + 8iv^* + 9iii^* + 10ii^* = 12n(C)$$

with $n(C)$ a positive integer.

To an elliptic surface $\pi : X \to \mathbf{P}^1$ corresponding to a point in $\mathcal{M}_n$ we can associate its (total) configuration of singular fibers $C(\pi)$. Then $C(\pi)$ satisfies Noether's condition, with $n(C(\pi)) = n$.

Assumption 3.2. *For the rest of the section, let C be a configuration of singular fibers satisfying Noether's condition with $n(C) = n$ and containing at least one fiber of type I_ν or I_ν^*, with $\nu > 0$.*

Lemma 3.3. *Suppose there exists a morphism $\varphi : \mathbf{P}^1 \to \mathbf{P}^1$, such that the ramification indices are as follows:*

- *Above 0*
  - *there are precisely $ii(C) + iv^*(C)$ points with ramification indices 1 modulo 3 and*
  - *there are precisely $iv(C) + ii^*(C)$ points with ramification indices 2 modulo 3.*
- *Above 1728 there are precisely $iii(C) + iii^*(C)$ points with ramification indices 1 modulo 2.*
- *Above ∞ there are for every $\nu > 0$ precisely $i_\nu(C) + i_\nu^*(C)$ points with ramification index ν.*

Then there exists an elliptic surface such that $C(\pi) = C$.

Conversely, if there exists an elliptic surface with $C(\pi) = C$, then $j(\pi)$ satisfies the above mentioned conditions.

Proof. The last part of the statement follows from [**47**, Lemma IV.4.1].

To prove the existence of π: Let $\pi_1 : X_1 \to \mathbf{P}^1$ be an elliptic surface with $j(\pi_1) = t$. (For example one can take the elliptic surface associated to $y^2 + xy = x^3 - 36/(t-1728)x - 1/(t-1728)$.)

Let X_2 be the base-change of X_1 and $\mathbf{P}^1$ with respect to φ and π_1. Let π_2 be the induced elliptic fibration on X_2. Then $i_\nu(C(\pi_2)) + i_\nu^*(C(\pi_2)) = i_\nu(C) + i_\nu^*(C)$, for $\nu > 0$, and $ii(C(\pi_2)) + iv^*(C(\pi_2)) = ii(C) + iv^*(C)$, and similar relations for (iii, iii^*) and (iv, ii^*).

It is easy to see that there exists a twist π_3 of π_2, changing all *-fibers in non-*-fibers, such that $C(\pi_3) - C = \epsilon I_0^*$, with $\epsilon \in \{0, 1\}$. Since both configurations satisfy Noether's condition, it follows that $\epsilon = 0$. Hence π_3 is the desired Jacobian elliptic surface. □

LEMMA 3.4. *Assume that there exists an elliptic surface* $\pi' : X' \to \mathbf{P}^1$ *with* $C(\pi') = C$, *then*

$$\dim\{[\pi : X \to \mathbf{P}^1] \in \mathcal{M}_{n(C)} \mid j(\pi) = j(\pi'), C(\pi) = C\} = i_0^*(C).$$

PROOF. Fix one $\pi_0 : X_0 \to \mathbf{P}^1$, with $C(\pi_0) = C$ and $j(\pi_0) = j(\pi')$.

Fix $i_0^*(C)$ points $P_i \in \mathbf{P}^1$, none of them in $j(\pi_0)^{-1}(\{0, 1728, \infty\})$, such that $\pi^{-1}(P_i)$ is smooth for all i. Let Q_i be the points over which the fiber of π is of type I_0^*. Then twisting π by the points $\{P_i, Q_i\}$ gives an elliptic surface π with $j(\pi) = j(\pi')$ and $C(\pi) = C$ (see Section 1.5). If two such twists are isomorphic then the set of points $\{P_i\}$ are the same. So

$$\dim\{[\pi : X \to \mathbf{P}^1] \in \mathcal{M}_{n(C)} | j(\pi) = j(\pi'), C(\pi) = C\} \geq i_0^*(C).$$

If one fixes the points over which the fibers are of type I_0^*, then there are only finitely many twists with the same configuration of singular fibers, proving equality. □

LEMMA 3.5. *Assume that there exists an elliptic surface* $\pi' : X' \to \mathbf{P}^1$ *with* $C(\pi') = C$. *Then the constructible locus* $L(C)$ *in* $\mathcal{M}_{n(C)}$ *corresponding to all elliptic surfaces with* $C(\pi) = C(\pi')$ *has dimension*

$$\#\textit{singular fibers} + \#\{\textit{fibers of type } II^*, III^*, IV^*, I_\nu^*\} - 2n(C) - 2.$$

PROOF. From the above lemmas it follows that $L(C)$ is a finite union of Zariski open subsets U_i in line bundles L_i of rank $i_0^*(C)$ over some $\mathcal{H}(\{e_{i,j}\}_{i,j})$. This proves the constructibility of $L(C)$.

Combining the above two lemmas with Corollary 2.12 gives that the dimension of the locus $L(C)$ is

$$i_0^* + q - d - 2,$$

where $d = \sum_\nu \nu(i_\nu + i_\nu^*)$ is the degree of $j(\pi)$ and

$$q = ii + iv^* + iv + ii^* + (d - ii - iv^* - 2iv - 2ii^*)/3 + iii + iii^* + (d - iii - iii^*)/2 + \sum_{\nu>0}(i_\nu + i_\nu^*).$$

This yields

$$\begin{aligned} q - d - 2 &= \frac{1}{12}\left(8ii + 8iv^* + 6iii + 6iii^* + 4iv + 4ii^* + 12\sum_{\nu>0}(i_\nu + i_\nu^*) - 2d - 24\right) \\ &= \sum_{\nu>0}(i_\nu + 2i_\nu^*) + ii + iii + iv + i_0^* + 2iv^* + 2iii^* + 2ii^* - 2 - 2n, \end{aligned}$$

where we used Noether's condition. This implies the lemma. □

PROPOSITION 3.6. *Let* C *be a configuration of singular fibers, containing at least one* I_ν *or* I_ν^**-fiber* ($\nu > 0$) *and such that there exists an elliptic surface* $\pi' : X' \to \mathbf{P}^1$ *with* $C(\pi') = C$. *Then*

$$\dim\{[\pi : X \to \mathbf{P}^1] \in \mathcal{M}_{n(C)} | C(\pi) = C\} = h^{1,1}(X') - \rho_{tr}(\pi) - \#\{\textit{fibers of type } II, III \textit{ or } IV\}.$$

PROOF. Apply

$$h^{1,1}(X') - \rho_{tr}(\pi) = 2n(C) - 2 - \#\{\text{multiplicative fibers}\} - 2\#\{\text{additive fibers}\}$$

(from e.g. Proposition 1.2.16) to Lemma 3.5. □

4. The lower bound

In this section we prove a lower bound for the dimension of NL_r.

THEOREM 4.1. *Let r be integer such that $2 \leq r \leq 10n$. Let L_r be the (constructible) locus of Jacobi elliptic surfaces in $\mathcal{M}_n$ such that $\rho_{tr} \geq r$ and the j-invariant is non-constant. Then*

$$\dim L_r = 10n - r.$$

PROOF. It suffices to prove for every $2 \leq r \leq 10n$ that there exists an elliptic surfaces without II, III and IV fibers, such that $\rho_{tr}(\pi) = r$. From Proposition 3.6 it follows that such a surface lies on a component of L_r of dimension $10n - r$. We construct some elliptic surfaces $\pi_{n',r'} : X_{n',r'} \to \mathbf{P}^1$ such that $\rho_{tr}(\pi_{n',r'}) = r'$, $p_g(X_{n',r'}) = n' - 1$ and π' has no fibers of type II, III, IV.

Fix a Jacobian rational elliptic surface $\pi_{1,10} : X_{1,10} \to \mathbf{P}^1$ with four singular fibers such that all fibers are multiplicative. (See [**9**] for the existence of such surfaces. Up to isomorphism there exist six of these surfaces.)

Let $\pi_{n,10n}$ be a cyclic base change of degree n of $\pi_{1,10}$ ramified at two points where the fibers are singular.

Since $\pi_{1,10} : X_{1,10} \to \mathbf{P}^1$ satisfies $\rho_{tr}(\pi_{1,10}) = h^{1,1}(X_{1,10})$, we obtain by Example 1.6.7

$$\rho_{tr}(\pi_{n,10n}) = h^{1,1}(X_{n,10n}).$$

By the deformation of the j-map ([**48**, Remark after Corollary 3.5]) we can construct an elliptic surface $\pi_{n,r} : X \to \mathbf{P}^1$ with $2n + 2 + (10n - r)$ singular fibers, all multiplicative, where r is an integer between 2 and $10n$. Such a surface has $\rho_{tr} = r$. This finishes the proof. □

COROLLARY 4.2. *Let r be an integer such that $2 \leq r \leq 10$. Then*

$$\dim NL_r \geq 10n - r.$$

Another consequence of Theorem 4.1 is the following:

COROLLARY 4.3. *Let $MK3$ be the moduli space of algebraic $K3$ surfaces. Let $2 \leq r \leq 20$ be an integer. Let S_r be the locus in $MK3$ corresponding to $K3$ surfaces with $\rho(X) \geq r$. Then*

$$\dim S_r \geq 20 - r.$$

PROOF. It is well-known that a Jacobian elliptic surface $\pi : X \to \mathbf{P}^1$ with $p_g(X) = 1$, is a $K3$ surface. Hence there is a morphism $\mathcal{M}_2 \to MK3$, which forgets the elliptic fibration. This morphism is finite onto its image (see [**76**]). Let C be a component of L_r in $\mathcal{M}_2$ of dimension $20 - r$. The image of C is contained in S_r and is of dimension $20 - r$. □

REMARK 4.4. Using the surjectivity of the period map for algebraic $K3$ surfaces one finds an alternative proof for the above result. Using the global Torelli theorem for $K3$ surfaces one obtains even equality.

REMARK 4.5. Note that one can give a proof of Corollary 4.3 that is completely algebraic. Our proof is almost completely algebraic, except that in the proof of Proposition 2.10 a transcendental result (the Riemann existence theorem) is used. To obtain a lower bound for the dimension of Hurwitz spaces one can also do a parameter-equation count, which is algebraic.

Moreover Corollary 4.3, with this algebraic proof, holds over every field of characteristic different from 2 and 3.

5. Constant j-invariant

In this section we study the components of NL_r corresponding to elliptic surfaces with constant j-invariant. In this section we assume that $\pi : X \to \mathbf{P}^1$ is an elliptic surface with constant j-invariant different from 0 or 1728, and $p_g(X) > 0$. The case $p_g(X) = 0$ is rather subtle, for example the surfaces with precisely two I_0^* fibers do not correspond to points in $\mathcal{M}_1$. The problem is that if one constructs $\mathcal{M}_1$ as in [**46**] (using G.I.T.), there are unstable points (like our surfaces with two I_0^* fibers). If $n \geq 2$, this problem does not occur: if $n = 2$ then all Jacobi elliptic surfaces (over $\mathbf{P}^1$) give rise to semi-stable or stable points, and if $n > 2$ then all Jacobi elliptic surfaces (over $\mathbf{P}^1$) are stable.

A Jacobi elliptic surface with constant j-invariant, different from 0 and 1728, has only I_0^* fibers, and their number is exactly $2n$. Such a surface is completely determined by the $2n$ points with a I_0^* fiber and the j-invariant. Conversely given a set S of $2n$ points on $\mathbf{P}^1$ and a number $j_0 \in \mathbf{C} - \{0, 1728\}$ one can find a unique elliptic surface (up to isomorphism) with $\pi : X \to \mathbf{P}^1$ with $j(\pi) = j_0$ and $Sing(\pi) = S$. Hence the dimension of the (constructible) locus of all elliptic surface with $2n$ I_0^*-fibers in $\mathcal{M}_n$ is $2n - 2$, if $n \geq 2$.

Remark 5.1. Let $\pi : X \to \mathbf{P}^1$ be an elliptic surface satisfying our assumptions. Then to π we associate a hyperelliptic curve $\varphi : C \to \mathbf{P}^1$ such that the ramification points of φ are the points over which π has a singular fiber. Let E be an elliptic curve with the same j-invariant as the fibers of π. Then the minimal desingularization of $(C \times E)/(\iota \times [-1])$ is isomorphic to X.

Remark 5.2. Consider the elliptic surface $\pi : X \to \mathbf{P}^1$, with

$$X = \frac{\widetilde{C \times E}}{\langle \iota \times [-1] \rangle}$$

and the morphism π is induced by the projection $C \times E \to C$. (One easily sees that $C/\langle \iota \rangle = \mathbf{P}^1$.)

A section $s : \mathbf{P}^1 \to X$ is seen to come from a morphism $\mu : C \to E$ and s maps a point $c \bmod \langle \iota \rangle$ to $(c, \mu(c)) \bmod \langle \iota \times [-1] \rangle$. Conversely a morphism μ defines a section if and only if μ maps the fixed points of ι to fixed points of $[-1]$. A constant morphism $\mu : C \to \{P\} \subset E$ yields a section if and only if P has order at most 2. This gives a contribution $(\mathbf{Z}/2\mathbf{Z})^2$ to $MW(\pi)$. Using [**47**, Corollary VII.3.3] one can show that $MW(\pi)_{\text{tor}} = (\mathbf{Z}/2\mathbf{Z})^2$. If $MW(\pi) \neq (\mathbf{Z}/2\mathbf{Z})^2$ then a non-constant morphism $C \to E$ exists with the above mentioned property.

Lemma 5.3. *Let E be a curve of genus 1. Then the locus $L(E)$ corresponding to hyperelliptic curves C admitting a non-constant morphism $C \to E$ in H_g, the moduli space of hyperelliptic curves of genus g, has dimension $g - 1$.*

Proof. From [**62**, Lemma 1.1] it follows that for any non-constant morphism $\psi : C \to E$, there exists an elliptic involution on E induced by the hyperelliptic involution of

C, i.e., such that the following diagram is commutative

$$\begin{array}{ccc} E & \leftarrow & C \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \leftarrow & \mathbf{P}^1, \end{array}$$

where the vertical arrows are obtained by dividing out the (hyper)elliptic involution.

Fix λ a Legendre parameter for E. Any function $f : \mathbf{P}^1 \to \mathbf{P}^1$ gives rise to a hyperelliptic curve $C = \widetilde{E \times_{\mathbf{P}^1} \mathbf{P}^1}$. The genus of C is determined by f, i.e., $2g(C) + 2$ equals the number of points with odd ramification index above $0, 1, \lambda$ and ∞.

From this we obtain that $\dim L(E)$ equals the maximum over all d of the dimension of the Hurwitz space corresponding to functions of degree d, such that above $0, 1, \lambda, \infty$, there are precisely the $2g + 2$ points with odd ramification index. By Corollary 2.12 this space has dimension $2 \cdot 4 - g - 1 + 2g + 2 - 2 \cdot 4 - 2 = g - 1$. □

THEOREM 5.4. *Let $n > 1$. The locus L in $\mathcal{M}_n$ of elliptic surfaces with $2n$ I_0^*-fibers has dimension $h^{1,1} - \rho_{tr} = 2n - 2 = 2p_g$. The locus L_1 of elliptic surfaces with $2n$ I_0^* fibers and positive Mordell-Weil rank has dimension p_g. The locus L_2 of elliptic surfaces with $2n$ I_0^* fibers and Mordell-Weil rank at least 2 has dimension p_g or $p_g - 1$.*

PROOF. A fiber of type I_0^* has 4 components not intersecting the zero-section, so from the Shioda-Tate formula 1.2.11 it follows that $\rho_{tr} = 8n + 2$. The first assertion follows from the correspondence between L and sets of $2n$ distinct points in $\mathbf{P}^1$ together with a j-invariant $j_0 \in \mathbf{C}$ as mentioned above.

For the second assertion it suffices to prove that the locus of elliptic surfaces with constant j-invariant and positive rank has dimension at most p_g. Then from general theorems on the period map it follows that the locus corresponding to elliptic surfaces with positive rank has dimension precisely p_g, one need to exploit the following fact: for a class in $\xi \in H^2(X, \mathbf{Z})$ to be in $H^{1,1}(X)$ gives p_g conditions, namely $\xi \cdot \omega = 0$, for every $\omega \in H^0(X, \Omega_X^2)$. Since $h^0(X, \Omega_X^2) = p_g$ this gives p_g conditions.

If $MW(\pi)$ is strictly bigger than $(\mathbf{Z}/2\mathbf{Z})^2$ then by Remark 5.2 there is a non-constant morphism $C \to E$, with C and E as in Remark 5.1. Hence for a fixed $j_0 \in \mathbf{C}$, the locus of elliptic surfaces with $j(\pi) = j_0$ and $MW(\pi)$ infinite has by Lemma 5.3 dimension at most $g(C) - 1$. Hence L_1 has dimension at most $g(C) = p_g(X)$.

If the fixed j_0 corresponds to a curve with complex multiplication, then the Mordell-Weil rank is even, so L_2 has dimension at least $p_g - 1$. □

6. j-invariant 0 or 1728

In this section we will prove that $\dim NL_r - (10n - r)$ can be arbitrarily large.

PROPOSITION 6.1. *Let $n \geq 2$. Fix an integer k such that $6n/5 \leq k \leq 6n$. Then there exists an elliptic surface $\pi : X \to \mathbf{P}^1$ with $j(\pi) = 0$, $p_g(X) = n - 1$ and k singular fibers. Moreover, the locus of elliptic surfaces with $j(\pi') = 0$ and $C(\pi') = C(\pi)$ has dimension $k - 3$ in $\mathcal{M}_n$. If m is an integer such that $m > 6n$ or $m < 6n/5$ then there exists no elliptic surface with $j(\pi') = 0$ and m singular fibers.*

PROPOSITION 6.2. *Let $n \geq 2$. Fix an integer k such that $4n/3 \leq k \leq 4n$. Then there exists an elliptic surface $\pi : X \to \mathbf{P}^1$ with $j(\pi) = 1728$, $p_g(X) = n - 1$ and k singular fibers. Moreover, the locus of elliptic surfaces with $j(\pi') = 1728$ and $C(\pi') = C(\pi)$ has* *dimension $k-3$ in $\mathcal{M}_n$. If m is an integer such that $m > 4n$ or $m < 4n/3$ then there exists no elliptic surface with $j(\pi') = 1728$ and m singular fibers.*

PROOF OF PROPOSITIONS 6.1 AND 6.2. Without loss of generality we may assume that all elliptic surfaces under consideration have a smooth fiber over ∞.

There exists an elliptic surface with k singular fibers, $p_g(X) = n-1$ and $j(\pi) = 0$ if and only if there exists a polynomial f of degree $6n$ with k distinct zeroes, and every zero has multiplicity at most 5. To any such a polynomial $f(t)$ we can associate an elliptic surface with Weierstrass equation $y^2 = x^3 + f(t)$, and vice-versa, an elliptic surface with j-invariant 0 gives rise to a Weierstrass equation of the above form.

Hence an elliptic surface with k singular fibers exists if and only if $6n/5 \leq k \leq 6n$. Modulo the action of $\mathrm{Aut}(\mathbf{P}^1)$ we obtain a $k-3$ dimensional locus in $\mathcal{M}_n$.

The case of $j(\pi) = 1728$ is similar except for the fact that the polynomial g is of degree $4n$, and the highest possible multiplicity is 3. The associated surfaces is then given by $y^2 = x^3 + gx$. □

PROPOSITION 6.3. *Let $n \geq 2$. Let $r \leq 1 + \frac{24}{5}n$ be a positive integer. Then the locus of elliptic surfaces with j-invariant 0 and $\rho_{tr}(X)$ at least $2r$ has dimension*

$$6n - r - 2$$

PROOF. If $j(\pi)$ is constant and π has k singular fibers then the number of components of singular fibers not intersecting the zero-section equals $12n - 2k$. Hence $\rho_{tr}(\pi) = 2 + 12n - 2k$. From this it follows that $\rho_{tr}(\pi) \geq 2r$ if and only if $k \leq 6n - r + 1$. We want to apply Proposition 6.1 for $k = 6n - r + 1$. The condition on k is equivalent to the above assumption on r. Then Proposition 6.1 implies that the dimension of the locus is $k - 3 = 6n - r - 2$. □

REMARK 6.4. A similar result holds in the case that $j(\pi) = 1728$. In that case one should take $r \leq \frac{14}{3}n + 1$.

REMARK 6.5. All loci L described in the Sections 4 and 5 satisfied $\dim L + \rho(X) \leq 10n$, for an X corresponding to a generic point of L. In the above theorem, one can choose $r = 1 + 4n + \lfloor 4n/5 \rfloor$, with $\lfloor \alpha \rfloor$ denoting the largest integer, not larger then α. One obtains

$$\dim L + \rho(X) = 6n - r - 2 + 2r = 10n + \left\lfloor \frac{4}{5}n \right\rfloor - 1$$

The excess term $\lfloor 4n/5 \rfloor - 1$ can be arbitrarily large.

COROLLARY 6.6. *Suppose $n \in \{2, 3, 4, 5\}$. Then $\dim NL_{10n} = n - 2$.*

PROOF. From Corollary 1.4.4 and Theorem 4.1 it follows that

$$\{[\pi : X \to \mathbf{P}^1] \in NL_{10n} \mid \rho_{tr}(\pi) < 10n \text{ or } j(\pi) \text{ not constant}\}$$

is a discrete set. If $j(\pi) \in \mathbf{C} - \{0, 1728\}$ then $\rho_{tr}(\pi) < 10n$, hence we only have to consider elliptic surfaces with $\rho_{tr}(\pi) = 10n$ and constant j-invariant 0 or 1728. Since $1 + \lfloor 24n/5 \rfloor = 5n$ for the n under consideration, we may apply Proposition 6.3 with $r = 5n$. This yields $\dim NL_{10n} = n - 2$. □

REMARK 6.7. From this Corollary we deduce that for $n \in \{3, 4, 5\}$, there exist positive dimensional loci $L \subset \mathcal{M}_n$, such that any surface X corresponding to a point in L satisfies

$\rho(X) = h^{1,1}(X)$. The image of the period map restricted to L has discrete image. This contradicts several Torelli-type properties. (see also Theorem 1.4.8.)

7. Upper bound

As in [**18**] we study the Noether-Lefschetz loci using the identification of $H^{1,1}, H^{2,0}$ and $H^{0,2}$ with several graded pieces of a Jacobian ring R. We choose to give a more algebraic presentation than in [**18**].

To be precise, given a Weierstrass minimal equation $F = 0$ for $\pi : X \to \mathbf{P}^1$ we can construct a hypersurface Y in the weighted projective space $\mathbf{P} := \mathbf{P}(1, 1, 2n, 3n)$ (with projective coordinates x, y, z, w of weight $1, 1, 2n, 3n$ resp.):

$$0 = -w^2 + z^3 + P(x, y)z + Q(x, y) =: F,\ n = p_g(X) + 1, \deg(P) = 4n, \deg(Q) = 6n.$$

Here X and Y are birational; Y is obtained from X by contracting the zero-section and all fiber components not intersecting the zero-section.

Let $A := \mathbf{C}[x, y, z, w]$ with weights $1, 1, 2n, 3n$. Let $B = \mathbf{C}[x, y] \subset A$. The construction $(\pi : X \to \mathbf{P}^1, \sigma_0 : \mathbf{P}^1 \to X) \mapsto Y$ gives a nice description of the moduli space $\mathcal{M}_n$. (See the proof of Theorem 7.7.)

Let $J \subset A$ be the ideal generated by the partial derivatives of F. The Jacobi ring R is the quotient ring A/J. It is well known (see [**18**], [**19**], [**20**], [**74**]) that if all the fibers of π are irreducible then Y is quasi-smooth, i.e., the cone $(F = 0) \subset \mathbf{A}^4$ is smooth outside the origin.

For graded ring R' we denote by R'_d all elements of degree d. For a variety $Y' \subset \mathbf{P}$ we denote by $H^{1,1}(Y')_{\text{prim}} = \text{Im}(H^2(\mathbf{P}, \mathbf{C}) \to H^{1,1}(Y'))^{\perp}$. We have isomorphisms

$$H^{2,0} \cong R_{n-2},\ H^{1,1}_{\text{prim}} \cong R_{7n-2},\ H^{0,2} \cong R_{13n-2}.$$

In the case that π has reducible fibers the situation is very similar. This follows from a special case of a recent result of Steenbrink [**75**].

THEOREM 7.1 (Steenbrink [**75**]). *Let $Y' \subset \mathbf{P}$ be a surface of degree $6n$, with along $\mathbf{P}_{\text{sing}}$ only singularities induced by $\mathbf{P}_{\text{sing}}$ and outside $\mathbf{P}_{\text{sing}}$ only isolated rational double points as singularities. Let R' be the Jacobi-ring of R. Then there is a natural isomorphism $H^{2,0}(X) \cong R'_{n-2}$ and an injective map*

$$H^{1,1}(Y')_{\text{prim}} \to R'_{7n-2}.$$

LEMMA 7.2. *We have*

$$H^{1,1}(Y)_{\text{prim}} \cong H^{1,1}(X)/(T(\pi) \otimes \mathbf{C}).$$

In particular, $\dim H^{1,1}(Y)_{\text{prim}} = 10n - \rho_{tr}$.

PROOF. The isomorphism follows from the fact that $\varphi : X \to Y$ is a resolution of singularities, φ contracts all fiber components not intersecting the zero-section and the fact that a general hyperplane section $H \cap Y$ is a fiber of π. ☐

COROLLARY 7.3. *There is a natural isomorphism $H^{2,0}(X) \cong R_{n-2}$ and an injective map*

$$H^{1,1}(X)/(T(\pi) \otimes \mathbf{C}) \to R_{7n-2}.$$

PROOF. This is a combination of Theorem 7.1 and Lemma 7.2. ☐

We prove now some elementary technical results. For a polynomial P, we use a subscript (like P_x) to indicate the derivative with respect to the variable in the subscript.

