Course program of Teoria delle Funzioni 1

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Almost all material presented during this lecture course can be found in [1].

1. Preliminaries and notation[1]**.** Basic facts from the theory of L^p-spaces in open subsets of $\mathbb{R}^N$. Minkowski's inequality for integrals. Convolution in $\mathbb{R}^N$. Young's inequality for convolution. Generalized Young's inequality for convolution (without proof). Mollifiers. Properties of mollifiers: pointwise convergence, uniform convergence, convergence in L^p. Density of $C_c^\infty(\Omega)$ in $L^p(\Omega)$ for $p \in [1, \infty[$. Fundamental Lemma of the Calculus of Variations.

2. Weak derivatives. Motivation: integration by parts in $\mathbb{R}^N$ and weak formulation of a differential problem (the case of the Poisson problem for the Laplace operator). Definition of weak derivatives via integration by parts. Equivalent definitions: definition of weak derivatives via approximation by means of regular functions, definition of weak derivatives in $\mathbb{R}$ via absolutely continuous functions. Weak differentiation under integral sign. Weak derivatives and convolution. Existence of intermediate weak derivatives.

3. Sobolev Spaces. Definition of the Sobolev Space $W^{l,p}(\Omega)$ and its variants $\widetilde{W}^{l,p}(\Omega)$, $w^{l,p}(\Omega)$. Completeness. Basic examples: an example of a function in the Sobolev space $W^{1,p}(\Omega)$ which is unbounded in any neighborhood of any point[2]. Equivalent norms. The Nikodym's domain[3]. The notion of differential dimension of a function space and the differential dimension of the Sobolev space $w^{l,p}(\mathbb{R}^N)$. Lipschitz continuous functions[4]: classic derivatives and weak derivatives, extensions of Lipschitz continuous functions, the Rademacher's Theorem.

4. Approximation theorems. Density of $C_c^\infty(\mathbb{R}^N)$ in $W^{l,p}(\mathbb{R}^N)$ for $p \in [1, \infty[$. The space $W_0^{l,p}(\Omega)$. Partition of unity and density[5] of $C^\infty(\Omega) \cap W^{l,p}(\Omega)$ in $W^{l,p}(\Omega)$ for $p \in [1, \infty[$. Counterexamples to the density of $C^\infty(\bar{\Omega})$ in $W^{l,p}(\Omega)$. An important consequence: characterization of weak derivatives via functions which are absolutely continuous in almost all lines parallel to the coordinate axes.

5. Integral representations. Star-shaped domains with respect to a ball and domains satisfying the cone condition. Taylor's formula in $\mathbb{R}^N