LEMMA 7.4. *Let $\pi : X \to \mathbf{P}^1$ be the elliptic surface associated to $w^2 = z^3 + Pz + Q$, with $P \in \mathbf{C}[x,y]_{4n}, Q \in \mathbf{C}[x,y]_{6n}$. Then $P_xQ_y - P_yQ_x = 0$ if and only if $j(\pi)$ is constant.*

PROOF (SEE [**18**]). The partial derivative to X or to Y of $j(\pi) = 1728 \cdot 4P^3/(4P^3 + 27Q^2)$ is identically zero if and only if $(P_xQ_y - P_yQ_x)PQ = 0$. If $PQ = 0$ then $P_xQ_y - P_yQ_x = 0$, which gives the lemma. □

LEMMA 7.5. *Fix a positive integer n. Let $F \in A = \mathbf{C}[x,y,z,w]$ be a weighted homogenous polynomial of degree $6n$. Suppose that the variety in $\mathbf{P}(1,1,2n,3n)$ defined by $F = 0$ is birational to an elliptic surface $\pi : X \to \mathbf{P}^1$, with π induced by $[x,y,z,w] \mapsto [x,y]$ and $F = 0$ is a Weierstrass minimal equation. Let J be the Jacobi-ideal of F. Let $M \subset A$ be the $\mathbf{C}[x,y]$-module generated by J_{6n}, the degree $6n$-part of the Jacobi-ideal. Then*

$$\operatorname{rank}(M) = \begin{cases} 7 & \textit{in the case } j(\pi) \textit{ not constant;} \\ 6 & \textit{in the case } j(\pi) \textit{ constant.} \end{cases}$$

Furthermore, we have $\dim R_{7n-2} = 10n - 2$.

PROOF. After applying an automorphism of $\mathbf{P}$ we may assume that $F = -w^2 + z^3 + Pz + Q$, with $P \in B_{4n}$ and $Q \in B_{6n}$. Then

$$J_{6n} = B_0w^2 + B_nwz + B_{3n}w + B_0(3z^3 + Pz) + B_{2n}(3z^2 + P) + B_1(P_xz + Q_x) + B_1(P_yz + Q_y).$$

The first five terms generate a B-module N of rank 5, and the quotient J_{6n}/N is generated by the classes of $P_xz + Q_x$ and $P_yz + Q_y$. This module has rank at least 1. It suffices to prove that it has rank precisely 1 if and only if $j(\pi)$ is constant.

Consider the elements $\alpha := Q_y(P_xz + Q_x) - Q_x(P_yz + Q_y) = (P_yQ_x - P_xQ_y)z$ and $\beta := P_y(P_xz + Q_x) - P_x(P_yz + Q_y) = P_yQ_x - P_xQ_y$.

Suppose $j(\pi)$ is constant then we obtain by Lemma 7.4 the relation $P_y(P_xz + Q_x) - P_x(P_yz + Q_y) = 0$, proving that $\operatorname{rank}(J/N) \le 1$.

Suppose $j(\pi)$ is not constant. Then Lemma 7.4 implies that α and β are non-zero, hence are independent modulo N, proving the assertion for this case.

For the final assertion: An easy calculation shows that $\dim A_{7n-2} = 23n - 7$. From

$$\begin{aligned} J_{7n-2} &= B_{n-2}w^2 + B_{2n-2}wz + B_{4n-2}w + B_{n-2}(3z^3 + Pz) + B_{3n-2}(3z^2 + P) + \\ &\quad + B_{n-1}(P_xz + Q_x) + B_{n-1}(P_yz + Q_y). \end{aligned}$$

it follows that $\dim J_{7n-2} \le 13n-5$. It suffices to prove equality, because then $\dim R_{7n-2} = \dim A_{7n-2} - \dim J_{7n-2} = 10n - 2$. Suppose that $\dim J_{7n-2} < 13n - 5$, then there exist polynomials $F, G \in \mathbf{C}[x,y]_{n-1}$, such that $(F,G) \neq (0,0)$ and

$$\begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = 0. \tag{2}$$

In particular, $P_xQ_y - Q_xP_y = 0$. From Lemma 7.4 it follows that the j-invariant is constant. There are three cases to consider, namely $P = 0$, $Q = 0$ and $P^3 = cQ^2$, where $c \in \mathbf{C}^*$, $PQ \neq 0$.

Suppose first that $P = 0$. Since $F = 0$ is a Weierstrass minimal equation if follows that for all finite places T, we have $v_T(Q) \leq 5$. From $6nQ = xQ_x + yQ_y$, we obtain that

$$\deg \gcd(Q_x, Q_y) \leq 6n - \#\{ \text{ zeroes of } Q\} \leq 5n$$

This implies that there does not exist a non-zero form G of degree $n-1$ such that the rational function $-GQ_y/Q_x$ is a polynomial, which contradicts the fact that this should equal F. So $P \neq 0$. The case $Q = 0$ can be dealt in a similar way.

Suppose that $P^3 = cQ^2$ with $c \in \mathbf{C}^*$ and $PQ \neq 0$. Then one can easily show that $\deg \gcd(Q_x, Q_y) = 4n$. From this one obtains that the only solution to (2) is $F = G = 0$. Hence in all cases we obtain $\dim R_{7n-2} = 10n - 2$. □

Remark 7.6. The B-module generated by A_{6n} has rank 7. So J_{6n} has the same rank as A_{6n} (considered as B-modules) if and only if $j(\pi)$ is not constant.

Theorem 7.7 together with the results in the previous sections will provide a proof for Theorem 1.1.

Theorem 7.7. *Let* $2 \leq r \leq 10n$. *Let* $U \subset \mathcal{M}_n$ *be the locus of elliptic surfaces with non-constant* j*-invariant. Then* $\dim NL_r \cap U$ *is at most* $10n - r$.

Proof. Take $t \in \mathbf{Z}_{\geq 0}$. Let $M \subset A_t$ be a linear subspace. We denote with M_k the image of the multiplication map $M \otimes B_k \to A_{t+k}$. This makes $M_\bullet := \oplus_{k\geq 0} M_k$ into a B-module.

We prove the theorem by induction. Assume that it is true for all r', $r < r' \leq 10n$. Define C by $NL_r = NL_{r+1} \coprod C$. Let $\pi : X \to \mathbf{P}^1$ correspond to a point p in $C \cap U$. We want to calculate the dimension of the tangent space of NL_r at p. Let $Y \subset \mathbf{P}$ be the corresponding surface in the weighted projective space $\mathbf{P}$. Write $s = \operatorname{rank} MW(\pi)$.

The moduli-space $\mathcal{M}_n$ can be obtained in the following way: Let

$U := \{f \in A_{6n} \mid f = 0$ is birational to an elliptic surface and f is Weierstrass minimal. $\}$

then $U/\operatorname{Aut}(\mathbf{P}) = \mathcal{M}_n$ (see [**46**]). Let $L \subset A_{6n}$ be the pre-image of a component containing $\pi : X \to \mathbf{P}^1$ of $NL_r \subset \mathcal{M}_n$.

Since $\operatorname{Aut}(\mathbf{P})$ is a reductive group, we have that the codimension of $L/\operatorname{Aut}(\mathbf{P})$ in $\mathcal{M}_n$ equals $\operatorname{codim}(L, A_{6n})$. From this if follows that it suffices to show that L has codimension at least $r-2$ in A_{6n}. Let $\mathcal{T} \subset A_{6n}$ be the tangent space of L at Y, considered as a point in A_{6n}.

Consider the multiplication map

$$\varphi : \mathcal{T} \otimes A_{n-2} \to A_{7n-2}.$$

Let ψ be the composition of φ with the projection onto R_{7n-2}. Using Corollary 7.3 we obtain that ψ corresponds to the map $\mathcal{T} \otimes H^{2,0} \to H^{1,1}_{\text{prim}}$ induced by the Period map. Hence the image of φ is contained in the subspace $W \subset A_{7n-2}$, the pre-image of $H^{1,1}_{\text{prim}} \hookrightarrow R_{7n-2}$ (using Corollary 7.3).

From $J_{7n-2} \subset W$, the quotient W/J_{7n-2} has dimension $10n - \rho_{tr}$ and $\dim R_{7n-2} = 10n - 2$ (see Lemma 7.2), we obtain $\operatorname{codim}(W, A_{7n-2}) = \rho_{tr} - 2$.

Let O be a suitable neighborhood of Y in $L \subset A_{6n}$. Since $Y' \mapsto H^2(Y', \mathbf{Z})$ is a constant sheaf on O and $Y' \mapsto H^{1,1}(Y', \mathbf{C})_{\text{prim}} \subset H^2(Y', \mathbf{Z})_{\text{prim}} \otimes \mathbf{C}$ has a constant subsheaf of complex dimension at least s, we obtain that $\operatorname{codim}(\mathcal{T}_{n-2}, W) \geq s$, hence $\operatorname{codim}(\mathcal{T}_{n-2}, A_{7n-2}) \geq r - 2$.

Hence it suffices to show that every linear subspace $V \subset A_{6n}$ containing J_{6n} satisfies $\operatorname{codim}(V, A_{6n}) \geq \operatorname{codim}(V_{n-2}, A_{7n-2})$.

Let $\tilde{J}$ and $\tilde{V}$ be the B-modules generated by J_{6n}, respectively, V.

Let $d_k := \dim \tilde{J}_{k+1} - \dim \tilde{J}_k$. Then d_k is decreasing for $k \geq 0$. By Lemma 7.5 we obtain that $\tilde{J}$ has rank 7, hence $d_k \geq 7$. This implies that $\dim \tilde{V}_{k+1} - \dim \tilde{V}_k$ is at least 7. Hence $\dim V_{n-2} \geq \dim V + 7(n-2)$. Since $\dim A_{7n-2} = \dim A_{6n} + 7(n-2)$, we obtain that $\operatorname{codim}(V, A_{6n}) \geq \operatorname{codim}(V_{n-2}, A_{7n})$, which finishes the proof. □

8. Concluding remarks

REMARK 8.1. The argument used in the proof of Theorem 7.7 cannot work for elliptic surfaces with constant j-invariant. First of all, in this case Lemma 7.5 gives that $d_k \geq 6$, which only implies $\operatorname{codim}(V, A_{6n}) \geq \operatorname{codim}(V_{n-2}, A_{7n}) - (n-2)$. Moreover, it is not hard to give linear subspaces $V \subset A_{6n}$, such that $J_{6n} \subset V$, $V \neq J_{6n}$ and $\operatorname{codim}(V_{n-2}, A_{7n-2}) > \operatorname{codim}(V, A_{6n})$. One needs to show that such a space V does not occur as the tangent space to a component of NL_r, or V is a tangent space to one of the components described in Section 6. By the results of Section 5 we know that such a V would have a large codimension in A_{6n}, but these results are not sufficient to prove the theorem in the case of constant j-invariant.

REMARK 8.2. There is still an interesting issue open. In the theory of Noether-Lefschetz loci there is the notion of special components and of general components. Special components are the components of NL_3 with codimension in NL_2 less then p_g. In the case of elliptic surfaces there is only one special component (see [**18**]). For higher Noether-Lefschetz loci, one can define the special components as the components in NL_r with codimension less then $(r-2)p_g$. Then one finds infinitely many special components. One can also define special components as the components of NL_r such that the maximal codimension in NL_{r-1} is less then p_g. By base-changing families of elliptic $K3$ surfaces we can find again infinitely many special components, even when we fix the component of NL_{r-1} in which these components are contained.

REMARK 8.3. Suppose $\mathcal{M}$ is a moduli space for some class of smooth surfaces. We would like to obtain $\operatorname{codim}(NL_r, \mathcal{M}) \geq r - \rho_{\text{gen}}$, where ρ_{gen} stands for the generic Picard number and $NL_r = \{X \in \mathcal{M} \mid \rho(X) \geq r\}$.

To give a proof similar to the proof of Theorem 7.7 it suffices to assume

- Griffiths-Steenbrink holds for the moduli-problem. I.e., there exists a threefold X, such that for all points $p \in \mathcal{M}$ there exists a surface $Y_p \subset X$, satisfying the conditions of [**75**]). Moreover, if $\tilde{Y}_p$ is the normalization of Y, then $[\tilde{Y}_p] \in \mathcal{M}$ is the point p.
- All surfaces are linearly equivalent (as divisors on X), i.e., fix a point $p \in \mathcal{M}$. Let X and Y_p as above, then there is a dense open $U \subset H^0(X, \mathcal{O}_X(Y_p))$ and a surjective morphism $U \to \mathcal{M}$, sending a divisor Y' to the class of its minimal desingularization.
- The following multiplication conditions hold. Let K be the kernel of ψ_2. Let $K(m)$ be the image of $K^{\otimes m}$ in $H^0(X, K_X^{\otimes m}(2mY))$. Then for all $m \geq 2$ we have

$$\dim K(m) - \dim K(m-1) \geq \dim H^0(X, K_X^{\otimes m}(2mY)) - \dim H^0(X, K_X^{\otimes (m-1)}(2(m-1)Y)).$$

REMARK 8.4. In [**25**] the following statement is proven. Let $d > 3$ be an integer, let $U \subset \mathbf{C}[x, y, z, w]_d$ be the set of homogenous polynomials F such that $F = 0$ defines a smooth surface. Let $NL \subset U$ be the locus of surfaces with Picard number at least 2. Then $\operatorname{codim}(NL, U) = d - 3$.

The strategy used in the proof is very similar to the strategy used in the proof of Theorem 7.7. However, in this case this strategy does not seem to work for larger Picard number. If one applies a reasoning as in the proof of Theorem 7.7 one obtains that $\operatorname{codim}(NL_r, U) \geq d - 1 - r$. Griffiths and Harris [**26**, page 208] *conjecture* that for $3 \leq r \leq d$ we have

$$\operatorname{codim}(NL_r, U) = (r-1)(d-3) - \begin{pmatrix} r-3 \\ 2 \end{pmatrix}.$$

and they claim that it is easy to prove that we can replace the equality sign by a less or equal sign.

There is still a gap between these two bounds for $\operatorname{codim}(NL_r, U)$.

CHAPTER 3

Kuwata's surfaces

1. Introduction

In this chapter we assume that K is a field of characteristic 0.

We introduce elliptic surfaces $\pi_n : X_n \to \mathbf{P}^1$ defined over K, for every integer $n \geq 1$. In the special case $n \equiv 0 \bmod 2$, the function field extension $K(X_n)/K(\mathbf{P}^1)$ is $K(x,z,t)/K(t)$ where $x^3 + ax + b - t^n(z^3 + cz + d) = 0$, for some $a, b, c, d \in K$. For every $n \geq 1$ the surface X_n is birational to a base-change $X_1 \times_{\mathbf{P}^1} \mathbf{P}^1$, where $\mathbf{P}^1 \to \mathbf{P}^1$ is an n-cyclic cover.

Kuwata [**41**] computed the rank of the Mordell-Weil groups $MW(\pi_i) := MW_{\overline{K}}(\pi_i)$ of π_i, for $i = 1, \ldots, 6$, and he gave a strategy for computing a rank 12 subgroup of the rank 16 group $MW(\pi_6)$. This strategy for finding generators of $MW(\pi_6)$ is already mentioned in an email correspondence between Jasper Scholten and Masato Kuwata. We describe and extend their ideas in order to describe generators of the Mordell-Weil groups $MW(\pi_i)$ for $i = 2, 3, 4$ and 6. Most of the computations we use, can be found in the Maple worksheet [**32**]. We indicate how one can find in special cases generators for the Mordell-Weil group of π_5. Finally, in the case $K = \mathbf{Q}$ we discuss how large the part of the Mordell-Weil group consisting of $\mathbf{Q}$-rational sections can be.

For $i \leq 6$ the $\pi_i : X_i \to \mathbf{P}^1$ are elliptic $K3$ surfaces. There are very few examples of such surfaces where generators for the Mordell-Weil group have been found.

2. Notation and results

Fix two elliptic curves E and F, with $j(E) \neq j(F)$. Let $\iota : E \times F \to E \times F$ be the automorphism sending $(P, Q) \to (-P, -Q)$. The minimal desingularization of $(E \times F)/\langle\iota\rangle$ is a $K3$ surface which we denote by Y. It is called the Kummer surface of $E \times F$.

The surface Y possesses several elliptic fibrations. For a generic choice of E and F, all fibrations on Y are classified by Oguiso ([**52**]). Following Kuwata [**41**], we concentrate on a particular fibration $\psi : Y \to \mathbf{P}^1$ having two fibers of type IV^* and 8 other irreducible singular fibers. Assume that the fibers of type IV^* are over 0 and ∞. Let $\pi_6 : X_6 \to \mathbf{P}^1$ be the cyclic degree 3 base-change of ψ ramified over 0 and ∞. Assume that E is given by $y^2 = x^3 + ax + b$ and F is given by $y^2 = x^3 + cx + d$. Set $B(t) = \Delta(F)t + 864bd + \Delta(E)/t$, with $\Delta(E)$ the discriminant of E, i.e., $\Delta(E) = -16(4a^3 + 27b^2)$, and $\Delta(F) = -16(4c^3 + 27d^2)$. Then a Weierstrass equation for π_6 is

$$y^2 = x^3 - 48acx + B(t^6).$$

Define $\pi_i : X_i \to \mathbf{P}^1$ as the elliptic surface associated to the Weierstrass equation

$$y^2 = x^3 - 48acx + B(t^i).$$

Clearly, interchanging the role of E and F corresponds to the automorphism $t \mapsto 1/t$ on X_i. One has that $j(\pi_2) = j(\psi)$, but X_2 and Y are *not* isomorphic as fibered surfaces.

By construction, it is clear that $MW(\pi_n)$ can be regarded as a subgroup of $MW(\pi_{nm})$ for $m \geq 1$.

We recall the following result from [**41**], most of which will be reproven in the course of Section 4.

THEOREM 2.1 (Kuwata, [**41**, Theorem 4.1]). *The surface X_i is a $K3$ surface if and only if $i \leq 6$. Set*

$$h = \begin{cases} 0 & \text{if } E \text{ and } F \text{ are not isogenous,} \\ 1 & \text{if } E \text{ and } F \text{ are isogenous and } E \text{ does not admit complex multiplication,} \\ 2 & \text{if } E \text{ and } F \text{ are isogenous and } E \text{ admits complex multiplication.} \end{cases}$$

Suppose $j(E) \neq j(F)$, then

$$\operatorname{rank} MW(\pi_i) = \begin{cases} h & \text{if } i = 1, \\ 4+h & \text{if } i = 2, \\ 8+h & \text{if } i = 3, \\ 12+h & \text{if } i = 4, \\ 16+h & \text{if } i = 5, \\ 16+h & \text{if } i = 6. \end{cases}$$

In the case that $j(E) = j(F)$ then the ranks of $MW(\pi_i)$ tend to be lower (see [**41**, Theorem 4.1]). Kuwata in [**41**] restricts himself to the X_i with $i \leq 6$. We observe the following corollary of his results.

COROLLARY 2.2. *The rank of $MW(\pi_{60})$ is at least $40 + h$.*

PROOF. It can be easily seen that the rank $40 + h$ group

$$MW(\pi_1) \oplus \frac{MW(\pi_2)}{MW(\pi_1)} \oplus \frac{MW(\pi_4)}{MW(\pi_2)} \oplus \frac{MW(\pi_5)}{MW(\pi_1)} \oplus \frac{MW(\pi_6)}{MW(\pi_2)}.$$

injects into $MW(\pi_{60})$. (The summands correspond to different eigenspaces for the induced action of $t \mapsto \zeta_{60}t$ on $MW(\pi_{60})$.) $\square$

REMARK 2.3. The highest known Mordell-Weil rank for an elliptic surface $\pi : X \to \mathbf{P}^1$, such that $j(\pi)$ is non-constant is 56, a result due to Stiller [**78**]. The geometric genus of our example is very low compared to for example Stiller's examples: one can easily show that $p_g(X_{60}) = 19$, while Stiller's examples have $p_g + 1$ divisible by 210.

Our aim is to provide explicit equations for generators of $MW(\pi_i)$, for several small values of i, in terms of the parameters a, b, c, d of the two elliptic curves

$$E : y^2 = x^3 + ax + b \text{ and } F : y^2 = x^3 + cx + d.$$

The main idea used in the sequel is the following. Let $\pi : S \to \mathbf{P}^1$ be a $K3$ surface. Let σ be an automorphism of finite order of S, such that we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S \\ \downarrow \pi & & \downarrow \pi \\ \mathbf{P}^1 & \xrightarrow{\tau} & \mathbf{P}^1 \end{array}$$

and the order of τ equals the order of σ. We obtain an elliptic surface $\psi : S' := \widetilde{X/\langle\sigma\rangle} \to \mathbf{P}^1/\langle\tau\rangle \cong \mathbf{P}^1$. In the case that S' is a rational surface, it is in principle possible to find explicit equations for generators of $MW(\psi)$. One can pull back sections of ψ to sections of π, which establishes $MW(\psi)$ as a subgroup of $MW(\pi)$. In the case that σ is of even order then there exists a second elliptic surface, $\psi'' : S'' \to \mathbf{P}^1$, such that $MW(\psi) \oplus MW(\psi'')$ modulo torsion injects into $MW(\pi)$ modulo torsion. (This process is called twisting and is discussed in more detail in Section 1.5.)

In [**41**, Section 5] it is indicated how one can find generators for a rank 12 subgroup of $MW(\pi_6)$ by using the above strategy for the involutions $(x, y, t) \mapsto (x, y, -t)$, and $(x, y, t) \mapsto (x, y, \alpha/t)$, for some α satisfying $\alpha^3 = \Delta(F)/\Delta(E)$. Taking a third involution $(x, y, t) \mapsto (x, y, \alpha\zeta_3/t)$ yields a rank 16 subgroup of $MW(\pi_6)$, isomorphic to $MW(\pi_6)/MW(\pi_1)$. (This is discussed in more detail in the Subsections 4.3 and 4.6.) In Subsection 4.6 we also describe the field of definition of sections generating $MW(\pi_6)$ modulo $MW(\pi_1)$. It turns out that our description corrects a mistake in Kuwata's description of the minimal field of definition.

RESULTS 2.4. On the Kuwata surfaces $X_1, X_2, \ldots, X_6$ one has several automorphisms such that the obtained quotients are rational surfaces. This is used in Section 4 to give the following results:

- We give sufficient conditions on E, F and K to have a rank 1 or a rank 2 subgroup of $MW_K(\pi)$.
- We give an indication how one can find explicit generators of $MW(\pi_2)/MW(\pi_1)$. The precise generators can be found in [**32**].
- We give an indication how one obtains a degree 24 polynomial such that its zeroes determine a set of generators of $MW(\pi_3)/MW(\pi_1)$. The explicit polynomial can be found in [**32**].
- We give explicit generators of a subgroup of finite index in $MW(\pi_4)/MW(\pi_2)$. (See Corollary 4.12.)
- We give an indication how one obtains a degree 240 polynomial such that its zeroes determine a set of generators of $MW(\pi_5)/MW(\pi_1)$. We did not manage to write down this polynomial. The degree of this polynomial and the number of variables is too high to write it down explicitly. For several choices of a, b, c, d it can be found in [**32**].
- We give an algorithm for determining a set of generators of $MW(\pi_6)/MW(\pi_3)$. The explicit generators are given in [**32**].

If E and F are not isogenous then $MW(\pi_1) = 0$ (see Theorem 2.1). If E and F are isogenous then the degree of the x-coordinate of a generator of $MW(\pi_1)$ seems to depend on the degree of the isogeny between E and F. This seems to give an obstruction for obtaining explicit formulas for all cases.

Before providing the explicit equations, we mention the following result, which can be proven without knowing explicitly the generators of $MW(\pi_i)$.

PROPOSITION 2.5. *Suppose that E and F are defined over $\mathbf{Q}$. One has that*

$$\operatorname{rank} MW_{\mathbf{Q}}(\pi_i) \leq \begin{cases} 1 & \text{if } i = 1, \\ 5 & \text{if } i = 2, \\ 7 & \text{if } i = 3, \\ 9 & \text{if } i = 4, \\ 5 & \text{if } i = 5, \\ 11 & \text{if } i = 6. \end{cases}$$

The total contribution of $\pi_1, \pi_2, \pi_3, \pi_4, \pi_6$ *to* $MW_{\mathbf{Q}}(\pi_{12})$ *is bounded by 15. The maximum total contribution of* $\pi_1, \dots, \pi_6$ *to* $MW_{\mathbf{Q}}(\pi_{60})$ *is bounded by 19.*

PROOF. If $\operatorname{rank} MW_{\mathbf{Q}}(\pi_1)$ equals 2 then the Shioda-Tate formula 1.2.11 implies that $NS(X)$ has rank 20 and $NS(X)$ is generated by divisors defined over $\mathbf{Q}$. It is well-known that this is impossible ([**71**]).

Suppose that $i \geq 2$. Let S be the image of a section of π_i, such that S is not the strict transform of the pull back of a section of π_j, for some j dividing i, and $j \neq i$. It is easy to check that if we push forward S to X_1 and then pull this divisor back to X_i, we obtain a divisor D consisting of i geometrically irreducible components, and one of these components is S.

Set $M_j := MW(\pi_j)$. Set $M := M_i / \sum_{j>0, j|i, j\neq i} M_j$. Take a minimal set of sections $S_m, m = 1, \dots, \ell$, satisfying the following property: denote $S_{m,n}$ the components of the pull back of the push forward of S_m, then the $S_{m,n}$, with $m = 1, \dots, \ell$ and $n = 1, \dots, i$, generate $M \otimes \mathbf{Q}$. Consider the ℓn-dimensional $\mathbf{Q}$-vector space F of formal linear expression in the $S_{m,n}$. By definition of the S_n it follows that the natural map $\Psi : F \to M \otimes \mathbf{Q}$ is surjective. Fix a generator σ of the (birational) automorphism group of the rational map $X_i \to X_1$. We can split $F \otimes \mathbf{C}$ and $M_j \otimes \mathbf{C}$ into eigenspaces for the action of σ. Set $E_{\zeta_i^k} \subset F \otimes \mathbf{C}$ the eigenspace for the eigenvalue ζ_i^k. Set $V' := \oplus_{j|\gcd(i,j)=1} E_{\zeta_i^j}$. Since the $S_{i,j}$ form a minimal set, we have that that $\Psi|_{V'}$ is injective. It is easy to see that if $\gcd(i,k) \neq 1$ then the eigenspace $E_{\zeta_i^k}$ is contained in the kernel of Ψ, hence we obtain that $\Psi|_{V'}$ is an isomorphism. This implies that $\dim V' = \varphi(i)\ell$, where φ is the Euler φ-function.

Fix m such that $1 \leq m \leq \ell$. Let G be the subgroup of $MW(\pi_i)$ generated by the $S_{m,n}$ for $n = 1, \dots i$. There is a faithful action of the Galois group $\operatorname{Gal}(\mathbf{Q}(\zeta_i)/\mathbf{Q})$ on G. This implies that the sections defined over $\mathbf{Q}$ form a rank at most 1 subgroup of G (if i is odd) or a rank at most two subgroup of G (if i is even). From this one obtains that

$$M_{\mathbf{Q}} := MW_{\mathbf{Q}}(\pi_i) / \oplus_{j>0, j|i, j\neq i} MW_{\mathbf{Q}}(\pi_j)$$

has rank at most ℓ (when i is odd) or 2ℓ (when i is even). This combined with Proposition 2.1 provides the upper bounds for $MW_{\mathbf{Q}}(\pi_i)$ for $i = 2, 3, 5, 6$. The upper bound for $MW_{\mathbf{Q}}(\pi_4)$ follows from the fact that the above mentioned Galois action is non-trivial, hence for each i there is an element g in G not fixed under the action of $\operatorname{Gal}(\mathbf{Q}(\zeta_i)/\mathbf{Q})$ and g is not mapped to 0 in M. One easily obtains from this that $M_{\mathbf{Q}}$ has co-rank at least ℓ in M, which gives the case $i = 4$.

The final assertions follow from a reasoning as in the proof of Corollary 2.2. □

For some of the π_i the bounds in Proposition 2.5 are sharp, for some of the others we do not know:

- We have $\operatorname{rank} MW_{\mathbf{Q}}(\pi_1) = 1$ when E and F are isogenous over $\mathbf{Q}$. (see 4.1.)
- We have that $\operatorname{rank} MW_{\mathbf{Q}}(\pi_2) \geq 4$ if and only if E and F have complete two-torsion over $\mathbf{Q}$, and, moreover, if E and F are isogenous then $\operatorname{rank} MW_{\mathbf{Q}}(\pi_2) = 5$. (see 4.2.)
- There exists E and F such that $\operatorname{rank} MW_{\mathbf{Q}}(\pi_4) \geq 8$. (see 4.4.)
- There exists E and F such that $\operatorname{rank} MW_{\mathbf{Q}}(\pi_6) \geq 6$.
- If one can find a rational solution of $uv^2(u^4 - 1)(v^4 - 1)^2 = w^3$, satisfying
  - $w(u - v) \neq 0$ and

– if $p_1 = 4u^2/(u^2-1)$ and $p_2 = 4v^2/(v^2-1)$ then $p_1 \notin \{p_2, 1/p_2, 1-p_2, 1-1/p_2, p_2/(1-p_2), 1/(1-p_2)\}$.

then one can make examples with rank $MW_{\mathbf{Q}}(\pi_{12}) \geq 10$. (see 4.6.)

REMARK 2.6. The highest known rank over $\mathbf{Q}$ for an elliptic surface $\pi : X \to \mathbf{P}^1$ over $\mathbf{Q}$ is 14 [**30**]. This example is a member of a family of elliptic surface introduced by Mestre [**44**].

The actual aim of this chapter is to produce explicit equations for the generators of $MW(\pi_i)$ modulo $MW(\pi_1)$, for $i = 2, \ldots, 6$. For convenience, we always assume that E and F have complete 2-torsion over the base field. It is not so hard to deduce from our results, these equations for the case that E and F have 2-torsion points defined over a larger field.

The organization of this chapter is as follows. In Section 3 we indicate an algorithm for finding generators of $MW(\pi)$ in the case that of a rational elliptic surface $\pi : X \to \mathbf{P}^1$ with an additive fiber. In Section 4 we prove the results stated in 2.4. To find generators for the $MW(\pi_i)$ we often rely on the results of Section 3.

3. Finding sections on a rational elliptic surface with an additive fiber

Fix a field K of characteristic 0 and a rational elliptic surface $\pi : X \to \mathbf{P}^1$ over K. Such a surface can be represented by a Weierstrass equation

$$(3)\quad y^2 = x^3 + (\alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0)x + \beta_6 t^6 + \beta_5 t^5 + \beta_4 t^4 + \beta_3 t^3 + \beta_2 t^2 + \beta_1 t + \beta_0,$$

with $\alpha_i, \beta_j \in K$. Assume that over $t = \infty$ the fiber is singular and of additive type. This happens if and only if $\alpha_4 = \beta_6 = 0$.

From [**53**, Theorem 2.5] we know that $MW(\pi)$ is generated by sections of the form

$$(4)\qquad x = b_2 t^2 + b_1 t + b_0, y = c_3 t^3 + c_2 t^2 + c_1 t + c_0.$$

Substituting (4) in (3) and using $\alpha_4 = \beta_6 = 0$ yields that

$$c_3^2 = b_2^3.$$

Set $c_3 = p_1^3, b_2 = p_1^2$. First we search for solutions with $p_1 \neq 0$. The equation for the coefficient of t^5 is an equation of the from $p_1^2 c_2 + f$ with f a polynomial in $p_1, b_1, \alpha_i, \beta_j$. Hence we can express c_2 in terms of the $p_1, b_1, \alpha_i, \beta_j$. Similarly, we can express c_1 and c_0 in terms of $p_1, b_k, \alpha_i, \beta_j$. One easily shows that if this procedure fails, then $p_1 = 0$.

Unfortunately, the three remaining equations $F_m = 0$ in $b_1, b_0, p_1, \alpha_i, \beta_j$ are not linear in any of the variables, but the degree of the F_i in b_1, b_0, p_1 is sufficiently low to compute the resultant

$$R(p_1) := \operatorname{res}_{b_1}(\operatorname{res}_{b_0}(F_1, F_3), \operatorname{res}_{b_0}(F_2, F_3))$$

in concrete examples (i.e., after substituting constants for the α_i, β_j). Calculating $R(p_1)$ and finding zeroes of it gives all possibilities for p_1. Substituting such a value of p_1 in $\operatorname{res}(F_1, F_3, b_0)$ and in $\operatorname{res}(F_2, F_3, b_0)$ yields two polynomials in b_1. Calculating the g.c.d. of these polynomials gives a list of possible values for b_1. Then substituting all possibilities for (p_1, b_0) in F_1, F_2, F_3 gives all possibilities for p_1, b_0, b_1.

Consider the case $p_1 = 0$, hence $c_3 = b_2 = 0$. The coefficient of t^5 is zero if and only if $\alpha_5 = 0$. If this is the case then the coefficient of t^4 is of the form $-c_2^2 + \beta_3 b_1 + \alpha_4$.

If $\beta_3 \neq 0$, then one eliminates b_1, b_0, c_0 as is done above, yielding two polynomial equation in two unknowns. In this case it suffices to compute only one resultant.

If $\beta_3 = 0$ then we can substitute $c_2 = \pm\sqrt{\alpha_4}$. The coefficient of t^3 is of the form $-2\sqrt{\alpha_4}c_1 + f$, with f a polynomial in $c_0, \alpha_i, \beta_j, \sqrt{\alpha_4}$. We can solve c_1 and obtain $-2\sqrt{\alpha_4}c_0 + g$ as a coefficient for t^3, where g is a polynomial in $c_0, \alpha_i, \beta_j, \sqrt{\alpha_4}$.

This fails when $\alpha_4 = 0$. In that case $c_2 = 0$, and one has four polynomial relations in b_0, c_0, c_1, b_1. We can eliminate c_1 as above, yielding three polynomial relations in three unknowns which can be solved as above. In this way we find all possible sections of the form (4).

Top [**80**, Section 5] discusses the following elliptic surface:

THEOREM 3.1. *Let $\pi : X \to \mathbf{P}^1$ be the elliptic surface*

$$y^2 = x^3 + 108(27t^4 - 74t^3 + 84t^2 - 48t + 12).$$

Then $MW_{\mathbf{Q}}(\pi)$ has rank 3 and is generated by sections σ_i with x-coordinates

$$x(\sigma_1) = 6t,\ x(\sigma_2) = 6t - 8,\ x(\sigma_3) = -12t + 9.$$

and fix a primitive cube root of unity ω. Let τ_i be obtained from σ_i by multiplying the x-coordinate with ω. Then the σ_i and τ_i generate a subgroup of finite index of $MW(\pi)$.

Top found the sections $\pm\sigma_1, \pm\sigma_2, \pm\tau_1$ and $\pm\tau_2$; an explicit description of the third independent section σ_3 seems not to be present in the literature.

PROOF. The elliptic surface $\pi : X \to \mathbf{P}^1$ has 4 fibers of type II and at $t = \infty$ a fiber of type IV. From this it follows that $MW(\pi)$ has rank 6. Top [**80**, Section 5] observes that the Mordell-Weil group is generated by a subset of the 18 sections of the form $(\omega x_i, \pm y_i), i = 1, 2, 3$, with $\omega^3 = 1$ and x_i is a polynomial of degree 1.

The form of the generators imply that, in terms of the above discussion, we are looking for sections with $p_1 = 0$. After eliminating all the c_i and b_j, except for b_1, we obtain a polynomial of degree 27. It can be factored as the product of $(b_1 - 6)^2(b_1 + 12)$ times 12 polynomials of degree 2, where all the degree 2 factors have discriminant -3.

This implies that $x = 6t$, $x = 6t - 8$ and $x = -12t + 9$ are the only x-coordinates of degree 1 defined over $\mathbf{Q}$. These sections are disjoint from the zero-section, and intersect at $t = \infty$ the singular fiber in the non-identity component. One can choose the y-coordinates of the three sections in such a way that the third sections is disjoint from the first two. This implies that the height pairing (see [**70**, Definition 8.5]) yields the following intersection matrix

$$\begin{pmatrix} 4/3 & 2/3 & 2/3 \\ 2/3 & 4/3 & 2/3 \\ 2/3 & 2/3 & 4/3 \end{pmatrix}.$$

The determinant of this matrix is non-zero. This implies that these three sections generate a rank 3 subgroup G_1, and $G_1 = MW_{\mathbf{Q}}(\pi)$.

Multiplying the x-coordinates with a cube root of unity will yield another rank 3 subgroup G_2 corresponding to a different eigenspace of the action of complex multiplication hence $G_1 \oplus G_2$ generate a rank 6 subgroup of the Mordell-Weil group. □

4. Explicit formulas

Fix a field K of characteristic 0. Let $E : y^2 = x^3 + ax + b$ and $F : y^2 = x^3 + cx + d$ be elliptic curves, with complete two-torsion over K. Assume that E and F have distinct

j-invariant. Fix λ, μ, ν and ξ such that E is isomorphic to $y^2 = x(x-\lambda)(x-\mu)$ and F is isomorphic to $y^2 = x(x-\nu)(x-\xi)$. Then we may assume that

$$a = \frac{1}{3}(\lambda\mu - \lambda^2 - \mu^2),\ b = \frac{1}{27}(3\lambda\mu(\lambda+\mu) - 2(\lambda^3+\mu^3)),$$

and similar equations for c and d.

The Kummer surface Y is birational to the surface $S \subset \mathbf{A}^3$ given by $(x^3+ax+b)t^2 = z^3+cz+d$, and the fibration $\psi : Y \to \mathbf{P}^1$ is corresponds to the map $(x,z,t) \mapsto t$.

LEMMA 4.1. *The groups $MW(\pi_i), i \geq 1$ are torsion-free.*

PROOF. Kuwata [**41**, Theorem 4.1] shows that π_6 has smooth fibers over $t = 0, \infty$ and only singular fibers of type I_1 or II. Hence the same holds for π_{6i}. This fact together with [**47**, Corollary VII.3.1] implies that the group $MW(\pi_{6i})$ is torsion-free. Since $MW(\pi_i)$ is a subgroup of $MW(\pi_{6i})$ it is also torsion-free. □

We now discuss how to find explicit formulas for generators of $MW(\pi_i), i = 1, \dots 6$.

4.1. $\pi_1 : X_1 \to \mathbf{P}^1$. It is not easy to find explicit equations for sections on π_1, since there are infinitely many cases, depending on the minimal degree of an isogeny $E \to F$. Instead we give a sufficient condition to have rank 1 or 2 over K.

LEMMA 4.2. *The group $MW_K(\pi_1)$ has rank at most 2.*

- *If E and F are isogenous over K then $MW_K(\pi_1)$ has positive rank.*
- *If E and F are isogenous over K and E admit complex multiplication then $MW_K(\pi_1)$ has rank 2.*
- *If $MW_K(\pi_1)$ has positive rank then there exists a degree at most two extension L/K such that E and F are isogenous.*
- *if $MW_K(\pi_1)$ has rank 2 then there exists a degree at most two extension L'/L such that E admits complex multiplication over L'.*

PROOF. Since E and F have complete two torsion it follows that $\operatorname{rank} MW_K(\pi_1) = r$ if and only if $18 + r = \operatorname{rank} NS_K(X_2) = \operatorname{rank} NS_K(Y)$, with Y the Kummer surface of E and F.

It is easy to see that if E and F satisfy the first, resp., second assumption then the rank of $NS_K(Y)$ is at least 19, resp., 20.

If $NS_K(Y) \geq 19$ then E and F are isogenous over some extension of K. Let Γ be the graph of the isogeny. Then the push-forward of Γ on Y is Galois-invariant. This implies that $([-1]\times[-1])^*\Gamma + \Gamma$ is Galois-invariant. From this it follows that E and F are isogenous over a degree 2 extension.

If $NS_K(Y) = 20$ then E and F are isogenous over some extension of K and E has potential complex multiplication. Let Γ be the graph of an isogeny, let Γ' be the graph of the isogeny composed with complex multiplication. Then the push-forward of both Γ and Γ' are Galois-invariant. As above, this implies that E admits complex multiplication over a degree 2 extension of L. □

REMARK 4.3. Suppose E and F are elliptic curves, not isogenous over K, but isogenous over a degree 2 extension L/K. Suppose that $\operatorname{End}(E) = \mathbf{Z}$. Let $\varphi : E \to F$ be an isogeny defined over L. It is an easy exercise to show that the divisor

$$D := \{(P, \varphi(P)) | P \in E\} \cup \{(P, -\varphi(P)) \mid P \in E\} \subset E \times F$$

is invariant under the action of $\mathrm{Gal}(L/K)$. Hence the push-forward of D onto the Kummer is invariant under the action of $\mathrm{Gal}(\overline{K}/K)$. The argument used in the above proof gives that $MW_K(\pi_1)$ has rank at least 1, while E and F are not isogenous over K.

4.2. $\pi_2 : X_2 \to \mathbf{P}^1$**.** Since $MW(\pi_2)$ is torsion-free, we have that $MW(\pi_1') \oplus MW(\pi_1)$ is of finite index in $MW(\pi_2)$, where $\pi_1' : X_1' \to \mathbf{P}^1$ is the twist of π_1 at 0 and ∞. (For more on this topic see Section 1.5.)

One can easily show that $MW(\pi_1') \subset MW(\pi_3')$, where π_3' is the twist of π_3 at 0 an ∞. Since $MW(\pi_3')$ can be considered in a natural way as a subgroup of $MW(\pi_6)$, we refer to that subsection for a discussion of the results. The explicit equations for generators of $MW(\pi_2')$ are given in [**32**]. If we drop for a moment the condition that E and F have complete 2-torsion over K, the we can use the result in [**32**] to prove that $MW_K(\pi_1')$ has rank 4 if and only if E and F have complete two-torsion over K.

4.3. $\pi_3 : X_3 \to \mathbf{P}^1$**.** A Weierstrass equation for π_3 is

$$y^2 = x^3 - 48acx + (\Delta(F)t^3 + 864bd + \Delta(E)t^{-3}).$$

Setting $s = (t + \alpha_i/t)$, with $\alpha_i^3 = \Delta(E)/\Delta(F)$ and $i = 1, 2, 3$, will give an equation for the rational elliptic surface $\psi_i : S_i \to \mathbf{P}^1$, given by

$$y^2 = x^3 - 48acx - \Delta(F)(s^3 - 3\alpha_i s) + 864bd.$$

This surface has a fiber of type I_0^* at $s = \infty$. For each choice of α_i we obtain an isomorphic surface over $K(\zeta_3)$, but the pullback of the sections to π_3 depends on the choice of α_i:

LEMMA 4.4. *For $i \neq j$ we have $MW(\psi_i) \cap MW(\psi_j) = \{\sigma_0\}$, where $MW(\psi_i)$ and $MW(\psi_j)$ are considered as subgroups of $MW(\pi_3)$.*

PROOF. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Set $L := K(\zeta_3, \alpha_1)$. Let $G \subset \mathrm{Aut}(L(t))$ be generated by $\varphi_1 : t \mapsto \alpha_1/t$ and $\varphi_2 : t \mapsto \alpha_1\zeta_3/t$. From $\varphi_1 \circ \varphi_2 : t \mapsto t\zeta_3$ it follow that $\#G \geq 6$. Set $s' = t^3 + \alpha_1^3/t^3$. Then s' is fixed under G. We have the following inequalities

$$6 \leq \#G = [L(t)^G : L(t)] \leq [L(s') : L(t)] = 6.$$

These inequalities give that $L(t)^G = L(s')$. This implies that a section in $MW(\psi_i) \cap MW(\psi_j)$ is the pull back of a section of the elliptic surface ψ' with Weierstrass equation

$$y^2 = x^3 - 48acx - \Delta(F)s' + 864bd.$$

This is an equation of rational elliptic surface with a II^*-fiber. In this case the Shioda-Tate formula 1.2.11 implies that $MW(\psi')$ has rank 0. Since $MW(\psi')$ is a subgroup of the torsion-free group $MW(\pi_3)$ (see Lemma 4.1) it follows that $\#MW(\psi') = 1$. □

LEMMA 4.5. *We have that* $\mathrm{rank}\, MW(\psi_i) = 4$.

PROOF. From the equation of ψ_i one easily sees that it a rational elliptic surface, with a fiber of type I_0^* over $s = \infty$, and no other reducible singular fibers. Hence the Shioda-Tate formula 1.2.11 implies that $\mathrm{rank}\, MW(\psi_i) = 4$. □

LEMMA 4.6. *For $i \neq j$ we have $(MW(\psi_i) \oplus MW(\psi_j)) \cap MW(\pi_1) = \{\sigma_0\}$, considered as subgroups of $MW(\pi_3)$.*

PROOF. From Lemma 4.4 and Lemma 4.5 it follows that $MW(\psi_i) \oplus MW(\psi_j)$ injects into $MW(\pi_3)$. Consider the vector spaces $V = (MW(\psi_i) \oplus MW(\psi_j)) \otimes \mathbf{C}$ and $W = MW(\pi_1) \otimes \mathbf{C}$. The automorphism $\sigma : (x, y, t) \mapsto (x, y, \zeta_3 t)$ induces a trivial action on W. We now prove that σ maps V to itself and all eigenvalues of this action are different from 1.

Assume for the moment that λ, μ, ν and ξ are algebraically independent over $\mathbf{Q}$. This defines a Kuwata elliptic surface

$$\Pi_k : \mathcal{X}_k \to \mathbf{P}^1_{\mathbf{Q}(\lambda,\mu,\nu,\xi,\sqrt[3]{\lambda(\lambda-\mu)\mu/\nu(\nu-\xi)\xi})}$$

and, similarly,

$$\Psi_k : \mathcal{S}_k \to \mathbf{P}^1_{\mathbf{Q}(\lambda,\mu,\nu,\xi,\sqrt[3]{\lambda(\lambda-\mu)\mu/\nu(\nu-\xi)\xi})}.$$

Since $MW(\Pi_1) = 0$, and $MW(\Psi_i) \oplus MW(\Psi_j)$ has rank 8, it follows that a lift $\tilde{\sigma}$ of σ acts on $\tilde{V} := MW(\Psi_i) \oplus MW(\Psi_j)$, hence σ acts on V. If $\tilde{\sigma}$ would have an eigenvalue 1 on $\tilde{V}$, then $MW(\Pi_1)$ would be non-trivial. Hence also σ acts without eigenvalue 1. This implies that $V \cap W = 0$, yielding the lemma. □

From the Lemmas 4.4, 4.5 and 4.6 and Theorem 2.1 it follows that for $i \neq j$ we have that $MW(\psi_i) \oplus MW(\psi_j)$ generates a subgroup of finite index in $MW(\pi_3)/MW(\pi_1)$. Hence if one wants to describe the Galois representation on $MW(\pi_3)/MW(\pi_1)$ it suffices to describe the Galois representation on $MW(\psi_i)$.

Since $\psi_i : S_i \to \mathbf{P}^1$ are rational elliptic surfaces with an additive fiber at $s = \infty$ we can apply Section 3 to calculate expressions for the generators. The relevant formulae for this may be found in [**32**].

4.4. $\pi_4 : X_4 \to \mathbf{P}^1$**.** Let $\pi_2' : X_2' \to \mathbf{P}^1$ be the twist of π_2 by the points 0 and ∞. (Compare Section 1.5.) The results of Section 1.5 imply that the group $MW(\pi_2) \oplus MW(\pi_2')$ is of finite index in $MW(\pi_4)$. Since the fibers over 0 and ∞ of π_2 are of type IV^* and all other fibers are of type I_1 or II, it follows from the results mentioned in Section 1.5 that π_2' has only fibers of type II or I_1. Moreover, $\pi_2' : X_2' \to \mathbf{P}^1$ defines a rational elliptic surface, hence $\operatorname{rank} MW(\pi_2') = 8$.

LEMMA 4.7. *Let $\pi : X \to \mathbf{P}^1$ be a Jacobian elliptic surface such that all singular fibers are irreducible. Let $Z := \sigma_0(\mathbf{P}^1)$. Let S be the image of a section $\sigma : \mathbf{P}^1 \to X$. Denote S_n the image of $(n\sigma) : \mathbf{P}^1 \to X$.*

Then the equality $(S_n \cdot Z) = (S_m \cdot Z)$ holds if and only if $n = \pm m$.

PROOF. Let $\langle \cdot, \cdot \rangle$ denote the height paring on $MW(\pi)$ (see [**70**, Definition 8.5]). In this case $\langle T, T \rangle = 2(T \cdot Z) - 2\chi(X)$, hence

$$2(S_n \cdot Z) - 2\chi(X) = \langle S_n, S_n \rangle = \langle nS_1, nS_1 \rangle = n^2 \langle S_1, S_1 \rangle = 2n^2(S_1 \cdot Z) - 2n^2\chi(X).$$

It follows that $(S_n \cdot Z) = n^2((S \cdot Z) - \chi(X)) + \chi(X)$, which yields the lemma. □

Assume, for the moment, that λ, μ, ν and ξ are algebraically independent over $\mathbf{Q}(t)$. We can consider the elliptic surface π_2' as an elliptic curve A over $K' := \mathbf{Q}(\lambda, \mu, \nu, \xi, t)$.

Then on A we have that

$$x = \frac{2(\sqrt{\lambda} + \sqrt{\mu})\lambda\mu}{\sqrt{\xi} + \sqrt{\xi - \nu}} \frac{1}{t} + \left(-\frac{4}{3}(2\xi - \nu)(\lambda + \mu) + 4\sqrt{\lambda\mu\xi(\xi - \nu)} \right) + \frac{2(\sqrt{\xi} + \sqrt{\xi - \nu})\xi(\xi - \nu)}{\sqrt{\lambda} + \sqrt{\mu}} t$$

is a x-coordinate of a point P_1, hence giving rise to 2 different points on A (see [**32**]). If we plug this x-coordinate in the equation for A, then

$$y^2 = 2(\sqrt{\xi} + \sqrt{\xi - \nu})(\sqrt{\lambda} + \sqrt{\mu})h(t)^2$$

for some $h \in \mathbf{Q}(t, \sqrt{\lambda}, \sqrt{\mu}, \sqrt{\xi}, \sqrt{\nu - \xi})$ (see [**32**]).

Using P_1 we now show how to find points $P_2, \dots P_8$, such that if we specialize, the P_i generate $MW(\pi_2')$. Let P_2 be a section such that $x(P_2)$ is obtained from $x(P_1)$ by replacing $\sqrt{\mu}$ by $-\sqrt{\mu}$. (The existence follows from the fact that the map $\sqrt{\mu} \mapsto -\sqrt{\mu}$ fixes the equation of π_2'.)

LEMMA 4.8. *The points P_1 and P_2 are independent in $A(\overline{K})$.*

PROOF. Since $x(P_1) \neq x(P_2)$ it follows that $P_1 \neq \pm P_2$. Hence the sections S_1 and S_2 on π_2' satisfy $(S_1 \cdot Z) = (S_2 \cdot Z)$ and $S_1 \neq \pm S_2$. It follows from Lemma 4.7 that S_1 and S_2 generate a subgroup of $MW(\pi_2')$ of rank 2, hence P_1 and P_2 are independent. □

Note that we have $P_1 \in A(K_+)$ and $P_2 \in A(K_-)$, where

$$K_\pm := K'(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\xi}, \sqrt{\xi - \nu}, \sqrt{2(\sqrt{\xi} + \sqrt{\xi - \nu})(\sqrt{\mu} \pm \sqrt{\lambda})}).$$

The automorphism σ of K', fixing $\mathbf{Q}$ and further given by

$$(t, \lambda, \mu, \nu, \xi) \mapsto \left(\frac{1}{t}, \nu, \xi, \lambda, \mu\right)$$

is an automorphism of A, (i.e., swapping E and F, and applying the $\mathbf{P}^1$-automorphism $t \mapsto 1/t$). This yields to different points $P_3 := \sigma(P_1), P_4 := \sigma(P_2)$ on A.

LEMMA 4.9. *The points $P_1, \dots P_4$ generate a rank 4 subgroup of $A(\overline{K})$.*

PROOF. The points P_3 and P_4 are not in $A(K_+K_-)$, but are contained in $A(K'_\pm)$, where

$$K'_\pm = K'(\sqrt{\mu}, \sqrt{\mu - \lambda}, \sqrt{\nu}, \sqrt{\xi}, \sqrt{2(\sqrt{\mu} + \sqrt{\mu - \lambda})(\sqrt{\xi} \pm \sqrt{\nu})}).$$

Applying [**63**, Lemma 1.3.2] yields that the P_i generate a rank 4 subgroup. □

Let τ' automorphism of K' fixing $\mathbf{Q}(t)$ and mapping $(\xi, \lambda) \leftrightarrow (\nu, \mu)$ (i.e., we are interchanging the role of the two-torsion points of both E and F). Set $P_i' = \tau(P_i)$.

LEMMA 4.10. *The points $P_1, \dots, P_4, P_1', \dots, P_4'$ generate a subgroup of rank 8 of $A(\overline{K'})$.*

PROOF. The points $P_1', \dots, P_4'$ are defined over different fields (i.e., they are defined over $K_\pm(i)$ or $K'_\pm(i)$, but not over $K_\pm$ or $K'_\pm$.) From [**63**, Lemma 1.3.2] it follows then that the rank of the subgroup generated by the S_i, S_i' is 8. □

PROPOSITION 4.11. *The sections of π_2' associated to the points $P_1, \dots P_4, P_1', \dots P_4'$ of A generate a rank 8 subgroup of $MW(\pi_2')$.*

PROOF. Let $\Pi_2' : \mathcal{X}_2' \to \mathbf{P}^1_{\mathbf{Q}(\lambda,\mu,\nu,\xi)}$ be the elliptic surface corresponding to A/K'. The points P_i, P_i' define sections $\mathcal{S}_i, \mathcal{S}_i'$ of Π_2', which generate a rank 8 subgroup of $MW(\Pi_2')$. This implies that for a general specialization the specialized sections S_i, S_i' generate a rank 8 subgroup of $MW(\pi_2')$. Since the intersection numbers are constant under deformation, it follows that if the S_i, S_i' do not generate $MW(\pi_2')$, then π_2' has at least one reducible fiber. From [**41**, Theorem 4.3] it follows that then E and F are isomorphic, which contradicts our assumption at the beginning of this section. □

COROLLARY 4.12. *The sections of π_2' associated to the points $P_1, \dots, P_4, P_1', \dots, P_4'$ generate a subgroup of finite index in $MW(\pi_4)/MW(\pi_2)$.*

PROOF. This follows from the above Proposition together with the observation that the direct sum of $MW(\pi_2')$ and $MW(\pi_2)$ generates a subgroup of finite index of $MW(\pi_4)$. $\square$

We will now take special values for λ, μ, ν, ξ such that several sections are defined over $\mathbf{Q}$.

COROLLARY 4.13. *For infinitely many pairs of elliptic curves (E, F) over $\mathbf{Q}$, the group $MW_{\mathbf{Q}}(\pi_4)$ has rank 8 or 9.*

PROOF. We have that $\operatorname{rank} MW_{\mathbf{Q}}(\pi_4) = \operatorname{rank} MW_{\mathbf{Q}}(\pi_2) + \operatorname{rank} MW_{\mathbf{Q}}(\pi_2')$. From the results in 4.1 and 4.2 we know that $\operatorname{rank} MW_{\mathbf{Q}}(\pi_2)$ is either 4 or 5.

As mentioned in Section 2, one can prove that the rank over $\mathbf{Q}$ of π_2' is at most 4. To obtain rank 4 it suffices to choose λ, μ, ν, ξ such that S_1, S_2, S_3 and S_4 are defined over $\mathbf{Q}$. In order to obtain this set $\lambda = l^2, \mu = m^2, \nu = n^2$, $\xi = k^2$, $k^2 - n^2 = n_2^2$ and $m^2 - l^2 = l_2^2$.

Then

$$2(k + n_2)(m \pm l), 2(m + l_2)(k \pm n)$$

have to be a non-zero square for all choices of $\pm$. One easily computes that this occurs precisely when

$$l_2 n_2 = \frac{u^2(\rho - 1)(\tau^2 - 1)}{4(\rho + 1)\tau^2}$$

for some $u \in \mathbf{Q}^*$ and

$$k = n_2 \frac{\tau^2 + 1}{\tau^2 - 1}, \quad n = n_2 \frac{2\tau}{\tau^2 - 1}, \quad m = l_2 \frac{\rho^2 + 1}{\rho^2 - 1}, \quad l = l_2 \frac{2\rho}{\rho^2 - 1}.$$

From these last equations we can obtain our original λ, μ, ν, ξ.

The associated Legendre parameters are

$$\left(\frac{2\tau}{\tau^2 + 1} \right)^2 \text{ and } \left(\frac{2\rho}{\rho^2 + 1} \right)^2.$$

One easily finds ρ, τ such that the corresponding j-invariants are different. $\square$

For later use, we remark that $\Delta(E)/\Delta(F)$ is a third power if and only if

$$\frac{\tau(\tau^4 - 1)}{\rho(\rho^4 - 1)}$$

is a third power. A necessary condition for obtaining curves with different j-invariant is $\rho \neq \tau$. The only solution we found with these properties is $(\rho, \tau) \in \{(2, 3), (3, 2)\}$. This gives rise to two curves having both Legendre parameter 25/9, hence does not give an interesting solution.

4.5. $\pi_5 : X_5 \to \mathbf{P}^1$**.** An equation for π_5 is

$$y^2 = x^3 - 48acx + (\Delta(F)t^5 + 864bd + \Delta(E)t^{-5})$$

Setting $s = (t+\alpha_i/t)$, with $\alpha_i^5 = \Delta(E)/\Delta(F)$, gives a rational elliptic surface $\psi_i : S_i \to \mathbf{P}^1$.

We can now copy the strategy used for of π_3. We have that $MW(\pi_5)/MW(\pi_1)$ has rank 16, the $MW(\psi_i)$ have rank 8, the intersection $MW(\psi_i) \cap MW(\psi_j) = \{\sigma_0\}$, considered as subgroups of $MW(\pi_5)$ and $(MW(\psi_i) \oplus MW(\psi_j)) \cap MW(\pi_1) = \{\sigma_0\}$. This combined with the fact that all the ψ_i are isomorphic over $K(\zeta_5)$ implies that it suffices to find sections of only one of the ψ_i. One can show that ψ_i is a rational elliptic surface with an additive fiber at $t = \infty$. Hence we can apply Section 3 to find an expression for the sections of $MW(\pi_5)$. In all choices for λ, μ, ν, ξ we tried the final resultant is a product of two polynomials of degree 120. (See [**32**])

4.6. $\pi_6 : X_6 \to \mathbf{P}^1$**.** Since π_6 is torsion-free we have that $MW(\pi_3) \oplus MW(\pi_3')$ is of finite index in $MW(\pi_6)$, where $\pi_3' : X_3' \to \mathbf{P}^1$ is the twist of π_3 at 0 and ∞.

The group $MW(\pi_3)$ is described above. We discuss here how to find generators for $MW(\pi_3')$. We follow Kuwata [**41**, Section 5]. Kuwata observes that X_3' is birational over $\mathbf{Q}$ to the cubic surface in $\mathbf{P}^3$

$$C : Z^3 + cZY^2 + dY^3 = X^3 + aXW + bW^3,$$

and that the strict transforms of the 27 lines of C to X_3' generate $MW(\pi_3')$.

It has been known for a long time how to find the 27 lines on a cubic surface of the above form, see for example [**61**]. Due to our special situation we give a somewhat different approach to find all 27 lines.

In our case C is isomorphic to

$$(*) \qquad Z(Z-\nu Y)(Z-\xi Y) = X(X-\lambda W)(X-\mu W).$$

One finds 9 lines defined over $\mathbf{Q}$, namely the intersections of $Z-\alpha Y=0$ and $X-\beta W=0$, with $\alpha \in \{0,\nu,\xi\}$ and $\beta \in \{0,\lambda,\mu\}$. We give now the equations for the strict transforms of these lines on X_3:

LEMMA 4.14. *Set*

$$A = (a_{ki}) = \begin{pmatrix} \nu-\xi & \xi-\nu & \nu \\ -\xi & -\nu & \xi \\ -\mu & -\lambda & \lambda \\ \lambda-\mu & \mu-\lambda & \mu \end{pmatrix}.$$

Then

$$x = 4a_{1i}a_{2i}t^2 + \frac{4}{3}(a_{1i}+a_{2i}+a_{3j}+a_{4j})t + 4a_{3j}a_{4j}$$

and

$$y = 4a_{1i}a_{2i}(a_{1i}+a_{2i})t^3 + 8a_{1i}a_{2i}(a_{3j}+a_{4j})t^2 + 8a_{3j}a_{4j}(a_{1i}+a_{2i})t + 4a_{3j}a_{4j}(a_{3j}+a_{4j})$$

for $1 \le i,j \le 3$ *are sections of* π_3'.

PROOF. This is a straightforward computation. See [**32**]. ☐

The usual strategy to find the other 18 lines is to take pencils of planes through the lines we found above. We choose a different strategy. The image of a section on X_3' can be pulled back to a divisor on X_6. We can take the push-forward to Y of this divisor and then push-forward in a natural way to $\mathbf{P}^1 \times \mathbf{P}^1$. We give 6 divisors on $\mathbf{P}^1 \times \mathbf{P}^1$, such that the pull back of such a divisor to X_6 consists of three components, and the 18 divisors obtained in this way correspond to the 18 lines on C. Consider $\mathbf{P}^1 \times \mathbf{P}^1$ with projective coordinates X, W and Y, Z. Set

$$P_1 := [0,1], P_2 := [\lambda, 1], P_3 := [\mu, 1], Q_1 := [0,1], Q_2 := [\nu, 1], Q_3 := [\xi, 1].$$

Fix a permutation $\sigma \in S_3$. Let $C_\sigma \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the $(1,1)$-curve going through $(P_i, Q_{\sigma(i)})$. For example, C_{id} is given by

$$G(X,Y,Z,W) := \nu\xi(\lambda-\mu)XW + \mu\lambda(\xi-\nu)WZ + (\nu\mu - \xi\lambda)XZ = 0.$$

We describe what the corresponding divisor on C is. This can be done by dehomogenizing first, say, we set $W = Y = 1$. Then we can express Z in terms of X. Substitute this expression for Z in $(*)$. We obtain a rational expression in X (a quotient of a polynomial of degree 6 and a polynomial of degree 3). The numerator is zero if and only if $X = 0, X = \lambda, X = \mu$ or X satisfies a degree 3 polynomial f. For example if $\sigma = \mathrm{id}$ then

$$\left(\frac{\mu\lambda(\nu-\xi) + \sqrt[3]{\mu\lambda(\mu-\lambda)\nu^2\xi^2(\nu-\xi)^2}}{\nu\mu - \xi\lambda}\right)\zeta_3^k =: \gamma_k$$

are the zeroes of f. Let H be the hyperplane $X = \gamma_k Y$. Let H' be $G(X,Y,\gamma_k,1) = 0$. Then $H \cap H'$ is contained in C and defines a line $\ell_{\sigma,k}$. One can easily show that for all $\sigma \in H$ one obtains three such hyperplanes, and that one obtains 18 lines in total.

Remark 4.15. One can show that 6 of these 18 lines are defined over

$$K(E[2], F[2], \sqrt[3]{\Delta(E)/\Delta(F)}),$$

and the 12 others over

$$K(E[2], F[2], \mu_3, \sqrt[3]{\Delta(E)/\Delta(F)}).$$

Moreover, one shows easily that if one of the 18 lines is defined over $K(E[2], F[2], \mu_3)$, then $\Delta(E)/\Delta(F) \in K(E[2], F[2])^{*3}$. This contradicts the claim in [**41**, Section 5], which states that all 27 lines are defined over $K(E[2], F[2], \mu_3)$.

Remark 4.16. One can easily find elliptic curves E, F over $\mathbf{Q}$ with complete two-torsion defined over $\mathbf{Q}$ such that $\Delta(E)/\Delta(F) \in \mathbf{Q}^{*3}$: one needs to find solutions of

$$\lambda(\lambda-\mu)\mu = \tau^3\nu(\nu-\xi)\xi.$$

This defines a conic over $\mathbf{Q}(\tau, \mu, \xi)$ containing the rational point $\lambda = 0$ and $\nu = 0$. Hence one can parameterize this conic and find solution giving rise to elliptic curves. For example $(\lambda, \mu, \nu, \xi, \tau) = (16, 1, 6, 1, 2)$ gives an example.

Remark 4.17. Suppose one can find a solution in $\mathbf{Q}^3$ of

$$\frac{\tau(\tau^4-1)}{\rho(\rho^4-1)} = \sigma^3$$

satisfying the conditions mentioned at the end of 4.4. Then we obtain examples such that $\operatorname{rank} MW_{\mathbf{Q}}(\pi_6) \geq 6$ and $\operatorname{rank} MW_{\mathbf{Q}}(\pi_4) \geq 8$, yielding $\operatorname{rank} MW_{\mathbf{Q}}(\pi_{12}) \geq 10$.

CHAPTER 4

Elliptic $K3$ surfaces with Mordell-Weil rank 15

1. Introduction

The Mordell-Weil rank r of a Jacobian elliptic surface $\pi : X \to C$ is defined as the rank of the group of sections of π. If X is a $K3$ surface, then it follows easily that $C = \mathbf{P}^1$. If one works over a field of characteristic 0, then it is well known that $0 \leq r \leq 18$. (In positive characteristic we know that $0 \leq r \leq 20$.)

By a result of Cox [**17**] there exists a Jacobian elliptic $K3$ surface defined over $\mathbf{C}$ with any given Mordell-Weil rank r, with r an integer, $0 \leq r \leq 18$. Actually, using a similar reasoning as in [**17**] one can show there are infinitely many $18-r$-dimensional families of Jacobian elliptic $K3$ surfaces defined over $\mathbf{C}$, with Mordell-Weil rank r. The examples constructed in the proof of Cox are not explicit: the existence of such examples follows from properties of a so-called period map.

Kuwata [**41**] has given a list of explicit Weierstrass equations for elliptic $K3$ surfaces defined over $\mathbf{Q}$ with Mordell-Weil rank r (over $\overline{\mathbf{Q}}$) for any r between 0 and 18, except for the case $r = 15$.

The aim of this chapter is to produce an explicit three-dimensional family of elliptic $K3$ surfaces with Mordell-Weil rank 15.

Theorem 1.1. *Let K be an algebraically closed field, with* $\mathrm{char}(K) \neq 2, 3$. *Let $E_{a,b,c}$ be the curve defined over $K(s)$ given by the Weierstrass equation*

$$y^2 = x^3 + A_{a,b,c}(s)x + B_{a,b,c}(s),$$

with

$$A_{a,b,c}(s) = 4a^3b^3((b-a)cs^8 + (2ac + 2bc + 4ab)s^4 + (b-a)c$$

and

$$B_{a,b,c} = 16a^5b^5s^2((b-a)s^8 + 2(b+a)s^4 + (b-a)).$$

Then for a general $(a, b, c) \in K^3$ this defines an elliptic $K3$ surface with 24 fibers of type I_1 and Mordell-Weil rank at least 15. Moreover if $K = \mathbf{C}$, then a generic member of this family has Mordell-Weil rank 15.

2. Construction

Let K be an algebraically closed field of characteristic different from 2 and 3.

In this chapter we use very often the notion of twisting an elliptic surface by $2n$ points. For more information on this see Section 1.5. We recall that if P is one of the $2n$ distinguished points, then the fiber of P changes in the following way:

$$I_\nu \leftrightarrow I_\nu^*(\nu \geq 0) \qquad II \leftrightarrow IV^* \qquad III \leftrightarrow III^* \qquad IV \leftrightarrow II^*$$

and the type of fiber in any point different from the $2n$ distinguished point remains the same.

$$
\begin{array}{ccccccc}
 & & & & Y & \stackrel{\varphi}{\rightarrow} & \mathbf{P}^1 \\
 & & & \swarrow & \downarrow & & \downarrow g_2' \\
\mathbf{P}^1 & \stackrel{\tilde{\pi}_2}{\leftarrow} & \tilde{X}_2 & & X_2 & \stackrel{\pi_2}{\rightarrow} & \mathbf{P}^1 \\
 & & & \swarrow & \downarrow & & \downarrow g_2 \\
\mathbf{P}^1 & \stackrel{\tilde{\pi}_1}{\leftarrow} & \tilde{X}_1 & & X_1 & \stackrel{\pi_1}{\rightarrow} & \mathbf{P}^1 \\
 & & & \swarrow & \downarrow & & \downarrow f \\
\mathbf{P}^1 & \stackrel{\tilde{\pi}}{\leftarrow} & \tilde{X} & & X & \stackrel{\pi}{\rightarrow} & \mathbf{P}^1
\end{array}
$$

TABLE 1. Overview of all maps used in this section ($g = g_2 \circ g_2'$).

Let $\pi : X \to C$ be a Jacobian elliptic surface, $P_1, \dots P_{2n} \in C$ points. Let $\tilde{\pi} : \tilde{X} \to C$ be a twist by the P_i. Let $\varphi : C_1 \to C$ be a double cover ramified at the P_i, such that the minimal models of base-changing φ and $\tilde{\varphi}$ by π are isomorphic. Denote this model by $\pi_1 : X_1 \to C_1$.

Recall that

$$\text{rank}(MW(\pi_1)) = \text{rank}(MW(\pi)) + \text{rank}(MW(\tilde{\pi})). \tag{5}$$

Moreover, the singular fibers change as follows

Fiber of π at P_i	I_ν or I_ν^*	II or IV^*	III or III^*	IV or II^*
Fiber of π_1 at $\varphi^{-1}(P_i)$	$I_{2\nu}$	IV	I_0^*	IV^*

The following result will be used several times. It is a direct consequence of the Shioda-Tate formula 1.2.11.

THEOREM 2.1 ([**70**, Theorem 10.3]). *Let $\pi : X \to \mathbf{P}^1$ be a rational Jacobian elliptic surface, then the rank of the Mordell-Weil group is 8 minus the number of irreducible components of singular fibers not intersecting the identity component.*

Consider the following construction:

CONSTRUCTION 2.2. Let $\pi : X \to \mathbf{P}^1$ be a Jacobian elliptic surface whose singular fibers are three fibers of type I_1 and one fiber of type III^*. Let $f \in K(t)$ be a function of degree two, such that the fibers of π over the critical values of f are non-singular.

Let $\alpha, \beta \in \mathbf{P}^1$ be the two distinct points such that $f(\alpha) = f(\beta)$ is the point which fiber is of type III^*. Let g be a degree 4 cyclic covering, with only ramification over α, β. Let $\varphi : Y \to \mathbf{P}^1$ be the non-singular relatively minimal model of the fiber product $X \times_{\mathbf{P}^1} \mathbf{P}^1$ with respect to π and $f \circ g : \mathbf{P}^1 \to \mathbf{P}^1$.

PROPOSITION 2.3. *The Mordell-Weil rank of φ (of Construction 2.2) is at least 15, and is precisely 15 if and only if the rank of the twist of π by the two critical values of f is 0.*

PROOF. The assumptions imply that X is a rational surface and hence using Theorem 2.1 we have that $\text{rank}\, MW(\pi) = 1$. Let $\pi_1 : X_1 \to \mathbf{P}^1$ be the fiber product $X \times_{\mathbf{P}^1} \mathbf{P}^1$ with respect to $f : \mathbf{P}^1 \to \mathbf{P}^1$ and π. Let $\tilde{\pi} : \tilde{X} \to \mathbf{P}^1$ be the twist of π by the two critical values of f. Then by (5) and Theorem 2.1

$$\text{rank}(MW(\pi_1)) = \text{rank}(MW(\pi)) + \text{rank}(MW(\tilde{\pi})) = 1 + \text{rank}(MW(\tilde{\pi})).$$

Note that π_1 has two fibers of type III^* and six fibers of type I_1. Let P_1 and P_2 be the points with a fiber of type III^*.

Let $g_2 : \mathbf{P}^1 \to \mathbf{P}^1$ be the degree two function, with critical values P_1 and P_2. Define $\pi_2 : X_2 \to \mathbf{P}^1$ to be the non-singular relatively minimal model of the fiber product $X \times_{\mathbf{P}^1} \mathbf{P}^1$ with respect to π_1 and g_2.

Let $\tilde{\pi}_1$ be the twist of π_1 by P_1 and P_2. Then $\tilde{\pi}_1$ has two fibers of type III and 6 fibers of type I_1, hence the corresponding surface is rational and by Theorem 2.1 $\tilde{\pi}_1$ has Mordell-Weil rank 6. From this it follows that $\operatorname{rank}(MW(\pi_2)) = 7 + \operatorname{rank}(MW(\tilde{\pi}))$. Furthermore, π_2 has two fibers of type I_0^*, and 12 fibers of type I_1.

Let $\tilde{\pi}_2$ be the twist of π_2 by the two points with fiber of type I_0^*. Then $\tilde{\pi}_2$ has 12 fibers of type I_1 and the corresponding surface is rational with Mordell-Weil rank 8. So

$$\operatorname{rank}(MW(\varphi)) = \operatorname{rank}(MW(\pi_2)) + \operatorname{rank}(MW(\tilde{\pi}_2)) = 15 + \operatorname{rank}(MW(\tilde{\pi})).$$

□

Remark 2.4. If we suppose that $\operatorname{rank}(MW(\tilde{\pi})) = 0$, then it is relatively easy to find explicit generators for $MW(\varphi)$. In that case the pull-backs of the generators of $MW(\pi)$, $MW(\tilde{\pi}_1)$, $MW(\tilde{\pi}_2)$ generate a subgroup of $MW(\varphi)$ of index 2^m, for some $m \geq 0$. Since all these three surfaces are rational, we can take a specific Weierstrass model for these surfaces such that all Mordell-Weil groups are generated by polynomials of degree at most 2. (See [**53**].)

Remark 2.5. In the case $K = \mathbf{C}$ there exists another proof. Since Y and $\tilde{X}$ are both $K3$ surfaces, and there exists a finite map between them, the Picard numbers of both surfaces coincide (see [**29**, Corollary 1.2]). From an easy exercise using Kodaira's classification of singular fibers it follows that the configuration of singular fibers of φ is the one mentioned in the Theorem. By Kodaira's classification of singular fibers and the Shioda-Tate formula 1.2.11 we conclude

$$2 + 15 + \operatorname{rank}(MW(\tilde{\pi})) = \rho(X) = \rho(Y) = 2 + \operatorname{rank}(\varphi).$$

Proposition 2.3 enables us to prove the main theorem of this chapter.

Proof of Theorem 1.1. Let $c \in K^*$ such that $c^2 \neq -1$. Then the rational elliptic surface E'_c associated to the Weierstrass equation

$$y^2 = x^3 + t^3(t-c)x + t^5$$

has a fiber of type III^* and three fibers of type I_1. One easily shows that if $E'_c \cong E'_{c'}$ then $c'^2 = c^2$. (If $E'_c \cong E'_{c'}$ then there exist be an automorphism $h : \mathbf{P}^1 \to \mathbf{P}^1$ fixing 0 and ∞, and a constant $\lambda \in K$, verifying $h(t)^3(h(t)-c) = \lambda^4 t^3 (t-c')$ and $h(t)^5 = \lambda^6 t^5$. This implies that $\lambda^2 = 1$ and $c' = \lambda c$.)

Let $a \neq b$ and

$$f_{a,b}(s) = \frac{4abs}{(a-b)s^2 - 2(a+b)s + a - b}.$$

The critical values of $f_{a,b}$ are a and b, and $f^{-1}(0) = \{0, \infty\}$. Hence by Proposition 2.3 the elliptic surface associated to the Weierstrass equation

$$y^2 = x^3 + f_{a,b}(s^4)^3(f_{a,b}(s^4) - c)x + f_{a,b}(s^4)^5$$

satisfies the properties stated in the theorem. After a coordinate change, which clears denominators, we obtain the equation of $E_{a,b,c}$.

This family contains a three-dimensional sub-family of non-isomorphic elliptic surfaces, because it is a finite base change of a three-dimensional family of non-isomorphic elliptic surfaces.

Assume now that $K = \mathbf{C}$. Let $U \subset M_2$ be the set of elliptic surfaces with non-constant j-invariant. (Notation from Chapter 2.)

Suppose that a generic twist of E'_c would have positive Mordell-Weil rank. Then the constructed family $E_{a,b,c}$ would map to a 3-dimensional component C of NL_{18}, moreover the general member of the family $E_{a,b,c}$ has non-constant j-invariant, hence $\dim C \cap U = 3$. From Theorem 2.1.1 it follows that $\dim NL_{18} \cap U \leq 2$, a contradiction. From Proposition 2.3 it follows that the generic member of $E_{a,b,c}$ has Mordell-Weil rank precisely 15. □

REMARK 2.6. Kuwata [**41**] gave explicit examples defined over $\mathbf{Q}$. Theorem 1.1 does not suffice to conclude that there exist examples of $K3$ surfaces defined over $\mathbf{Q}$ with Mordell-Weil rank 15. One can show that the elliptic surface

$$y^2 = x^3 + 2(t^8 + 14t^4 + 1)x + 4t^2(t^8 + 6t^4 + 1)$$

has Mordell-Weil rank 15 over $\overline{\mathbf{Q}}$.

The methods used to prove this result are completely different from the methods we use in this chapter. We intend to develop this in a future publication [**36**].

CHAPTER 5

Classification of all Jacobian elliptic fibrations on certain $K3$ surfaces

1. Introduction

In this chapter we classify all possible bad fiber configurations on Jacobian elliptic fibrations on the $K3$ surface X, which is the minimal model of the double cover of $\mathbf{P}^2$ ramified along six lines in 'general position'. When we say that six lines are in 'general position' we mean that the rank of the Néron-Severi group $NS(X)$ is 16.

The strategy we use is purely geometric, and very similar to Oguiso's classification of Jacobian elliptic fibrations on the Kummer surface of the product of two non-isogenous elliptic curves ([**52**]). It seems possible to extend this method to the case of $K3$ surfaces, which are birational to a double cover of $\mathbf{P}^2$ ramified along a sextic curve, such that the Néron-Severi group of the $K3$ surface is generated by the (reduced) components of the pull-back of the branch divisor together with all divisors obtained by resolving singularities on the double cover and the pull-back of a general line in $\mathbf{P}^2$.

Here we give the full list of possibilities.

THEOREM 1.1. *Let X be as before. Suppose $\pi : X \to \mathbf{P}^1$ be an elliptic fibration with positive Mordell-Weil rank. Then the configuration of singular fibers is contained in the following list.*

Class	*Configuration of singular fibers*		*MW-rank*
1.1	$I_{10}\ I_2\ aII\ bI_1$	$2a+b=12$	4
1.2	$I_8\ I_4\ aII\ bI_1$	$2a+b=12$	4
1.3	$2I_6\ aII\ bI_1$	$2a+b=12$	4
1.4	$IV^*\ I_4\ aII\ bI_1$	$2a+b=12$	5

Conversely, for each class there exists a, b such that these fibrations occur.

We did not establish the possible values of a and b. By Proposition 5.1 we know that generically $a \leq 4$ in $1.1, 1.2$ and 1.3, $a \leq 5$ in 1.4. We believe that for each of the classes $1.1 - 1.4$ we have that generically $a = 0$.

THEOREM 1.2. *Let X be as before. Let $\pi : X \to \mathbf{P}^1$ be a Jacobian elliptic fibration with finite Mordell-Weil group. Then the configuration of singular fibers is contained in Table 1. (In this table we list all fiber types, different from III, I_2, II, I_1, the structure of $MW(\pi)$ and two quantities, namely $iii + i_2$ and $3iii + 2i_2 + 2ii + i_1$, where iii means the number of fibers of type III etc.) Conversely, for each class there exists a Jacobian elliptic fibration $\pi : X \to \mathbf{P}^1$ with the given configuration of singular fibers.*

COROLLARY 1.3. *Let X be as before, and let $\pi : X \to \mathbf{P}^1$ be an elliptic fibration with finite Mordell-Weil group. If the six lines are sufficiently general then the configuration*

Class	Configuration	$MW(\pi)$	$iii + i_2$	$3iii + 2i_2 + 2ii + i_1$
2.1	II^*	1	6	14
2.2	III^*	$\mathbf{Z}/2\mathbf{Z}$	7	15
2.3	$III^* I_0^*$	1	3	9
2.4	I_6^*	1	4	12
2.5	I_4^*	$\mathbf{Z}/2\mathbf{Z}$	6	14
2.6	$I_4^* I_0^*$	1	2	8
2.7	I_2^*	$(\mathbf{Z}/2\mathbf{Z})^2$	8	16
2.8	$I_2^* I_0^*$	$\mathbf{Z}/2\mathbf{Z}$	4	8
2.9	$2I_2^*$	1	2	8
2.10	$I_2^*\ 2I_0^*$	1	0	8
2.11	$2I_0^*$	$(\mathbf{Z}/2\mathbf{Z})^2$	6	12
2.12	$3I_0^*$	$\mathbf{Z}/2\mathbf{Z}$	2	6

TABLE 1. List of possible configurations

Class	Configuration	$MW(\pi)$	i_2	i_1
2.1	II^*	1	6	2
2.2	III^*	$\mathbf{Z}/2\mathbf{Z}$	7	1
2.3	$III^* I_0^*$	1	3	3
2.4	I_6^*	1	4	4
2.5	I_4^*	$\mathbf{Z}/2\mathbf{Z}$	6	2
2.6	$I_4^* I_0^*$	1	2	4
2.7	I_2^*	$(\mathbf{Z}/2\mathbf{Z})^2$	8	0
2.8	$I_2^* I_0^*$	$\mathbf{Z}/2\mathbf{Z}$	4	0
2.9	$2I_2^*$	1	2	4
2.10	$I_2^*\ 2I_0^*$	1	0	8
2.11	$2I_0^*$	$(\mathbf{Z}/2\mathbf{Z})^2$	6	0
2.12	$3I_0^*$	$\mathbf{Z}/2\mathbf{Z}$	2	2

TABLE 2. List of possible configurations on a general X

of singular fibers of π is contained in Table 2. Conversely, for each class there exists a Jacobian elliptic fibration $\pi : X \to \mathbf{P}^1$ with the given configuration of singular fibers.

This corollary is an immediate consequence from Theorem 1.2 and Proposition 5.1.

REMARK 1.4. Note that in the cases 2.7, 2.8, 2.10 and 2.11, the fiber configuration in Theorem 1.2 and Corollary 1.3 are the same.

The generality condition is very essential in our strategy: one of the key tools in this chapter uses the fact that the involution σ on X induced by the double-cover involution acts trivially on the Néron-Severi group $NS(X)$. Our definition of general position implies that σ acts trivial on $NS(X)$.

The organization of this chapter is as follows: In Section 2 we start with recalling some facts about curves on $K3$ surfaces. In Section 3 we recall some standard facts concerning singular fibers of elliptic fibrations and some special results on elliptic surfaces.

In Section 4 we study the Néron-Severi group of a double cover of $\mathbf{P}^2$, ramified along six lines in general position. In Section 5 we give a variant of Proposition 2.3.6, which implies that Corollary 1.3 follows from Theorem 1.2. In Section 6 we list all types of possible singular fibers for an elliptic fibration on X and we count the number of pre-images of the six lines contained in each fiber type. In Section 7 we classify all fibrations in which all special rational curves (the pre-images of the six branch lines) are contained in the singular fibers, thereby proving Theorem 1.1. In Section 8 we prove Theorem 1.2.

Every proof of the actual existence of a fibration presented in this chapter runs as follows: First we give an effective divisor D, with $D^2 = 0$ and such that there exists an irreducible curve $C \subset X$ with $D.C = 1$. It is easy to see that then $|D|$ defines an elliptic fibration $\pi : X \to \mathbf{P}^1$ (see Lemma 2.2), that $C \cong \mathbf{P}^1$, and that $(\pi|_C)^{-1} : \mathbf{P}^1 \to X$ is a section.

In this chapter all fibrations, sections and components of singular fibers are defined over the field of definition of the six lines.

2. Curves on $K3$ surfaces

In this section we give some elementary results on curves on $K3$ surfaces.

LEMMA 2.1. *Suppose D be a smooth curve on a $K3$ surface X. Then*

$$g(D) = 1 + \frac{D^2}{2}$$

PROOF. Since the canonical bundle K_X is trivial, the adjunction formula for a divisor on a $K3$ surface is $2p_a(D) - 2 = D^2$ (see [**28**, Proposition V.1.5]). Since $p_a(D) = g(D)$ for a smooth curve, this implies the result. □

LEMMA 2.2. *Suppose D is an effective divisor on a $K3$ surface X with $p_a(D) = 1$. Then $|D|$ defines an elliptic fibration $\pi : X \to \mathbf{P}^1$. Every effective connected divisor D' such that $D \cdot D' = D'^2 = 0$ is a fiber of π.*

PROOF. From the the adjunction formula [**28**, Proposition V.1.5] it follows that $D^2 = 0$.

Applying Riemann-Roch [**28**, Theorem V.1.6] yields

$$\dim H^0(X, \mathcal{O}_X(D)) + \dim H^0(X, \mathcal{O}_X(-D)) \geq 2.$$

Since $\dim H^0(X, \mathcal{O}_X(-D)) = 0$ (D is effective), we obtain that $\dim H^0(X, \mathcal{O}_X(D)) > 1$. This, combined with the fact that $D^2 = 0$, implies that $|D|$ is base-point-free. So we can apply Bertini's theorem [**28**, Theorem II.8.18 & Remark III.7.91], hence there is an irreducible curve F linearly equivalent to D. By Lemma 2.1 this curve has genus 1.

The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(F) \to \mathcal{O}_F \to 0,$$

together with the facts $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X(F)) = 0$ and $\dim H^2(X, \mathcal{O}_X) = 1$, gives that $H^1(X, \mathcal{O}_X(F)) = 0$, whence by Riemann-Roch [**28**, Theorem V.1.6] we obtain that the dimension of $H^0(X, \mathcal{O}_X(F))$ equals 2, so "the" morphism associated to $|D|$, $\pi : X \to \mathbf{P}^1$ is an elliptic fibration.

Since D' is effective and $D' \cdot D = 0$, we obtain that no irreducible component of D' is intersecting D. Since D' is connected we obtain that $\pi(D')$ is a point. From $D'^2 = 0$ it follows that $p_a(D') = p_a(D)$, hence D' is a fiber. □

3. Kodaira's classification of singular fibers

In this section we describe the possible fibers for a minimal elliptic surface $\pi : X \to \mathbf{P}^1$. The dual graph associated to a (singular, reducible) curve has a vertex for each irreducible component of the curve and has an edge between two vertices if and only if the two corresponding components intersect.

THEOREM 3.1 (Kodaira). *Let $\pi : X \to \mathbf{P}^1$ be an elliptic surface. Then the following types of fibers are possible:*

- *I_0: A smooth elliptic curve.*
- *I_1: A nodal rational curve. (The dual graph is $\tilde{A}_0$.)*
- *$I_\nu, \nu \geq 2$: A ν-gon of smooth rational curves. (The dual graph is $\tilde{A}_{\nu-1}$.)*
- *II: A cuspidal rational curve. (The dual graph is $\tilde{A}_0$.)*
- *III: Two rational curves, intersecting in exactly one point with multiplicity 2. (The dual graph is $\tilde{A}_1$.)*
- *IV: Three concurrent lines. (The dual graph is $\tilde{A}_2$.)*
- *$I_\nu^*, \nu \geq 0$: The dual graph is of type $\tilde{D}_{4+\nu}$.*
- *IV^*, III^*, II^*: The dual graph is of type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.*

For more on this see for example [**47**, Lecture I], [**72**, Appendix C, Theorem 15.2] or [**73**, Theorem IV.8.2].

In Figure 1 we give the dual graph for some of the fiber types. For a resolution $X \to Y$ of a fixed double covering $\varphi' : Y \to \mathbf{P}^2$ ramified along R we define we define "special" curve as the strict transform of a component of $\varphi'^{-1}(R)$, and a curve is called "ordinary" otherwise. This notion depends heavily on our situation: it gives information on the behavior of the double cover involution on the fiber components.

Using our classification of elliptic fibrations one can actually show that X can be obtained in an unique way as a double cover of $\mathbf{P}^2$ ramified along six lines up to automorphism, i.e., given two morphisms $\pi_i : X \to \mathbf{P}^2$ of degree 2, ramified along six lines $\ell_j^{(i)}$, with $j = 1 \dots 6$, there exist an automorphism of $\mathbf{P}^2$, mapping $\{\ell_j^{(1)}\}$ to $\{\ell_j^{(2)}\}$.

Moreover, for all fiber types, except III and I_2, one knows which components are special rational curves. In the dual graphs given here a vertex is drawn as a circle if the component is a ordinary rational curve; a vertex is drawn as a square then the corresponding component is a special rational curve.

4. Divisors on double covers of $\mathbf{P}^2$ ramified along six lines

In this section we study the Néron-Severi group of a double cover of $\mathbf{P}^2$ ramified along six lines. Most of the results are probably well known to the experts and are likely to be found somewhere in the literature.

NOTATION 4.1. Fix six distinct lines $L_i \subset \mathbf{P}^2$, such that no three of them are concurrent. Denote by $P_{i,j}$ the point of intersection of L_i and L_j.

Let $\varphi' : Y \to \mathbf{P}^2$ be the double cover ramified along the six lines L_i. Then Y has 15 double points of type A_1. Resolving these points gives a surface X, with fifteen exceptional divisors and a rational map $\varphi : X \to \mathbf{P}^2$. Denote by σ the involution on X induced by the double-cover involution on Y associated to φ'.

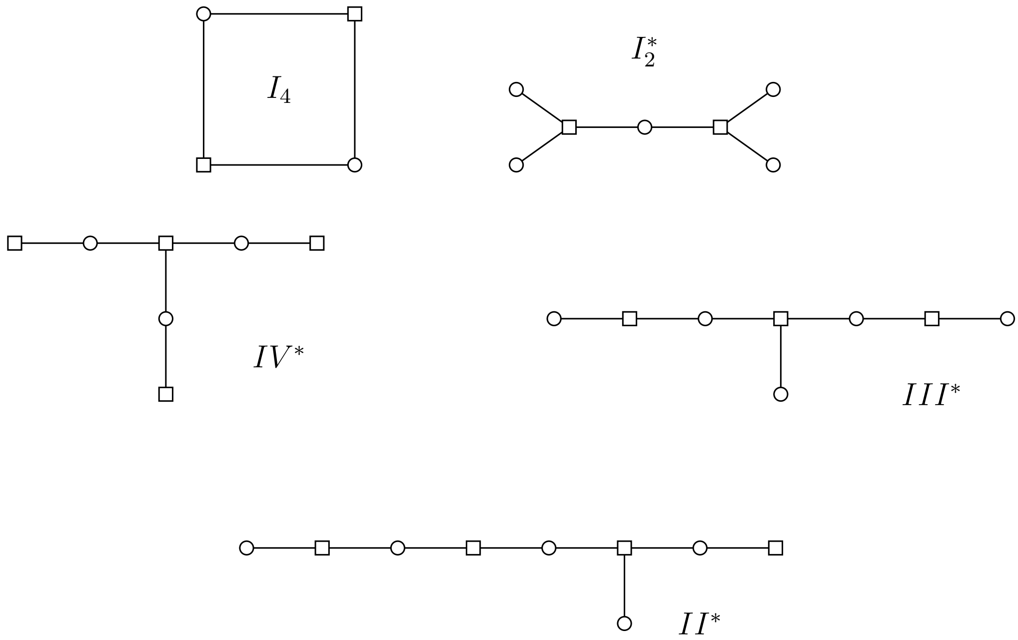

FIGURE 1. Examples of dual graphs of singular fibers

Let $\tilde{\mathbf{P}}$ be the blow-up of $\mathbf{P}^2$ at the points $P_{i,j}$. Then $X/\langle\sigma\rangle = \tilde{\mathbf{P}}$, and $\varphi : X \to \mathbf{P}^2$ factors through $\psi : X \to \tilde{\mathbf{P}}$. Then ψ is a degree 2 cover with branch locus $\tilde{B}$, the strict transform of the six lines L_i.

Denote by $\ell_{i,j} \subset X$ the divisor obtained by blowing up the point of Y above $P_{i,j}$ $(i < j)$. Let ℓ_i be such that $2\ell_i$ is the strict transform of $\varphi'^*(L_i)$.

Let $M^{i,j}_{k,m}$ be the line connecting $P_{i,j}$ and $P_{k,m}$ $(i < j, k < m, i < k$ and i, j, k, m pairwise distinct). Let $\mu^{i,j}_{k,m} \subset X$ be the strict transform of $\varphi'^* M^{i,j}_{k,m}$.

With these notations we have the following intersection results.

LEMMA 4.2. *We have* $\ell_i \cdot \ell_j = -2\delta_{i,j}$, $\ell_{i,j}\ell_{k,m} = -2\delta_{i,k}\delta_{j,m}$ *and* $\ell_i \cdot \ell_{k,m} = \delta_{i,k} + \delta_{i,m}$.

LEMMA 4.3. *The curve* $\mu^{i,j}_{k,m}$ *is irreducible if and only if* $M^{i,j}_{k,m}$ *does not intersect any of the 13 points* $P_{i',j'}$ *with* $(i', j') \neq (i, j), (k, m)$.

PROOF. The curve $\mu^{i,j}_{k,m}$ is reducible if and only if the strict transform $\tilde{M}$ of $M^{i,j}_{k,m}$ on $\tilde{\mathbf{P}}$ intersects the branch locus $\tilde{B}$ of $\psi : X \to \tilde{P}$ at every point of intersection with even multiplicity.

Let $n, n' \in \{1, \ldots, 6\} \setminus \{i, j, k, m\}$ such that $n' \neq n$. Let P be the intersection point of $M^{i,j}_{k,m}$ and L_n. Since the intersection multiplicity of $M^{i,j}_{k,m}$ and B at P is even this implies that P is also on $L_{n'}$, hence $P = P_{n,n'}$. Conversely, if $M^{i,j}_{k,m}$ passes through $P_{n,n'}$ then $\tilde{M}$ and $\tilde{B}$ are disjoint, hence the degree morphism from $\mu^{i,j}_{k,m}$ to the rational curve $\varphi'^*(M^{i,j}_{k,m})$ is unramified, hence $\mu^{i,j}_{k,m}$ is reducible. $\square$

Lemma 4.4. *Let N be the subgroup of $NS(X)$ generated by the ℓ_i's, the $\ell_{i,j}$'s and the $\mu^{i,j}_{k,m}$'s. Then $N \cong \mathbf{Z}^{16}$ as abelian groups. Moreover ℓ_1 and the $\ell_{i,j}$'s form a basis for $N \otimes \mathbf{Q}$.*

Proof. First we show that $\operatorname{rank} N \leq 16$. Since $2\ell_i + \sum_j \ell_{i,j}$ is linearly equivalent to the pull-back of a general line $L \subset \mathbf{P}^2$, we obtain that $\varphi^* L \in N$. Since $\varphi^* L$ is linear equivalent to $\mu^{i,j}_{k,m} + \sum a_{k,m}\ell_{k,m}$ for some choice of $a_{k,m}$, we obtain that the $\ell_2, \dots \ell_6, \mu^{i,j}_{k,m}$ are contained in

$$(\mathbf{Z}[\ell_1] \oplus \langle [\ell_{i,j}] \rangle) \otimes \mathbf{Q},$$

hence $\operatorname{rank}(N) \leq 16$.

Since all $\ell_{i,j}$ are disjoint and $\ell_{i,j}^2 = -2$, they form a subgroup N' of rank 15 of the Néron-Severi group. An easy computation shows that ℓ_1 is not linearly equivalent to a divisor contained in $N' \otimes \mathbf{Q}$, proving that $\operatorname{rank} N = 16$. Since the intersection pairing is non-degenerate we have that N is torsion-free. □

Lemma 4.5. *The action of σ on N is trivial.*

Proof. This follows from the fact that for all i, j the automorphism $\sigma|_{\ell_i}$ is the identity and σ fixes the curves $\ell_{i,j}$. □

Lemma 4.6. *Let $L \subset \mathbf{P}^2$ be a rational curve, $L \neq L_i$. Suppose $\varphi'^{-1}(L)$ has two components. Then $\rho(X) \geq 17$.*

Proof. Let $r_i, i = 1, 2$ be the strict transforms on X of the components of $\varphi'^{-1}(L)$. It suffices to show that $r_1 \notin N \otimes \mathbf{Q}$. Since $\sigma(r_1) = r_2$ and σ acts trivial on N, we obtain that r_1 is linearly equivalent to r_2. From Lemma 2.1 it follows that $r_1^2 = -2$. We have that any irreducible effective divisor $r \neq r_1$ linear equivalent to r_1 satisfies $\#r \cap r_1 = r.r_1 = -2$, hence the only effective irreducible divisor linearly equivalent to r_1 is r_1 itself. From this it follows that $r_1 = r_2$, contradicting our assumption. □

Lemma 4.7. *Suppose that there exists a permutation $\tau \in S_6$ such that $P_{\tau(1),\tau(2)}$, $P_{\tau(3),\tau(4)}$, $P_{\tau(5),\tau(6)}$ are collinear. Then $NS(X)$ has rank at least 17.*

Proof. Using Lemma 4.3 the assumption implies that at least one of the $\mu^{i,j}_{k,m}$ is reducible. Hence by Lemma 4.6 we have that $\operatorname{rank} NS(X) \geq 17$. □

Assumption 4.8. *For the rest of this chapter assume that the six lines in $\mathbf{P}^2$ are chosen in such a way that* $\operatorname{rank} NS(X) = 16$. *(Hence the $\mu^{i,j}_{k,m}$ are reduced and irreducible).*

Remark 4.9. Choosing 6 distinct lines in $\mathbf{P}^2$ gives 4 moduli. Hence the family G of $K3$ surfaces that can be obtained as a double cover of six lines in $\mathbf{P}^2$ is 4-dimensional. From the Torelli theorem for $K3$ surfaces ([**55**]) it follows that a general element $X \in G$ satisfies $\rho(X) \leq 16$. By Lemma 4.4 any $X \in G$ has Picard number at least 16. Hence a general $X \in G$ satisfies $\rho(X) = 16$. Therefore the above assumption makes sense.

Definition 4.10. Let C be an irreducible curve on X different from the ℓ_i's, with $C^2 = -2$. Then C is a called an *ordinary rational curve.* If C is one of the ℓ_i's then C is called a *special rational curve.*

A smooth rational curve C satisfies $p_a(C) = 0$. From Lemma 2.1 it follows that $C^2 = -2$, which explains one of the conditions in the above definition.

Let $B = \ell_1 + \ell_2 + \cdots + \ell_6$. Since $\psi : X \to \tilde{\mathbf{P}}$ is ramified along the strict transforms of the L_i, we obtain that the fixed locus X^σ of σ equals B.

LEMMA 4.11. *Let D be an ordinary rational curve. Then $D \cdot B = 2$.*

PROOF. Assumption 4.8 and Lemma 4.4 imply that the involution σ maps D to a linear equivalent divisor. Since $D^2 = -2$ the only effective irreducible divisor linear equivalent to D, it D itself. Hence σ acts non-trivially on $D \cong \mathbf{P}^1$, from which it follows that there are two fixed points. □

LEMMA 4.12. *Let D_1 and D_2 be two ordinary rational curves. Then $D_1 \cdot D_2 \equiv 0 \bmod 2$.*

PROOF. The proof of [**52**, Lemma 1.6] carries over in this case. □

5. A special result

PROPOSITION 5.1. *Suppose X is a double cover of $\mathbf{P}^2$ ramified along six lines. If the position of the six lines is sufficiently general then for every elliptic fibration $\pi : X \to \mathbf{P}^1$ the total number of fibers of π of type II, III or IV is at most* $\operatorname{rank}(MW(\pi))$.

PROOF. As explained in Remark 4.9 the general member of the family of all double covers of $\mathbf{P}^2$ ramified along 6 lines has Picard number 16, hence

$$20 = h^{1,1}(X) = 4 + \rho_{tr}(\pi) + \operatorname{rank} MW(\pi).$$

Using Proposition 2.3.6 we obtain that

$$\begin{aligned} 4 &\leq \dim\{[\psi : X \to \mathbf{P}^1] \in \mathcal{M}_2 | C(\psi) = C(\pi)\} \\ &= h^{1,1} - \rho_{tr}(\pi) - \#\{\text{fibers of type } II, III, IV\} \\ &\leq 4 + \operatorname{rank} MW(\pi) - \#\{\text{fibers of type } II, III, IV\}, \end{aligned}$$

which gives the desired inequality. □

REMARK 5.2. This proposition can be used to determine the number of fibers of type I_1 and I_2 in several cases of Oguiso's classification [**52**]. However, Oguiso classified all Jacobian elliptic fibrations on the Kummer surface of the product of two non-isogenous elliptic curves, while if one wants to apply Proposition 5.1 one obtains only the classification of Jacobian elliptic fibrations on a Kummer surface of a product of two general elliptic curves.

6. Possible singular fibers

In this section we classify all elliptic fibrations on the double cover of $\mathbf{P}^2$ ramified along six fixed lines ℓ_i in general position, where general position means that $\rho(X) = 16$.

DEFINITION 6.1. By a *simple component* D of a fiber F we mean an irreducible component D of F such that D occurs with multiplicity one in F.

PROPOSITION 6.2. *Let X be as before. Let $\pi : X \to \mathbf{P}^1$ be an elliptic fibration with a section. Then the Kodaira type of the singular fiber is contained in the following list. For each Kodaira type we list the number of components which are special rational curves, the number of simple components which are special rational curves, and the number of simple components which are ordinary rational curves, are contained in Table 3.*

Type	#Cmp. special rational curves	#Simple cmp. special rational curves	#Simple cmp. ordinary rational curves
I_1	0	0	0
I_2	0 or 1	0 or 1	2 or 1
I_4	2	2	2
I_6	3	3	3
I_8	4	4	4
I_{10}	5	5	5
I_0^*	1	0	4
I_2^*	2	0	4
I_4^*	3	0	4
I_6^*	4	0	4
II	0	0	0
III	0 or 1	0 or 1	2 or 1
IV^*	4	3	0
III^*	3	0	2
II^*	4	0	1.

TABLE 3. Number of special and ordinary components in singular fibers.

SKETCH OF PROOF. First of all we prove that no fiber F of type $I_{2k+1}, k > 0$ exists. Such a fiber F is a $2k+1$-gon of rational curves. Since two special rational curves do not intersect, and two ordinary rational curves have even intersection number (Lemma 4.12), it follows that every ordinary rational curve intersects two special rational curves, and each special rational curve intersects two ordinary rational curves. This forces the number of components in an n-gon to be even. Hence I_{2k+1} does not occur. For the same reasons no fiber of type IV or type I_{2k+1}^* occurs. Since there at most 6 special nodal curves, no fiber of type $I_{2k}, k > 6$ or of type I_{2k}^*, $k > 5$ occurs.

We prove now that no fiber F of type I_{12} exists. If it would exists, then this fiber would contain all special rational curves. Hence the zero section is an ordinary rational curve Z. From Lemma 4.11 it follows that then $1 = Z \cdot F \geq Z \cdot B = 2$, a contradiction. The non-existence of I_{10}^* follows similarly. A fiber of type I_8^* has four ordinary rational curves R_i with the property that R_i intersects only one other fiber component. Moreover, the R_i are the only simple components. Hence the zero-section Z intersects one of the R_i, say R_1, and Z has to be a special rational curve, by Lemma 4.12. The curve $R_i, i \neq 1$ intersect a special rational curve not contained in the fiber and different from Z, hence there are at most 4 special rational curves contained in F. Using Lemma 4.12 one obtains easily that F contains at least 5 special nodal curves, a contradiction.

Let D be a rational curve intersecting three other disjoint rational curves $D_i, i = 1, \dots, 3$. If D were ordinary, then by Lemma 4.12 the curves D_i would be special. This would imply that $D \cdot B \geq 3$, contradicting Lemma 4.11. Hence D is a special rational curve. This observation determines in many cases the number of special components in a singular fiber. □

7. Possible configurations I

In this section we study all Jacobian elliptic fibrations $\pi : X \to \mathbf{P}^1$ having the property that all special rational curves are fiber components. This section yields the proof of Theorem 1.1.

PROPOSITION 7.1. *Let X be as before, in particular* $\operatorname{rank} NS(X) = 16$. *Let $\pi : X \to \mathbf{P}^1$ be a Jacobian elliptic fibration. Suppose all special rational curves are contained in the fibers of π. Then one of the following occurs:*

Singular fibers		*Mordell-Weil rank*
$I_{10}\ I_2\ aII\ bI_1$	$2a+b=12$	4
$I_8\ I_4\ aII\ bI_1$	$2a+b=12$	4
$2I_6\ aII\ bI_1$	$2a+b=12$	4
$IV^*\ I_4\ aII\ bI_1$	$2a+b=12$	5

Conversely, for each case there exist a, b such that these fibration do occur.

PROOF. Since the zero section is a rational curve and all special rational curves are contained in some fibers, we have that the zero section Z is an ordinary rational curve. From Lemma 4.12 it follows that if Z intersects a reducible fiber, then it intersects in a simple component, which is an special nodal curve. From Lemma 4.11 and the fact that special nodal curves are smooth, it follows that there are precisely two reducible fibers. Using Proposition 6.2 we obtain that the possible reducible fibers are III, IV^* and I_{2k}, with $1 \leq k \leq 5$.

Since there are six special rational curves, the above possibilities are the only ones, except that the case $I_{10}\ III\ aII\ bI_1$ with $2a+b=11$ might occur. We prove that this one cannot exist: A fiber of type I_{10} contains precisely five special rational curves, say $\ell_1, \dots \ell_5$. Then ℓ_6 is a component of the III-fiber. The other component of this fiber is an ordinary rational curve D tangent to ℓ_6, not intersecting $\ell_i, i = 1, \dots, 5$. Consider $\psi : X \to \tilde{\mathbf{P}}$. Then $\psi(D)$ is a reduced curve tangent to $\psi(\ell_6)$ and not intersecting $\psi(\ell_i), i \leq 5$, hence intersects the branch locus with even multiplicity in each intersection point. Since $\psi(D)$ is a rational curve, this implies that $\psi^{-1}(\psi(D))$ has two components. Hence $\varphi^{-1}(\varphi(D))$ has two components. This contradicts Lemma 4.6 (note that we assumed that $\rho(X) = 16$).

The quantity $2a+b$ can be determined using Noether's formula (Theorem 1.2.7), the Mordell-Weil rank can be obtained using the Shioda-Tate formula (Theorem 1.2.11).

It remains to prove the existence of the remaining four cases.

Let $k \in \{3, 4, 5\}$. To prove the existence of $I_{2k} I_{12-2k} aII bI_1$, take $D = \ell_1 + \ell_{1,2} + \ell_2 + \ell_{2,3} + \cdots + \ell_k + \ell_{1,k}$. If $k = 3, 4$ then $D_1 = \ell_{k+1} + \ell_{k+1,k+2} + \cdots + \ell_6 + \ell_{k+1,6}$ is an effective divisor with $D_1^2 = 0$ and $D.D_1 = 0$. From Lemma 2.2 it follows that the fibration associated to $|D|$ has D and D_1 as fibers. They are of type I_{2k} and I_{12-2k}.

If $k = 5$, then from Lemma 2.2 it follows that $|D|$ defines a fibration with a I_{10} fiber, which proves the existence of the first case.

To prove the existence of the fibration with the IV^* and I_4 fiber, take $D = \ell_1 + 2\ell_{1,2} + 3\ell_2 + 2\ell_{2,3} + \ell_3 + 2\ell_{2,4} + \ell_4$. Then the fibration associated to $|D|$ has a fiber of type IV^*. Yielding the final case.

It remains to prove that the above fibrations are Jacobian. In all cases one easily shows that $D.\ell_{1,6} = 1$, hence $\ell_{1,6}$ is a section. □

8. Possible configurations II

In this section we consider fibrations on X such that at least one of the special rational curves is not a component of a singular fiber. This section yields the proof of Theorem 1.2.

Let $\pi : X \to \mathbf{P}^1$ be a Jacobian elliptic fibration, such that at least one of the special rational curves is not contained in a fiber.

LEMMA 8.1. *The group $MW(\pi)$ is finite.*

PROOF. The proof of [**52**, Lemma 2.4] carries over. □

LEMMA 8.2. *Suppose one of the singular fibers of π is of type I_{2k}^*, III^* or II^*. Then all sections are special rational curves.*

PROOF. A section intersects a reducible fiber in a simple component with multiplicity one. By Proposition 6.2 we know that all simple components in the above mentioned singular fibers are ordinary rational curves. From Lemma 4.12 we know that two ordinary rational curves intersect with even multiplicity, hence every section is a special rational curve. □

LEMMA 8.3. *Let $\pi : X \to \mathbf{P}^1$ be as above. Then the only fibers of type I_ν are of type I_1 and I_2, and no fiber of type I_2 or type III contains a special rational curve as a component.*

PROOF. From Proposition 6.2 we know that every fiber of type I_ν, $\nu > 1$ is of type I_{2k}, $k \geq 1$

Without loss of generality we may assume that ℓ_1 is not contained in a fiber of π, hence intersects every fiber. Let F be a singular fiber of type $I_{2k}, k \geq 1$ or of type III, containing a special rational curve.

Then ℓ_1 intersects a reducible fiber in an ordinary rational curve, say D. If F is of type I_{2k}, $k > 1$ then D intersects two other components, and by Lemma 4.12 these components are special rational curves. Let $B = \sum \ell_i$. Then $D \cdot B \geq 3$, contradicting Lemma 4.11. Hence it is not possible to have a fiber of type I_{2k}, $k > 1$ containing a special rational curve. By Proposition 6.2 every singular fiber of type I_{2k}, $k > 1$ contains a special nodal curve, hence such a fiber does not occur.

If $k = 1$ or the fiber is of type III, then D intersects the other component twice, and, by assumption, this component is special. Hence $D \cdot B \geq 3$, contradicting Lemma 4.11. □

From now on we study the possibilities for the fibration π. We distinguish eight cases. In each case we suppose that π has a fiber F of a certain given Kodaira type. In each case we study which other singular fibers can occur. Then we prove the existence of the configuration by giving a divisor D such that the linear system $|D|$ gives the desired fibration. First, we determine the fiber types of fibers containing a special rational curve. Observe that by Proposition 6.2 all singular fibers, not containing special rational curves, are of type III, I_2, II or I_1. Using the Shioda-Tate formula (Theorem 1.2.11) and Noether's formula (Theorem 1.2.7) we can determine the quantities $iii + i_2$ and $3iii + 2i_2 + 2ii + i_1$.

For existence proofs we use Proposition 6.2.

DEFINITION 8.4. An *end-component* C of a fiber F is a component C intersecting the support of $\overline{F \setminus C}$ transversally in one point.

In the sequel we use that if an end-component of a fiber is a ordinary rational curve, then this component has to intersect a special rational curve, not contained in the fiber. This follows immediately from Lemma 4.11.

In order to determine the Mordell-Weil group, we use Lemma 8.2.

8.1. II^***.** In this case four special rational curves are contained in F. Using Proposition 6.2 it follows that there are three end-components E_1, E_2, E_3, of which E_1 and E_2 are ordinary rational curves, and E_1 is a simple component. This means that the zero-section is a special nodal curve, say D_1, and E_2 intersects a special nodal curve, not contained in F and different from D_1, say D_2. This yields that $F \cdot D_1 = 1$, $F \cdot D_2 = 3$. All other fibers are of type II, III, I_1, I_2, yielding case 2.1. For example take $D = \ell_{1,5} + 2\ell_1 + 3\ell_{1,2} + 4\ell_2 + 5\ell_{2,3} + 6\ell_3 + 4\ell_{3,4} + 2\ell_4 + 3\ell_{3,6}$. Then ℓ_5 is a section and ℓ_6 is a trisection.

8.2. III^***.** In this case three special rational curves are contained in F. At least two special rational curves have positive intersection number with F.

Suppose that two special rational curves are sections, then the third special rational curve is a multisection and all other fibers are of type II, III, I_1, I_2. For example, take $D = \ell_{3,4} + 2\ell_3 + 3\ell_{1,3} + 4\ell_1 + 2\ell_{1,5} + 3\ell_{1,2} + 2\ell_2 + \ell_{2,6}$. Then ℓ_4 and ℓ_6 are sections. This gives the case 2.2.

If precisely one special rational curve is a section, then there is a special rational which is a multisection, and one special rational curve which is contained in some singular fiber. Since the multisection and the section intersect the fiber four times, this fiber cannot be of type III or I_2, so it is of type I_0^*. All other fibers are of type II, III, I_1, I_2. For example take $D = \ell_{3,4} + 2\ell_3 + 3\ell_{1,3} + 4\ell_1 + 2\ell_{1,5} + 3\ell_{1,2} + 2\ell_2 + \ell_{2,5}$. This gives case 2.3. In this case ℓ_4 is a section.

8.3. IV^***.** In this case F contains four special rational curves. The other two special rational curves do not intersect this fiber, so they are components of other singular fibers, hence all special rational curves are components, contradicting our assumptions.

8.4. I_6^***.** In this case F has four special rational curves as components. One special rational curve is a section and one a multisection. All other fibers are of type III, I_2, II and I_1. For example take $D = \ell_{1,5}+\ell_{1,6}+2\ell_1+2\ell_{1,2}+2\ell_2+2\ell_{2,3}+2\ell_3+2\ell_{3,4}+2\ell_4+\ell_{4,5}+\mu_{2,6}^{1,3}$. The curve ℓ_6 is a section. This gives case 2.4.

8.5. I_4^***.** In this case F has three special rational curves as components. Either three or two of the other special curves intersect any fiber.

If all three special curves intersect any fiber then all other singular fibers are of type III, I_2, II, I_1. For example take $D = \ell_{1,5} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + 2\ell_{2,3} + 2\ell_3 + \ell_{3,5} + \ell_{3,6}$. In this case ℓ_4 and ℓ_6 are sections. This gives case 2.5.

If two special curves intersect any fiber, then one of the special curves is again a component of a singular fiber. Since the two other special rational curves intersect any fiber four times, this fiber cannot be of type I_2 or III, so it is a fiber of type I_0^*. For example take $D = \ell_{1,5} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + 2\ell_{2,3} + 2\ell_3 + \ell_{3,5} + \mu_{2,4}^{1,6}$. In this case, ℓ_4 is a section. This gives case 2.6.

8.6. I_2^***.** In this case F fiber has 2 special curves as components.

If 4 special curves intersect any fiber then all other singular fibers are of type III, I_2, II, I_1. For example take

$$D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,5} + \ell_{2,6}.$$

Then $\ell_i, i = 3, \dots, 6$ are sections. From [**47**, Corollary VII.3.3] it follows that $MW(\pi) \not\cong \mathbf{Z}/4\mathbf{Z}$. This gives case 2.7.

If 3 special curves intersect any fiber then there is one special rational curve D_1 not intersecting the fiber and not contained in the F. From this it follows that D_1 is a component of a fiber of type I_0^*. All other fibers are of type II, III, I_1, I_2. For example take $D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,4} + \ell_{2,5}$. Then ℓ_3 and ℓ_5 are sections, This gives case 2.8.

If 2 special curves intersect any fiber then there are two remaining special rational curves D_1, D_2, with $F \cdot D_1 = F \cdot D_2 = 0$. If D_1 and D_2 are components of the same fiber, then this fiber is of type I_2^*. For example take $D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,4} + \mu_{3,6}^{1,5}$. Then ℓ_5, ℓ_6 and $\ell_{5,6}$ are components of another singular fiber, which has to be of type I_2^*. Then ℓ_3 is a section. This gives case 2.9.

If D_1 and D_2 are in different fibers then they are both components of fibers of type I_0^*. For example take $D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,4} + C'$, with C' the strict transform of $\varphi'^{-1}(C)$, with C the the conic through $P_{1,3}P_{1,5}P_{2,3}P_{4,6}P_{5,6}$. The curves $\ell_{3,5}, \ell_{3,6}, \ell_5, \ell_6$ are components of some singular fibers. Since $F \cdot \ell_3 = 1$, they are components of two distinct fibers. This gives case 2.10.

8.7. I_0^***.** In this case F contains only one special rational curve. Each ordinary component intersects only one special rational curve not contained in F. Hence there are at most 4 special curves intersecting F.

If there are four special rational curves D_i, with $F \cdot D_i > 0$, then the remaining special rational curve is a component of a fiber, which has to be of type I_0^*. There are still six components of fibers left. The only way to arrange them is with 6 I_2 fibers. For example take $D = 2\ell_1 + \sum_{k=2}^5 \ell_{1,k}$, then $D' = 2\ell_6 + \sum_{k=2}^5 \ell_{k,6}$ is another singular fiber. The rational curves $\ell_k, k = 2, \dots, 5$ are sections. From [**47**, Corollary VII.3.3] it follows that $MW(\pi) \not\cong \mathbf{Z}/4\mathbf{Z}$. This gives case 2.11.

If there are three special rational curves D_i, with $F \cdot D_i > 0$, then the two other special rational curves are components of some singular fiber. If both components are in the same fiber then that fiber is of type I_2^* (which is handled above), otherwise the fibers containing the special rational curves are of type I_0^*. So we have in total 3 fibers of type I_0^*. All other fibers are of type II, III, I_1, I_2. For example take $D = \mu_{5,6}^{2,3} + \ell_{1,4} + \ell_{1,5} + \ell_{1,6} + 2\ell_1$. Then none of $\ell_2, \ell_{2,4}, \ell_{2,5}, \ell_{2,6}, \ell_3, \ell_{3,4}, \ell_{3,5}, \ell_{3,6}$ intersect a fiber. Hence they are components of two I_0^* fibers. The curves ℓ_5 and ℓ_6 are sections. This gives case 2.12.

If there are two special rational curves D_i, with $F \cdot D_i > 0$, then the three other special curves D_i' are components of some singular fibers.

If all D_i''s are contained in the same fiber, then that fiber is either of type III^* (which we already handled) or of type I_4^* (which we also handled above).

If all D_i''s are contained in two singular fibers then we obtain one fiber of type I_2^* and one of I_0^*. This case we handled above.

If all D_i''s are contained in three singular fibers then all three fibers are of type I_0^*, which is impossible, since then the Picard number of X would be at least 18.

8.8. Only I_2 and III. From the Shioda-Tate formula (Theorem 1.2.11) it follows that there are $\rho(X) - 2 - \operatorname{rank} MW(\pi) = 14$ singular fibers of type I_2 or III. From Noether's formula 1.2.7 it follows that then $24 = 12p_g(X) + 12 = \sum v_p(\Delta_p) \geq 2 \cdot 14 = 28$. A contradiction. Hence this does not occur.

CHAPTER 6

The p-part of Tate-Shafarevich groups of elliptic curves can be arbitrarily large

A paper based on this chapter will appear in the Journal de théorie des nombres de Bordeaux [**35**].

1. Introduction

For the notations used in this introduction we refer to Section 3. Some of the general ideas behind this chapter are mentioned in the Introduction of this Thesis.

The aim of this chapter is to give a proof of

THEOREM 1.1. *There is a function g from the prime numbers to the positive integers, such that for every prime number p the dimension* $\dim_{\mathbf{F}_p} Ш(E/K)[p]$, *taken over number fields K of degree at most $g(p)$ and elliptic curves E/K, is unbounded.*

The proof of this theorem starts on page 85. Using Weil restriction of scalars, we obtain as an easy consequence (see page 87).

COROLLARY 1.2. *For every prime number p there exist abelian varieties $A/\mathbf{Q}$, with* $\dim A \leq g(p)$ *and A is simple over* $\mathbf{Q}$, *such that* $\dim_{\mathbf{F}_p} Ш(A/\mathbf{Q})[p]$ *is arbitrarily large.*

In fact, a rough estimate using the present proof reveals that $g(p) = O(p^4)$. It is an old open question whether $g(p)$ can be taken 1, i.e., for any p, the p-torsion of the Tate-Shafarevich groups of elliptic curves over $\mathbf{Q}$ are unbounded.

For $p \in \{2, 3, 5\}$, it is known that the group $Ш(E/\mathbf{Q})[p]$ can be arbitrarily large. (See [**11**], [**13**], [**23**] and [**40**].) So we may assume that $p > 5$, in fact, our proof only uses $p > 3$. The above mentioned papers (except [**23**]) use heavily the fact that one can find explicit equations for the homogeneous spaces associated to elements in the p-part Tate-Shafarevich group, if $p \in \{2, 3, 5\}$. We use a different strategy, which we mention below.

P.L. Clark communicated to the author that, after seeing an earlier version of this chapter, he proved by different methods that for every elliptic curve E/K, such that $E(K)[p] \cong (\mathbf{Z}/p\mathbf{Z})^2$ we have that $Ш(E/L)[p]$ is arbitrarily large if L runs over all extension of K of degree p, but E remains fixed (see [**16**]). This gives a sharper bound in the case that E has potential complex multiplication. The elliptic curves we describe in the proof of theorem 1.1 all have several primes $\mathfrak{p}$ of K for which the reduction at $\mathfrak{p}$ is split-multiplicative, namely a certain multiple of the k in the estimate of Theorem 1.1. Hence these curves do *not* have potential complex multiplication. Another difference is explained in Remark 4.6.

The proof of Theorem 1.1 is based on combining the strategy used in [**23**] to prove that the dimension (over $\mathbf{F}_5$) of $Ш(E/\mathbf{Q})[5]$ can be arbitrarily large and the strategy used in [**38**] to prove that $\dim_{\mathbf{F}_p} S^p(E/K)$ can be arbitrarily large, where E and K vary, but $[K : \mathbf{Q}]$ is bounded by a function depending on p of type $O(p)$.

In [**38**] the strategy was to find a field K, such that $[K : \mathbf{Q}]$ is small and a point $P \in X_0(p)(K)$ such that P reduces to one cusp for many primes $\mathfrak{p}$ and reduces to the other cusp for very few primes $\mathfrak{p}$. Then to P we can associate an elliptic curve E/K such that an application of a Theorem of Cassels [**14**] shows that $S^p(E/K)$ gets large.

The strategy of [**23**] can be described as follows. Suppose K is a field with class number 1. Suppose E/K has a K-rational point of order p, with $p > 3$ a prime number. Let $\varphi : E \to E'$ be the isogeny obtained by dividing out the group generated by a point of order p. Then one can define a $\mathbf{F}_p$-linear transformation T from some subgroup of K^*/K^{*p} to another subgroup of K^*/K^{*p}, such that the φ-Selmer group is isomorphic to the kernel of T, while the $\hat{\varphi}$-Selmer group is isomorphic to the kernel of an adjoint of T. One can then show that the rank of $E(K)$ and of $E'(K)$ is bounded by the number of split multiplicative primes minus twice the rank of T minus 1.

Moreover, one can prove that if the difference between the dimension of the domain of T and the domain of the adjoint of T is large, then the dimension of the p-Selmer group of one of E, E' is large. If one has an elliptic curve with two rational torsion points of prime order p and q respectively (or full p-torsion, if one wants to take $p = q$), one can hope that for one isogeny the associated transformation has high rank, while for the other isogeny the difference between the dimension of the domain of T and its adjoint is large. Fisher uses points on $X(5)$ to find elliptic curves $E/\mathbf{Q}$ with two isogenies, one such that the associated matrix has large rank, and the other such that the 5-Selmer group is large.

We generalize this idea to number fields, without the class number 1 condition. We can still express the Selmer group attached to the isogeny as the kernel of a linear transformation T. In general, the transformation for the dual isogeny turns out to be different from any adjoint of T. This is in contrast to the case where the class number of K is one. (As mentioned above, see also [**23**].)

Remark 1.3. Fix an element ξ of $S^p(E/K)$. Set $K' := K(E[p])$. Restrict ξ to to $H^1(K', E[p])$, the latter group is isomorphic to $\mathrm{Hom}(G_{K'}, (\mathbf{Z}/p\mathbf{Z})^2)$. Then ξ gives a Galois extension L of $K(E[p])$ of degree p or p^2, satisfying certain local conditions. (For the case of a cyclic isogeny, these conditions are made more precise in Proposition 3.5.) To check whether a given class in $H^1(K', E[p])$ comes from an element in $S^p(E/K)$ we need to check whether the Galois group of L/K' interacts in some prescribed way with the Galois group of K'/K.

The examples of elliptic curves with large Selmer and large Tate-Shafarevich groups in [**23**], [**38**] and in this chapter have one thing in common, namely that the representation of the absolute Galois group of K on $E[p]$ is reducible. In this case the conditions on the interaction of the Galois group of K'/K with the Galois group of L/K' almost disappear.

The level of difficulty to construct large p-Selmer groups (and large p-parts in the Tate-Shafarevich groups) seems to depend on the size of the image of the Galois representation on the full p-torsion of E.

To support this idea, we note that the above mentioned constructions give examples where the image of this representation is very small. Elliptic curves E/K with complex multiplication over a proper extension L/K have an irreducible Galois-representation on $E[p]$ for all but finitely many p, but the image of the representation is strictly smaller than $\mathrm{GL}_2(\mathbf{F}_p)$.

In view of the above remarks it seems that if one would like to produce examples of elliptic curves with large p-Selmer groups, and an irreducible representation of the Galois group on $E[p]$, one could start with the case of elliptic curves with complex multiplication over some proper extension. Unfortunately, we do not have a strategy to produce such examples.

The organization of this chapter is as follows: In Section 2 we discuss some properties of Tamagawa numbers. In Section 3 we prove several lower and upper bounds for the size of φ-Selmer groups, where φ is an isogeny with kernel generated by a rational point of prime order at least 5. In Section 4 we use the modular curve $X(p)$ and the estimates from Section 3 to prove Theorem 1.1.

2. Tamagawa numbers

In this section we study the behavior of Tamagawa numbers under isogeny. Let K be a number field. Let $\mathfrak{p}$ be a prime of K. Let $K_\mathfrak{p}$ be the completion of K with respect to $\mathfrak{p}$. Let $E/K_\mathfrak{p}$ be an elliptic curve.

NOTATION 2.1. Let $E_0(K_\mathfrak{p})$ denote the group of $K_\mathfrak{p}$-rational points with non-singular reduction with respect to a minimal Weierstrass equation for E over $K_\mathfrak{p}$ (see [**72**, Ch. VII]). Define the *Tamagawa number* $c_{E,\mathfrak{p}}$ to be $\#E(K_\mathfrak{p})/E_0(K_\mathfrak{p})$.

We denote the discriminant of a given Weierstrass equation of the elliptic curve E by $\Delta(E)$. We denote the minimal discriminant of E at the prime $\mathfrak{p}$ by $\Delta_\mathfrak{p}(E)$.

DEFINITION 2.2. Let $\tilde{E}$ be the irreducible curve obtained by reducing a Weierstrass minimal equation for E modulo $\mathfrak{p}$.

We say that

- E has good reduction if $\tilde{E}$ is smooth,
- E has additive reduction if $\tilde{E}$ is a cuspidal curve,
- E has split multiplicative reduction if $\tilde{E}$ is a nodal curve and the generalized tangents are defined over the residue field,
- E has non-split multiplicative reduction if $\tilde{E}$ is a nodal curve and the generalized tangents at the node are not defined over the residue field.

We assume that all valuations in this chapter are normalized, i.e., all valuations have $\mathbf{Z}$ as their image.

PROPOSITION 2.3. *The Tamagawa number* $c_{E,\mathfrak{p}}$ *equals* 1 *when* E *has good reduction at* $\mathfrak{p}$. *If the reduction is additive or non-split multiplicative then* $c_{E,\mathfrak{p}} \leq 4$. *If the reduction at* $\mathfrak{p}$ *is split multiplicative then* $c_{E,\mathfrak{p}} = v_\mathfrak{p}(\Delta_\mathfrak{p}(E)) = -v_\mathfrak{p}(j(E))$.

Suppose $\varphi : E \to E'$ *is an isogeny defined over* $K_\mathfrak{p}$, *Then*

$$\frac{c_{E,\mathfrak{p}}}{c_{E',\mathfrak{p}}} = \prod p_i^{n_i}$$

with p_i *prime,* p_i *divides* $\deg(\varphi)$ *and the* n_i *are integers.*

PROOF. All statements except the last one can be found in [**79**, p. 46]. For the last statement it suffices to show the statement for an isogeny φ of prime degree p. Then φ induces a homomorphism of groups $\overline{\varphi} : E(K_\mathfrak{q})/E_0(K_\mathfrak{q}) \to E'(K_\mathfrak{q})/E'_0(K_\mathfrak{q})$. Composing this homomorphism with the homomorphism induced by the dual isogeny gives us that

$\ker(\overline{\varphi}) \subset [p]^{-1}(E_0(K_\mathfrak{q})) \bmod E_0(K_\mathfrak{q})$. This implies that the $\#\ker\overline{\varphi} \in \{1, p, p^2\}$, and similarly for the dual isogeny. Since the co-kernel of multiplication by p has at most $1, p$ or p^2 elements, we obtain that the number of elements of the co-kernel of $\hat{\varphi}$ is also a p-th power. Hence the quotient of the numbers of elements is a p-th power. □

REMARK 2.4. If $\varphi : E \to E'$ is an isogeny of degree p^n (with p prime) and p is bigger than 3, then $c_{E,\mathfrak{p}} = c_{E',\mathfrak{p}}$ unless E has split multiplicative reduction at $\mathfrak{p}$.

PROPOSITION 2.5. *Let $\varphi : E \to E'$ be an isogeny of degree $p > 2$ over K. Suppose the reduction at $\mathfrak{p}$ is multiplicative. Then $v_\mathfrak{p}(\Delta_\mathfrak{p}(E))/v_\mathfrak{p}(\Delta_\mathfrak{p}(E'))$ equals p or $1/p$. The second case occurs if and only if $E(\overline{K_\mathfrak{p}})[\varphi]$ is contained in $E_0(\overline{K_\mathfrak{p}})$.*

PROOF. Suppose the reduction at $\mathfrak{p}$ is non-split multiplicative. Then there exists a degree 2 unramified extension $L/K_\mathfrak{p}$, such that E/L has split multiplicative reduction, and the minimal discriminant of E/L is the same as the minimal discriminant of $E/K_\mathfrak{p}$. So we may assume that the reduction is split multiplicative. Hence E is isomorphic to a Tate curve E_q (see [**73**, Section V.3]).

Let $q \in K_\mathfrak{p}$ be the Tate parameter for $E/K_\mathfrak{p}$, i.e., $E(\overline{K_\mathfrak{p}})$ is isomorphic to $\overline{K_\mathfrak{p}}^*/\langle q\rangle$ and $E_0(\overline{K_\mathfrak{p}})$ is isomorphic to $\mathcal{O}^*_{\overline{K_\mathfrak{p}}} \bmod \langle q\rangle$ as $\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})$-modules.

Over $K_\mathfrak{p}(\sqrt[p]{q}, \zeta_p)$ we have that E_q is isogenous to E_{q^p} and $E_{\sqrt[p]{q}\zeta_p^k}$, with $k = 0, \ldots, p-1$ and ζ_p is a fixed primitive p-th root of unity. In the case $E_q \to E_{q^p}$ the isogeny is induced by the map on $K_\mathfrak{p}^*$ given by $x \mapsto x^p$, in the other cases it is induced by projection. Hence the kernel is generated by ζ_p, respectively, $\sqrt[p]{q}\zeta_p^k$ and these isogenies are of degree p. From this it follows that in the case $E_q \to E_{q^p}$, the kernel is contained in $E_0(\overline{K_\mathfrak{p}})$, and in all other cases the kernel of the isogeny is not contained in $E_0(\overline{K_\mathfrak{p}})$.

Since there are at most $p+1$ curves isogenous of degree p to E, we covered all possibilities.

The statement follows from $v_\mathfrak{p}(\Delta_\mathfrak{p}(E_q)) = v_\mathfrak{p}(q)$. □

3. Selmer groups

In this section we give several upper and lower bounds for the p-Selmer group of an elliptic curve E/K with a K-rational point of order p, and $\zeta_p \in K$. We combine two of these bounds to obtain a lower bound for $\dim_{\mathbf{F}_p} Ш(E/K)[p]$ (see Lemma 3.15). This result is used in the next section.

Suppose K is a number field, E/K is an elliptic curve and $\varphi : E \to E'$ is an isogeny defined over K. Let $E[\varphi]$ denote the kernel of φ. Let $H^1(K, E[\varphi])$ be the first cohomology group of the Galois module $E[\varphi]$.

DEFINITION 3.1. The φ-Selmer group of E/K is

$$S^\varphi(E/K) := \ker H^1(K, E[\varphi]) \to \prod_{\mathfrak{p}\,\mathrm{prime}} H^1(K_\mathfrak{p}, E).$$

and the Tate-Shafarevich group of E/K is

$$Ш(E/K) := \ker H^1(K, E) \to \prod_{\mathfrak{p}\,\mathrm{prime}} H^1(K_\mathfrak{p}, E).$$

In this definition a prime means a finite or an archimedean prime. In the sequel we consider only isogenies of odd prime degree. In that case $H^1(K_\mathfrak{p}, E[\varphi]) = 0$ for all archimedean primes $\mathfrak{p}$, so we may exclude the archimedean primes.

NOTATION 3.2. For the rest of this section fix a prime number $p > 3$, a number field K such that $\zeta_p \in K$ and an elliptic curve E/K such that there is a non-trivial point $P \in E(K)$ of order p. Let $\varphi : E \to E'$ be the isogeny obtained by dividing out $\langle P\rangle$. Let $\hat{\varphi} : E' \to E$ be the dual isogeny.

To φ we associate three sets of primes. Let $S_1(\varphi)$ be the set of primes $\mathfrak{p} \subset \mathcal{O}_K$, such that $\mathfrak{p}$ does not divide p, the reduction of E is split multiplicative at $\mathfrak{p}$, and $P \in E_0(K_\mathfrak{p})$. Let $S_2(\varphi)$ be the set of primes $\mathfrak{p} \subset \mathcal{O}_K$, such that $\mathfrak{p}$ does not divide p, the reduction of E is split multiplicative at $\mathfrak{p}$, and $P \notin E_0(K_\mathfrak{p})$. Let $S_3(\varphi)$ be the set of all primes above p.

From Proposition 2.5 if follows that we a split multiplicative prime $\mathfrak{p}$, not dividing p, is in $S_1(\varphi)$ if and only if $v_p(\Delta_p(E)) < v_p(\Delta_p(E'))$. (Notation as in the previous section.) From this it follows that $S_1(\hat{\varphi}) = S_2(\varphi)$ and $S_2(\hat{\varphi}) = S_1(\varphi)$. (To define $S_i(\hat{\varphi})$ we need to start with a K-rational point P of order p. Since $\zeta_p \in K$, we have that $\#E'(K)[\hat{\varphi}] = p$, so we can take any generator P' of the kernel of $\hat{\varphi}$.) If no confusion arises we write S_1 and S_2 for $S_1(\varphi)$ and $S_2(\varphi)$.

NOTATION 3.3. Suppose $\mathcal{S}$ is a finite sets of finite primes. Let

$$K(\mathcal{S}, p) := \{x \in K^*/K^{*p} : \ \forall \mathfrak{p} \notin \mathcal{S}, \mathfrak{p} \text{ non-archimedean }, v_\mathfrak{p}(x) \equiv 0 \bmod p\}.$$

Let C_K denote the class group of K. Denote G_K the absolute Galois group of K. Let M be a G_K-module. Let $H^1(K, M; \mathcal{S})$ be the subgroup of $H^1(K, M)$ of all classes of cocycles not ramified outside $\mathcal{S}$, i.e., all cocycles such that the restriction to $H^1(I_\mathfrak{p}, M)$ is trivial for all archimedean primes $\mathfrak{p} \notin S$, where $I_p \subset \mathrm{Gal}(\overline{K}/K)$ is the inertia subgroup.

For any cocycle $\xi \in H^1(K, M)$ denote $\xi_\mathfrak{p} := \mathrm{res}_\mathfrak{p}(\xi) \in H^1(K_\mathfrak{p}, M)$. Let $\delta_\mathfrak{p}$ be the map

$$E'(K_\mathfrak{p})/\varphi(E(K_\mathfrak{p})) \to H^1(K_\mathfrak{p}, E[\varphi])$$

induced by the boundary map, coming from the short exact sequence

$$0 \to E[\varphi] \to E \to E' \to 0.$$

Recall the following proposition of Schaefer and Stoll:

PROPOSITION 3.4 ([**60**, Proposition 4.6]). *Let $\psi : E \to E'$ be an isogeny of degree p^t defined over K, for some $t \in \mathbf{Z}_{>0}$. Let S be the set of primes $\mathfrak{p}$ such that p divides $c_{E,\mathfrak{p}}c_{E',\mathfrak{p}}$ or $\mathfrak{p}$ divides (p). Then*

$$S^\psi(E/K) = \ker H^1(K, E[\psi]; S) \to \oplus_{\mathfrak{p}\in S}(H^1(K_\mathfrak{p}, E[\psi])/\operatorname{Im}(\delta_\mathfrak{p})).$$

SKETCH OF THE PROOF. Consider the factorization

$$H^1(K, E[\psi]) \to H^1(K_\mathfrak{p}, E[\psi]) \xrightarrow{\iota_\mathfrak{p}^*} H^1(K_\mathfrak{p}, E).$$

From the long exact sequence in Galois cohomology it follows that the kernel of $\iota_\mathfrak{p}^*$ is isomorphic to $E'(K_\mathfrak{p})/\psi(E(K_\mathfrak{p}))$.

Then one shows the followings results.

- The unramified subgroup of $H^1(K_\mathfrak{p}, E[\psi])$ has the same number of elements as the number of $K_\mathfrak{p}$-rational points in the kernel ψ.

- If P is a point of $E'_0(K_\mathfrak{p})$ and (p) does not divide $\mathfrak{p}$ then $\delta_\mathfrak{p}(P \bmod \psi(E(K_\mathfrak{p}))$ is in the unramified subgroup.
- $\#\operatorname{Im}(\delta_\mathfrak{p}) = \#E(K_\mathfrak{p})[\psi]\frac{c_{E',\mathfrak{p}}}{c_{E,\mathfrak{p}}}$.

A short argument suffices to show that if $\mathfrak{p} \nmid (p)$ and $p \nmid c_{E',\mathfrak{p}}c_{E,\mathfrak{p}}$ (i.e., $\mathfrak{p} \notin S$) then $\operatorname{Im}(\delta_\mathfrak{p})$ is the unramified subgroup of $H^1(K_\mathfrak{p}, E[\psi])$ which yields the proof. □

In the case of an isogeny of prime degree one obtains a stronger corollary, where S_1, S_2 and S_3 are as defined above:

Proposition 3.5. *Let $\psi : E \to E'$ be a cyclic isogeny then $S^\psi(E/K)$ is the kernel of*

$$H^1(K, E[\psi]; S_1 \cup S_3) \to \oplus_{\mathfrak{p}\in S_2} H^1(K_\mathfrak{p}, E[\psi]) \oplus_{\mathfrak{p}\in S_3} (H^1(K_\mathfrak{p}, E[\psi])/\operatorname{Im}(\delta_\mathfrak{p})).$$

Proof. Suppose $\mathfrak{p}$ is a prime such that p divides the Tamagawa number $c_{E,\mathfrak{p}}c_{E',\mathfrak{p}}$. By assumption we have $p \le \max\{c_{E,\mathfrak{p}}, c_{E',\mathfrak{p}}\}$ and $p > 4$, hence Proposition 2.4 implies that the reduction at $\mathfrak{p}$ is split multiplicative. Conversely, Proposition 2.5 shows that for all primes $\mathfrak{p}$ such that E has split multiplicative reduction we have that p divides $c_{E,\mathfrak{p}}c_{E',\mathfrak{p}}$. Combining these two facts, we obtain that the set S of Proposition 3.4 equals the union $S_1 \cup S_2 \cup S_3$.

By Proposition 3.4 it suffices to show that if $\mathfrak{p} \in S_1$ then $\operatorname{Im}(\delta_\mathfrak{p}) = H^1(K, E[\psi])$ and if $p \in S_2$ then $\operatorname{Im}(\delta_\mathfrak{p}) = 0$. (The latter statement implies in particular that every $\xi \in S^\psi(E/K)$ is unramified at every prime $\mathfrak{p} \in S_2$: since $\operatorname{Im}(\delta_\mathfrak{p}) = 0$, we have by Proposition 3.4 that $\operatorname{res}_\mathfrak{p}(\xi) = 0$, hence the restriction to the inertia subgroup is 0, so ξ is unramified at $\mathfrak{p}$.)

If $\mathfrak{p} \in S_1 \cup S_2$ then $\mathfrak{p} \nmid (p)$. From [**83**, Proposition 3] it follows that

$$\dim_{\mathbf{F}_p} H^1(K_\mathfrak{p}, E[\psi]) \le 2.$$

Using [**59**, Lemma 3.8] one obtains

$$\#\operatorname{Im}(\delta_\mathfrak{p}) = \#E(K_\mathfrak{p})[\psi]\frac{c_{E',\mathfrak{p}}}{c_{E,\mathfrak{p}}} = p\frac{c_{E',\mathfrak{p}}}{c_{E,\mathfrak{p}}}.$$

If $\mathfrak{p} \in S_1$ then Proposition 2.5 implies that $c_{E',\mathfrak{p}} = pc_{E,\mathfrak{p}}$, hence

$$p^2 = \#\operatorname{Im}(\delta_\mathfrak{p}) \le \#H^1(K, E[\psi]) \le p^2$$

from which we deduce that $\operatorname{Im}(\delta_\mathfrak{p}) = H^1(K, E[\psi])$.

If $\mathfrak{p} \in S_2$ then Proposition 2.5 implies that $pc_{E',\mathfrak{p}} = c_{E,\mathfrak{p}}$, hence $\operatorname{Im}(\delta_\mathfrak{p}) = 0$. □

Remark 3.6. Proposition 3.5 is false when the degree of the isogeny is 2 or 3. For degree 3 Proposition 3.4 is valid.

We will indicate why Proposition 3.5 is false for degree 2 and 3. First of all, if the degree is 2, one needs to include conditions for the archimedean primes. Moreover, one needs to give conditions for non-split multiplicative primes (if the degree is 2) and conditions for the additive primes (if the degree is either 2 or 3).

Consider for example the curve $y^2 = x(x^2 + ax + a)$, for some square-free odd integer a. Let ψ be the isogeny obtained by dividing out $\{O, (0, 0)\}$. Then S_2 is an empty set, and S_1 consists of a subset of all primes dividing $a - 4$. We can twist this curve such that S_2 remains empty and all multiplicative primes are split. If the above proposition were true for degree 2, then the size of the ψ-Selmer group would depend on the number of prime factors of $a - 4$. Using [**72**, Proposition X.4.9] one can produce a such that the ψ-Selmer group is much smaller than the kernel given in Proposition 3.5.

DEFINITION 3.7. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two disjoint finite sets of finite primes of K, such that none of the primes in these sets divides (p).

Let

$$T = T(\mathcal{S}_1, \mathcal{S}_2) : K(\mathcal{S}_1, p) \to \oplus_{\mathfrak{p}\in\mathcal{S}_2} \mathcal{O}_{\mathfrak{p}}^*/\mathcal{O}_{\mathfrak{p}}^{*p}$$

be the $\mathbf{F}_p$-linear map induced by inclusion. Let $m(\mathcal{S}_1, \mathcal{S}_2)$ be the rank of T. In the special case of an isogeny $\varphi : E \to E'$ with associated sets $S_1(\varphi)$ and $S_2(\varphi)$ as above we write $m(\varphi) := m(S_1(\varphi), S_2(\varphi))$.

LEMMA 3.8. *We have*

$$\dim_{\mathbf{F}_p} K(\mathcal{S}, p) = \frac{1}{2}[K : \mathbf{Q}] + \#\mathcal{S} + \dim_{\mathbf{F}_p} C_K[p]$$

Hence the domain of T is finite-dimensional.

PROOF. Since $\zeta_p \in K$ we have that K does not admit any real embedding. The above formula is a special case of [**56**, Proposition 12.6]. □

PROPOSITION 3.9. *We have*

$$S^{\varphi}(E/K) \subset \{x \in K(S_1 \cup S_3, p) \colon x = [1 \bmod K_{\mathfrak{p}}^{*p}] \text{ for all } \mathfrak{p} \in S_2\} = \ker T(S_1 \cup S_3, S_2)$$

and

$$S^{\varphi}(E/K) \supset \{x \in K(S_1, p) \colon x = [1 \bmod K_{\mathfrak{p}}^{*p}] \text{ for all } \mathfrak{p} \in S_2 \cup S_3\} = \ker T(S_1, S_2 \cup S_3).$$

PROOF. This follows from the identification $E[\varphi] \cong \mathbf{Z}/p\mathbf{Z} \cong \mu_p$, the fact $H^1(L, \mu_p) \cong L^*/L^{*p}$ for any field L of characteristic different from p (see [**64**, X.3.b]), and Proposition 3.5. □

PROPOSITION 3.10. *We have*

$$\begin{aligned} \#S_1 - \#S_2 + \dim_{\mathbf{F}_p} C_K[p] - \frac{3}{2}[K : \mathbf{Q}] &\leq \dim_{\mathbf{F}_p} S^{\varphi}(E/K) \\ &\leq \#S_1 + \dim_{\mathbf{F}_p} C_K[p] - m(\varphi) + \frac{3}{2}[K : \mathbf{Q}]. \end{aligned}$$

PROOF. Using Hilbert 90 [**64**, Proposition X.2] we have that

$$\dim \mathcal{O}_{\mathfrak{p}}^*/\mathcal{O}_{\mathfrak{p}}^{*p} = \dim H^1(K_{\mathfrak{p}}, \mu_p) - 1.$$

Using [**83**, Proposition 3] we obtain that $\dim H^1(K_{\mathfrak{p}}, \mu_p) = 2 + e(\mathfrak{p}/p)$, where $e(\mathfrak{p}/p)$ is the ramification index of $\mathfrak{p}/p$ if $\mathfrak{p}$ divides p and zero otherwise. This yields

$$\dim \oplus_{\mathfrak{p}\in S_3} \mathcal{O}_{\mathfrak{p}}^*/\mathcal{O}_{\mathfrak{p}}^{*p} = \sum_{\mathfrak{p}\in S_3} 1 + e(\mathfrak{p}/p) \leq 2[K : \mathbf{Q}].$$

The above bounds combined with Lemma 3.8 and Proposition 3.9 give us

$$\begin{aligned} \dim_{\mathbf{F}_p} S^{\varphi}(E/K) &\geq \dim_{\mathbf{F}_p} K(S_1, p) - \#S_2 - \#S_3 \geq -\frac{3}{2}[K : \mathbf{Q}] + \\ &+ \#S_1 + \dim_{\mathbf{F}_p} C_K[p] - \#S_2 \end{aligned}$$

For the other inequality, we obtain using Proposition 3.9

$$\dim_{\mathbf{F}_p} S^{\varphi}(E/K) \leq \dim_{\mathbf{F}_p} \ker T(S_1 \cup S_3, S_2) \leq \dim_{\mathbf{F}_p} K(S_1 \cup S_3, p) - m(\varphi).$$

Since $\#S_3 \leq [K:\mathbf{Q}]$, another application of Lemma 3.8 to the right hand side of this inequality yields

$$\dim_{\mathbf{F}_p} S^{\varphi}(E/K) \leq \#S_1 + \dim_{\mathbf{F}_p} C_K[p] - m(\varphi) + \frac{3}{2}[K:\mathbf{Q}].$$

□

LEMMA 3.11. *We have*

$$\operatorname{rank} E(K) \leq \#S_1(\varphi) + \#S_2(\varphi) + 2\dim_{\mathbf{F}_p} C_K[p] + 3[K:\mathbf{Q}] - m(\varphi) - m(\hat{\varphi}) - 1.$$

PROOF. Consider the following sequences of inequalities

$$\begin{aligned} 1 + \operatorname{rank} E(K) &\leq \dim_{\mathbf{F}_p} E(K)/pE(K) \\ &\leq \dim_{\mathbf{F}_p} S^p(E/K) \\ &\leq \dim_{\mathbf{F}_p} S^{\varphi}(E/K) + \dim_{\mathbf{F}_p} S^{\hat{\varphi}}(E'/K). \end{aligned}$$

The first inequality follows from the fact that $E(K)$ has p-torsion, the second one follows from the long exact sequence in cohomology associated to $0 \to E[p] \to E \to E \to 0$ and the third one follows from the exact sequence

$$0 \to E'(K)[\hat{\varphi}]/\varphi(E(K)[p]) \to S^{\varphi}(E/K) \to S^p(E/K) \to S^{\hat{\varphi}}(E'/K).$$

(See [**60**, Lemma 9.1].)

Applying Proposition 3.10 gives

$$\begin{aligned} &\dim_{\mathbf{F}_p} S^{\varphi}(E/K) + \dim_{\mathbf{F}_p} S^{\hat{\varphi}}(E'/K) \leq \\ &\leq \#S_1(\varphi) + \#S_1(\hat{\varphi}) + 2\dim_{\mathbf{F}_p} C_K[p] + 3[K:\mathbf{Q}] - m(\varphi) - m(\hat{\varphi}). \end{aligned}$$

□

By a theorem of Cassels we can compute the difference of the dimension of $S^{\varphi}(E/K)$ and $S^{\hat{\varphi}}(E'/K)$. We do not need the precise difference, but only an estimate, namely

LEMMA 3.12. *There is an integer t, with $|t| \leq 2[K:\mathbf{Q}] + 1$ such that*

$$\dim_{\mathbf{F}_p} S^{\hat{\varphi}}(E'/K) = \dim_{\mathbf{F}_p} S^{\varphi}(E/K) - \#S_1(\varphi) + \#S_2(\varphi) + t.$$

PROOF. This follows from [**14**] combined with Proposition 2.5. □

LEMMA 3.13.

$$\dim_{\mathbf{F}_p} S^{\varphi}(E/K) + \dim_{\mathbf{F}_p} S^{\hat{\varphi}}(E'/K) \geq |\#S_1 - \#S_2| + 2\dim_{\mathbf{F}_p} C_K[p] - 5[K:\mathbf{Q}] - 1.$$

PROOF. After possibly interchanging E and E' we may assume that $\#S_1 \geq \#S_2$. From Proposition 3.10 we know

$$\dim_{\mathbf{F}_p} S^{\varphi}(E/K) \geq \#S_1 - \#S_2 + \dim_{\mathbf{F}_p} C_K[p] - \frac{3}{2}[K:\mathbf{Q}].$$

From this inequality and Lemma 3.12 we obtain that

$$\begin{aligned} \dim_{\mathbf{F}_p} S^{\hat{\varphi}}(E'/K) &\geq \dim_{\mathbf{F}_p} S^{\varphi}(E/K) - 2[K:\mathbf{Q}] - 1 - \#S_1 + \#S_2 \\ &\geq \dim_{\mathbf{F}_p} C_K[p] - \frac{7}{2}[K:\mathbf{Q}] - 1. \end{aligned}$$

Summing both inequalities gives the Lemma. □

LEMMA 3.14. *Let* $s := \dim_{\mathbf{F}_p} S^{\varphi}(E/K) + \dim_{\mathbf{F}_p} S^{\hat{\varphi}}(E'/K) - 1$ *and* $r := \operatorname{rank} E(K)$, *then*

$$\max(\dim_{\mathbf{F}_p} Ш(E/K)[p], \dim_{\mathbf{F}_p} Ш(E'/K)[p]) \geq \frac{(s-r)}{2}.$$

PROOF. The exact sequence

$$0 \to E'(K)[\hat{\varphi}]/\varphi(E(K)[p]) \to S^{\varphi}(E/K) \to S^{p}(E/K) \to$$
$$\to S^{\hat{\varphi}}(E'/K) \to Ш(E'/K)[\hat{\varphi}]/\varphi(Ш(E/K)[p])$$

(See [**60**, Lemma 9.1]) implies

$$\dim_{\mathbf{F}_p} Ш(E'/K)[\hat{\varphi}] + \dim_{\mathbf{F}_p} S^{p}(E/K) \geq s - 1 + \dim_{\mathbf{F}_p} E(K)[p].$$

The lemma follows now from the following inequality coming from the long exact sequence in Galois cohomology

$$\dim_{\mathbf{F}_p} Ш(E'/K)[p] + \dim_{\mathbf{F}_p} Ш(E/K)[p] \geq$$
$$\geq \dim_{\mathbf{F}_p} Ш(E'/K)[\hat{\varphi}] + \dim_{\mathbf{F}_p} S^{p}(E/K) - r - \dim_{\mathbf{F}_p} E(K)[p].$$

□

LEMMA 3.15. *Let* $\psi : E_1 \to E_2$ *be some isogeny obtained by dividing out the group generated by a* K*-rational point of order* p*, with* E_1 *is* K*-isogenous to* E*. Then*

$$\max(\dim_{\mathbf{F}_p} Ш(E/K)[p], \dim_{\mathbf{F}_p} Ш(E'/K)[p]) \geq$$
$$\geq -\min(\#S_1(\varphi), \#S_2(\varphi)) - 5[K:\mathbf{Q}] - 1 + \frac{1}{2}(m(\psi) + m(\hat{\psi})).$$

PROOF. Use Lemma 3.11 for the isogeny ψ to obtain the bound for the rank of $E(K)$. Then combine this with Lemma 3.13 and Lemma 3.14 and use that

$$\#S_1(\varphi) + \#S_2(\varphi) = \#S_1(\psi) + \#S_2(\psi).$$

□

4. Modular curves

In this section we prove Theorem 1.1. We construct certain fields $K/\mathbf{Q}$ such that $X(p)(K)$ contains points with certain reduction properties. These reduction properties translate into certain properties of elliptic curves E/K admitting two cyclic isogenies φ, ψ such that $m(\psi)$ is much larger then $\min(\#S_1(\varphi), \#S_2(\varphi))$ (notation from the previous section). Then applying the results of the previous section one obtains a proof of Theorem 1.1.

The following result will be used in the proof of Theorem 1.1.

THEOREM 4.1 ([**27**, Theorem 10.4]). *Let* $f \in \mathbf{Z}[X]$ *be a polynomial of degree at least 1. Let* d *be the number of irreducible factors of* f*. Suppose that for every prime* ℓ*, there exists a* $y \in \mathbf{Z}/\ell\mathbf{Z}$ *such that* $f(y) \not\equiv 0 \bmod \ell$*. Then there exists a constant* n *depending on the degree of* f *such that there exist infinitely many primes* ℓ*, such that* $f(\ell)$ *has at most* n *prime factors. Moreover, let*

$$h(x) := \#\{y \in \mathbf{Z} \colon 0 \leq y \leq x \text{ and the number of prime factors of } f(y) \text{ is at most } n\}$$

then there exist $\delta > 0$, such that

$$h(x) \geq \delta \frac{x}{\log^d x}\left(1 + O\left(\frac{1}{\sqrt{\log(x)}}\right)\right)$$

as $x \to \infty$.

Any improvement on the constant n in the Theorem will give a better function $g(p)$ (notation from Theorem 1.1), but the new $g(p)$ will still be of type $O(p^4)$.

The proofs for most of the below mentioned properties of $X_0(p)$ and $X(p)$ can be found in [**68**] or [**81**]. See also [**22**, Chapter 4].

NOTATION 4.2. Denote by $X(p)/\mathbf{Q}$ the compactification of the curve parameterizing pairs $((E, O), f)$ where (E, O) is an elliptic curve and f is an isomorphism of group schemes $f : \mathbf{Z}/p\mathbf{Z} \times \mu_p \to E[p]$ with the property that $\zeta^{ax-by} = e(f(x, \zeta^a), f(y, \zeta^b))$, where $e : E[p] \times E[p] \to \mu_p$ is the Weil-pairing and ζ is a fixed primitive p-th root of unity. One regards $X(p)$ as the curve "parameterizing elliptic curves with a symplectic structure on the p-torsion". This definition forces $X(p)$ to be absolutely irreducible.

Let H be the $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-invariant Borel subgroup of $\mathrm{Aut}(X(p) \otimes \overline{\mathbf{Q}}) = \mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$, which fixes $((E, O), f|_{\mathbf{Z}/p\mathbf{Z}\times\{1\}})$. Denote by $X_0(p)/\mathbf{Q}$ the curve obtained by taking the quotient of $X(p)$ under the action of H. The curve $X_0(p)$ is a coarse moduli space for pairs $((E, O), \varphi)$ where $\varphi : E \to E'$ is an isogeny of degree p. (See for example [**43**, Chapter 2].)

Let $R_1 \in X_0(p)$ be the unramified cusp (classically called 'infinity'). Let $R_2 \in X_0(p)$ be the ramified cusp.

Let $\pi_i : X(p) \to X_0(p)$ be the morphism obtained by mapping (E, f) to (E, φ_i) where φ_i is the isogeny obtained by dividing out by $f(\mathbf{Z}/p\mathbf{Z} \times \{1\})$ when $i = 1$, and $f(\{0\} \times \mu_p)$ when $i = 2$. The maps π_i are defined over $\mathbf{Q}$.

Let $P \in X(p)$ be a point, which is not a cusp. Let E_P be the elliptic curve corresponding to P. For $i = 1, 2$ define the isogeny $\varphi_{P,i} : E_P \to E'_P$ in such a way that $(E_P, \varphi_{P,i})$ represents the point $\pi_i(P) \in X_0(p)$.

DEFINITION 4.3. Let T be a cusp of $X(p)$. We say that T is of type $(\delta, \epsilon) \in \{1, 2\}^2$ if $\pi_1(T) = R_\delta$ and $\pi_2(T) = R_\epsilon$.

Being of type (δ, ϵ) is invariant under the action of the absolute Galois group of $\mathbf{Q}$, since the morphisms π_i are defined over $\mathbf{Q}$ and the cusps on $X_0(p)$ are $\mathbf{Q}$-rational.

Suppose T is a cusp of type (δ, ϵ). Then for all number fields $K/\mathbf{Q}(\zeta_p)$ and all points $P \in X(p)(K)$ we have that if $\mathfrak{p} \nmid (p)$ is a prime of K such that $P \equiv T \bmod \mathfrak{p}$ then $\mathfrak{p} \in S_\delta(\varphi_{P,1})$ and $\mathfrak{p} \in S_\epsilon(\varphi_{P,2})$. This statement can be shown by an easy consideration on the behavior of the Tate-parameter q of the curve representing the point $P \in X(p)(K)$ and the relation between q and the j-invariant. (Compare the proof of Proposition 2.5.)

LEMMA 4.4. *$X(p)$ has $(p-1)/2$ cusps of each of the types $(2, 1)$ and $(1, 2)$. The other $(p-1)^2/2$ cusps are of type $(2, 2)$. All cusps of type $(1, 2)$ are $\mathbf{Q}$-rational.*

PROOF. A cusp of type $(1, 1)$ would give rise to elliptic curves $E/K_\mathfrak{p}$, with multiplicative reduction such that its reduction modulo $\mathfrak{p}$ has $(\mathbf{Z}/p\mathbf{Z})^2$ as a subgroup, but over an algebraically closed field L of characteristic p, we have $\#E(L)[p] \leq p$, a contradiction.

The ramification index of every point in $\pi_i^{-1}(R_1)$ is p, hence there are $(p-1)/2$ points in $\pi_i^{-1}(R_1)$. From this it follows that there exists $(p-1)/2$ cusps of type $(1,2)$ and $(2,1)$, respectively. The remaining cusps are of type $(2,2)$.

An argument as in [**57**, page 44 and 45] shows that there is a cusp of type $(1,2)$ that is $\mathbf{Q}$-rational. From this it follows that all cusps of type $(1,2)$ are $\mathbf{Q}$-rational. (See [**22**, Chapter 4].) □

Proof of Theorem 1.1. In this proof the word cusp is used to indicate points P on $X(p)$ such that $j(P)=\infty$ for the usual map $j : X(p) \to X(1)$.

Let D be a divisor on $X(p)$ satisfying the following properties:

- D is effective;
- D is invariant under $G_{\mathbf{Q}}$;
- the support of D is contained in the set of cusps of type $(1,2)$;
- the dimension of the linear system $|D|$ is at least 2;
- the morphism $\varphi_{|D|} : X(p) \to \mathbf{P}^n$ is injective at almost all geometric points of $X(p)$.

Let L be a 2-dimensional linear subsystem of $|D|$ containing D and such that the corresponding morphism is injective at almost all geometric points. Let $C \subset \mathbf{P}^2$ be the image of $X(p)$ given by L. We may assume that the intersection of the line $X=0$ with C is precisely D. An automorphism ψ of $\mathbf{P}^2$ fixing the line $X=0$, is of the form $[X,Y,Z] \mapsto [a_1X, b_1X+b_2Y+b_3Z, c_1X+c_2Y+c_3Z]$. It is easy to see that we can choose a_1, b_i, c_i in such a way that none of the cusps is on the line $Z=0$, and the function $x = X/Z$ takes distinct values at any pair of cusps with $x \neq 0$. So we may assume that we have a fixed (possibly singular) model $C/\mathbf{Q}$ for $X(p)$ in $\mathbf{P}^2$, such that the line $X=0$ intersects C only in cusps of type $(1,2)$ and no other points, all x-coordinates of other the cusps are distinct and finite, and all y-coordinates of the cusps are finite. Denote $H \in \mathbf{Z}[X,Y,Z]$ a defining polynomial of C. Set $h(x,y) := H(X,Y,1)$.

Let $f_{\delta,\epsilon} \in \mathbf{Z}[X]$ be the square-free polynomial with roots all x-coordinates of the cusps of type (δ,ϵ) of $X(p)$ and content 1. After a simultaneous transformation of the $f_{\delta,\epsilon}$ of the form $x \mapsto cx$, we may assume that $f_{2,1}(0)=1$ and $f_{2,1} \in \mathbf{Z}[X]$. Let n denote the constant of Theorem 4.1 for the polynomial $f_{2,1}$. The discriminant of $f_{1,2}f_{2,1}f_{2,2}$ is non-zero, since every cusp has only one type and all cusps have distinct x-coordinate.

Let $\mathcal{B}$ consist of p, all primes ℓ dividing the leading coefficient or the discriminant of

$$f_{1,2}f_{2,1}f_{2,2},$$

all primes ℓ smaller than the degree of $f_{2,1}$ and all primes dividing the leading coefficient of $\operatorname{res}(h, f_{2,2}, x)$, the resultant of h and $f_{2,2}$ with respect to x.

Let $\mathcal{P}_2$ be the set of primes ℓ not in $\mathcal{B}$ such that every irreducible factor of $f_{2,1}(x)(x^p - 1) \bmod \ell$ and every irreducible factor of $\operatorname{res}(h, f_{2,1}, x) \bmod \ell$ has degree 1. Note that by Frobenius' Theorem ([**77**]) the set $\mathcal{P}_2$ is infinite. The condition mentioned here, implies that if we take a triple (x_0, ℓ, y_0) with $x_0 \in \mathbf{Z}$, the prime $\ell \in \mathcal{P}_2$ divides $f_{2,1}(x_0)$ and y_0 is a zero of $h(x_0, y)$ then every prime $\mathfrak{q}$ of $\mathbf{Q}(\zeta_p, y_0)$ over ℓ satisfies $f(\mathfrak{q}/\ell)=1$, where $f(\mathfrak{q}/\ell)$ denotes the degree of the extension of the residue fields.

Fix $\mathcal{S}_1$ and $\mathcal{S}_2$ two finite, disjoint sets of primes, not containing a archimedean prime such that

$$m(\mathcal{S}_1, \mathcal{S}_2) \geq 2k + 2(n+5)\deg(h)(p-1) + 2,$$

$S_1 \cap \mathcal{B} = \emptyset$ and $S_2 \subset \mathcal{P}_2$, with $m(\mathcal{S}_1, \mathcal{S}_2)$ as defined in Section 3. (The existence of such sets follows from Dirichlet's theorem on primes in arithmetic progression and the fact that $\ell \in \mathcal{S}_2$ implies $\ell \equiv 1 \bmod p$.)

LEMMA 4.5. *There exists an $x_0 \in \mathbf{Z}$ such that*

- $x_0 \equiv 0 \bmod \ell$*, for all primes ℓ smaller than the degree of $f_{2,1}$ and all ℓ dividing the leading coefficient of $f_{2,1}$.*
- $x_0 \equiv 0 \bmod \ell$*, for all $\ell \in \mathcal{S}_1$,*
- $f_{2,2}(x_0) \equiv 0 \bmod \ell$*, for all $\ell \in \mathcal{S}_2$,*
- $f_{2,1}(x_0)$ *has at most n prime divisors.*
- $h(x_0, y)$ *is irreducible.*

PROOF. The existence of such an x_0 can be proven as follows. Take an $a \in \mathbf{Z}$ satisfying the above three congruence relations. Take b to be the product of all primes mentioned in the above congruence relations. Define $\tilde{f}(Z) = f_{2,1}(a+bZ)$. We claim that the content of $\tilde{f}$ is one. Suppose ℓ divides this content. Then ℓ divides the leading coefficient of $\tilde{f}$. From this one deduces that ℓ divides b. We distinguish several cases:

- If $\ell \in \mathcal{S}_i$ then $f_{i,2}(a) \equiv 0 \bmod \ell$ and ℓ does not divide the discriminant of the product of the $f_{\delta,\epsilon}$, so we have $\tilde{f}(0) \equiv f_{2,1}(a) \not\equiv 0 \bmod \ell$.
- If ℓ divides b and ℓ is not in $\mathcal{S}_1 \cup \mathcal{S}_2$ then $\tilde{f}(0) \equiv f_{2,1}(0) \equiv 1 \bmod \ell$.

So for all primes ℓ dividing b we have that $\tilde{f} \not\equiv 0 \bmod \ell$. This proves the claim on the content of $\tilde{f}$.

Suppose ℓ is a prime smaller than the degree of $\tilde{f}$, then $\tilde{f}(0) \equiv 1 \bmod \ell$ holds. If ℓ is different from these primes, then there is a coefficient of $\tilde{f}$ which is not divisible by ℓ and the degree of $\tilde{f}$ is smaller than ℓ. So for every prime ℓ there is an $z_\ell \in \mathbf{Z}$ with $\tilde{f}(z_\ell) \not\equiv 0 \bmod \ell$. From this we deduce that we can apply Theorem 4.1. The constant for $\tilde{f}$ depends only on the degree of the irreducible factors of $\tilde{f}$, hence equals n. The set

$$\{x_1 \in \mathbf{Z} \colon \tilde{f}(x_1) \text{ has at most } n \text{ prime divisors}\}$$

is not a thin set. So

$$\mathcal{H} := \{x_1 \in \mathbf{Z} \colon \tilde{f}(x_1) \text{ has at most } n \text{ prime divisors and } h(a+bx_1, y) \text{ is irreducible}\}$$

is not empty by Hilbert's Irreducibility Theorem [**65**, Chapter 9]. Fix such an $x_1 \in \mathcal{H}$. Let $x_0 = a + bx_1$. This proves the claim on the existence of such an x_0. □

Fix an x_0 satisfying the conditions of Lemma 4.5. Adjoin a root y_0 of $h(x_0, y)$ to $\mathbf{Q}(\zeta_p)$. Denote the field $\mathbf{Q}(\zeta_p, y_0)$ by K_1. Let P be the point on $X(p)(K_1)$ corresponding to (x_0, y_0). Let E/K_1 be the elliptic curve corresponding to P. Let $K = K_1(\sqrt{c_4(E)})$. Then if $\mathfrak{q}$ is a prime such that $E/K_\mathfrak{q}$ has multiplicative reduction then $E/K_\mathfrak{q}$ has split multiplicative reduction.

For every prime $\mathfrak{p}$ of K over $\ell \in \mathcal{S}_1$ we have that $P \bmod \mathfrak{q}$ is a cusp of type $(1,2)$. Over every prime $\ell \in \mathcal{S}_2$ there exists a prime $\mathfrak{q}$ such that $P \bmod \mathfrak{q}$ is a cusp of type $(2,2)$. From our assumptions on x_0 it follows that p does not divide $f(\mathfrak{q}/\ell)$. Let $\mathcal{T}_1$ consists of the primes of K lying over the primes in $\mathcal{S}_1$. Let $\mathcal{T}_2$ be the set of primes $\mathfrak{q}$ such that $\mathfrak{q}$ lies over a prime in $\mathcal{S}_2$ and $P \bmod \mathfrak{q}$ is a cusp of type $(2,2)$.

Note that the set of primes of K such that P reduces to a cusp of type $(2,1)$ has at most $n[K : \mathbf{Q}]$ elements.

We have the following diagram

$$\begin{array}{ccc} \mathbf{Q}(\mathcal{S}_1,p) & \rightarrow & \oplus_{\ell\in\mathcal{S}_2}\mathbf{Z}_\ell^*/\mathbf{Z}_\ell^{*p} \\ \downarrow & & \downarrow \\ K(\mathcal{T}_1,p) & \rightarrow & \oplus_{\mathfrak{q}\in\mathcal{T}_2}\mathcal{O}_{K_\mathfrak{q}}^*/\mathcal{O}_{K_\mathfrak{q}}^{*p}. \end{array}$$

Since $p\nmid f(\mathfrak{q}/\ell)$ for all $\ell\in\mathcal{S}_2$, the arrow in the right column is injective. This implies

$$m(\varphi_{P,1}/K)\geq m(\mathcal{T}_1,\mathcal{T}_2)\geq m(\mathcal{S}_1,\mathcal{S}_2)=2k+4(n+5)\deg(h)(p-1)+2.$$

Since $S_2(\varphi_{p,2}/K)\leq[K:\mathbf{Q}]n$ and $[K:\mathbf{Q}]\leq 2(p-1)\deg(h)$ we obtain by Lemma 3.15 that for some E' isogenous to E we have

$$\begin{aligned} \dim_{\mathbf{F}_p}\text{Ш}(E'/K)[p] &\geq -\#S_1(\varphi_{P,2})-5[K:\mathbf{Q}]-1+\frac{1}{2}m(\mathcal{S}_1,\mathcal{S}_2) \\ &\geq -(n+5)[K:\mathbf{Q}]-1+\frac{1}{2}m(\mathcal{S}_1,\mathcal{S}_2)=k. \end{aligned}$$

Note that $\deg(h)$ can be bounded by a function of type $O(p^3)$, hence $[K:\mathbf{Q}]$ can be bounded by a function of type $O(p^4)$. □

To finish, we prove Corollary 1.2.

Proof of Corollary 1.2. Let E/K be an elliptic curve such that

$$\dim_{\mathbf{F}_p}\text{Ш}(E/K)[p]\geq kg(p)$$

and $[K:\mathbf{Q}]\leq g(p)$.

Let R be the $[K:\mathbf{Q}]$-dimensional abelian variety $\mathrm{Res}_{K/\mathbf{Q}}(E)$ (Weil restriction of scalars of E). Then by [**45**, Proof of Theorem 1]

$$\dim_{\mathbf{F}_p}\text{Ш}(R/\mathbf{Q})[p]=\dim_{\mathbf{F}_p}\text{Ш}(E/K)[p]\geq kg(p).$$

From this it follows that there is a simple factor A of R, with $\dim_{\mathbf{F}_p}\text{Ш}(A/\mathbf{Q})[p]\geq k$. □

Remark 4.6. In this final remark we want to discuss a technical difference between our strategy and Clark's strategy [**16**] to construct elliptic curves E/K such that $\#\text{Ш}(E/K)[p]$ is unbounded. Both strategies use elliptic curves such that $\#E(K)[p]=p^2$. This condition implies that (compare the proof of Lemma 3.11)

$$\begin{aligned} \dim\text{Ш}(E/K)[p] &\leq \dim S^p(E/K)-2 \\ &\leq 3[K:\mathbf{Q}]+\#\{\mathfrak{p}\text{ prime: } E/K_\mathfrak{p}\text{ has bad reduction}\}+2\dim C_K[p]. \end{aligned}$$

To produce examples of large Tate-Shafarevich groups with bounded field extension degree one needs, first of all, either to produce families of elliptic curves in which the number of bad primes is unbounded or to produce examples of families of fields in which $\dim C_K[p]$ is unbounded. We chose to pursue the first strategy, while Clark's method allows one to bound the number of bad primes, hence he has to produce families of fields in which $\dim C_K[p]$ is unbounded.

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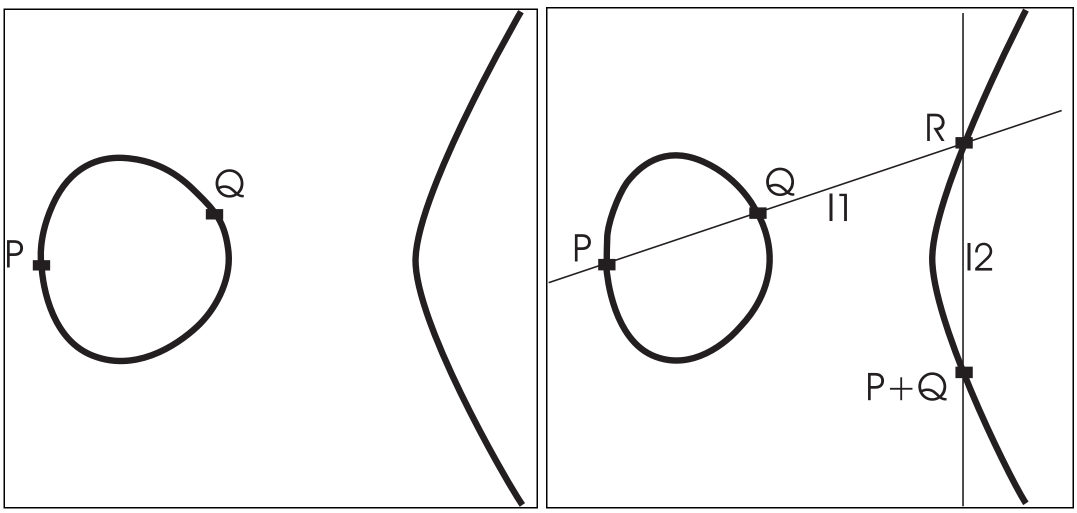
Samenvatting

In dit proefschrift worden elliptische krommen en elliptische oppervlakken bestudeerd. Een *elliptische kromme* E over de complexe getallen $\mathbf{C}$ wordt beschreven door een vergelijking

$$y^2 = x^3 + Ax + B$$

met $A, B \in \mathbf{C}$, zodat de bovenstaande vergelijking een gladde kromme definieert. Deze laatste eis is equivalent met $4A^3 + 27B^2 \neq 0$. Er zijn verschillende manieren om elliptische krommen te visualiseren: Een elliptische kromme (over $\mathbf{C}$) is continu te vervormen in een torus, of in alledaags Nederlands, een fietsband.

De punten van E met coördinaten in $\mathbf{C}$, dat wil zeggen alle $(x, y) \in \mathbf{C}^2$, waarvoor $y^2 = x^3 + Ax + B$ geldt, tezamen met een 'punt O op oneindig', vormen een abelse groep. Dit laatste betekent dat men punten van E kan optellen: kies als basispunt $O \in E$. De som van twee punten P, Q kan als volgt worden uitgerekend: laat ℓ_1 de lijn zijn die door P en Q gaat, als $P \neq Q$, en de raaklijn aan E bij P, als $P = Q$. De lijn ℓ_1 snijdt E in een derde punt R. Als R verschillend is van O, laat dan ℓ_2 de verticale lijn zijn door R. Als ℓ_2 geen raaklijn aan E is, dan is $P + Q$ het snijpunt van ℓ_2 ongelijk aan R en O, als ℓ_2 een raaklijn aan E is, dan is $P + Q = R$. Als $R = O$, dan is $P + Q = O$.



FIGUUR 1. Optellen op een elliptische kromme

Hoofdstuk 6 van het proefschrift gaat over een speciale klasse van elliptische krommen. Stel K is een getallenlichaam (bijvoorbeeld $K = \mathbf{Q}$, de rationale getallen). Neem een elliptische kromme E/K, dat wil zeggen er is een vergelijking voor E van de vorm $y^2 = x^3 + Ax + B$, met $A, B \in K$. De punten op E met coördinaten in K tezamen met het punt op oneindig vormen een ondergroep $E(K)$ van de groep van punten op E. Het is bekend (Mordell, 1922 en Weil, 1928) dat $E(K)$ het product van een eindige groep en $\mathbf{Z}^r$ is. Het getal r wordt de rang van E genoemd.

Om r te bepalen, probeert men vaak een zogenoemde p-Selmer groep $S^p(E/K)$ te berekenen, waarbij p een willekeurig priemgetal is. Deze eindige groep is relatief gemakkelijk uit te rekenen en er geldt $p^r \leq \#S^p(E/K)$. In hoofdstuk 6 wordt voor ieder priemgetal p een rij getallenlichamen K_n en elliptische krommen E_n/K_n geconstrueerd, met rang van $E_n(K_n)$ gelijk aan r_n, waarvoor het quotient

$$\frac{\#S^p(E_n/K_n)}{p^{r_n}}$$

onbegrensd is. Dit zijn dus voorbeelden van elliptische krommen waarvoor de bovengrens op de r gegeven door de p-Selmer groep slecht is. Voor de priemgetallen $p = 2$ en $p = 3$ is dit een klassiek resultaat ([**11**] ($p = 2$) en [**13**] ($p = 3$)) . Het geval $p = 5$ is recent bewezen door Fisher ([**23**]).

De auteur heeft samen met Schaefer ([**38**]) in 2003 bewezen dat voor ieder priemgetal p er een rij zoals boven bestaat, waarvoor $\#S^p(E_n/K_n)$ naar oneindig gaat. Het bovenstaande resultaat is hier een verdieping va. Recent heeft Clark een analoog resultaat bewezen, gebruikmakend van een andere methode en na een eerdere versie van Hoofdstuk 6 te hebben gezien.

De overige vijf hoofdstukken gaan over elliptische oppervlakken. Een *elliptisch oppervlak* is een 1-dimensionale familie van elliptische krommen. Iets preciezer: een elliptisch oppervlak is een oppervlak X waarvoor er een kromme C en een afbeelding $\pi : X \to C$ bestaan, zodat voor bijna alle punten $p \in C$ geldt dat $\pi^{-1}(p)$ een elliptische kromme is. Een elliptisch oppervlak kan worden beschreven door een vergelijking

$$y^2 = x^3 + Ax + B,$$

waarbij A en B nu functies op C zijn. Een voorbeeld is

$$y^2 = x^3 + t^5x + 7t^2 + 3,$$

waarbij π wordt gegeven door (x, y, t) naar t te sturen. Een elliptisch oppervlak kan tegelijkertijd worden beschouwd als oppervlak en als elliptische kromme over het functielichaam $K(C)$ van C.

Aan een gegeven elliptisch oppervlak $\pi : X \to C$ wordt een groep $NS(X)$ geassocieerd, de Néron-Severi groep. Deze groep bevat veel informatie over X. De elementen van $NS(X)$ zijn formele sommen van irreducibele krommen in X, waarbij twee krommen gelijk worden gesteld als zij bijna alle andere krommen even vaak snijden. Men kan aantonen dat $NS(X) \cong \mathbf{Z}^{\rho(X)}$, voor een zeker geheel getal $\rho(X)$. Het getal $\rho(X)$ wordt het Picard getal van X genoemd. De secties (of sneden) van π zijn alle afbeeldingen $\sigma : C \to X$, zodat $\pi\sigma$ de identiteit op C is.

In de eerste twee hoofdstukken van dit proefschrift bestuderen wij $\rho(X)$ als functie in X.

Er geldt $\rho(X) \leq h^{1,1}(X)$, waarbij $h^{1,1}(X)$ de dimensie (over $\mathbf{C}$) van de vectorruimte van de zogeheten $(1, 1)$ differentiaalvormen op X is. In hoofdstuk 1 worden elliptische oppervlakken bestudeerd waarvoor deze bovengrens wordt aangenomen, dat wil zeggen $\rho(X) = h^{1,1}$, met de extra eis dat er slechts eindig veel secties zijn. Een van de resultaten (Stelling 1.6.3 van het proefschrift) van dat hoofdstuk is dat bijna alle voorbeelden geïsoleerde voorbeelden zijn, maar er zijn ook een aantal families van dit soort oppervlakken (zie Paragraaf 1.3 van het proefschrift). In de literatuur is al een gedeelte van onze classificatie te vinden (zie [**51**]). De complete classificatie is evenwel nieuw. Een onmiddellijk gevolg van onze classificatie (Stelling 1.4.8 van het proefschrift) is al wel in de literatuur te vinden, maar slechts in een zwakkere ([**31**], [**58**]), danwel foutieve ([**15**], [**10**]) vorm.

In hoofdstuk 2 wordt voor ieder geheel getal r de collectie van elliptische oppervlakken bestudeerd met de eigenschap $\rho(X) \geq r$. In het bijzonder geven we aan (Stelling 2.1.1 van het proefschrift) hoe 'groot' deze collectie is, voor kleine waarden van r. Dit is een uitbreiding van het werk van David Cox ([**18**]) die deze 'grootte' kon bepalen in het speciale geval $r = 3$.

De secties van een elliptisch oppervlak $\pi : X \to C$ vormen een abelse groep, die de Mordell-Weil groep wordt genoemd en wordt genoteerd als $MW(\pi)$. Als we het elliptische oppervlak beschouwen als elliptische krommen E over $K(C)$, dan is $MW(\pi)$ hetzelfde als de punten in $E(K(C))$. In de hoofdstukken 3 en 4 wordt deze groep $MW(\pi)$ bestudeerd.

De rang van $MW(\pi)$ wordt begrensd door $\rho(X) - 2$. Het bepalen van de rang van $MW(\pi)$ is echter even moeilijk als het bepalen van $\rho(X)$. Daarentegen is het eenvoudig in te zien dat er collecties elliptische oppervlakken bestaan (met $C = \mathbf{P}^1$, de projectieve lijn) waarvoor de rang van $NS(X)$ onbegrensd is. Het is echter onbekend of er soortgelijke voorbeelden bestaan waarvoor de rang van $MW(\pi)$ onbegrensd is.

In hoofdstuk 3 worden voortbrengers gevonden voor $MW(\pi)$ voor een klasse van elliptische oppervlakken, geïntroduceerd door Kuwata. Kuwata bepaalde voor alle elliptische oppervlakken in deze klasse de rang van $MW(\pi)$, maar hij kon geen expliciete voortbrengers vinden.

In hoofdstuk 4 wordt een voorbeeld van een elliptisch oppervlak $\pi : X \to C$ met rang $MW(\pi) = 15$ gegeven. Dit is een oppervlak van een type waarvoor Kuwata ([**41**]) al voorbeelden had gegeven met rang $MW(\pi) \in \{0, 1, \ldots, 14, 16, 17, 18\}$. Een voorbeeld met rang 15 ontbrak nog in dit rijtje.

In het algemeen heeft een elliptisch oppervlak X een unieke structuur $\pi : X \to C$. Uitzondering hierop vormen de zogeheten elliptische $K3$ oppervlakken. In hoofdstuk 5 worden op een familie van elliptische $K3$ oppervlakken alle mogelijke structuren $\pi : X \to C$ gegeven.

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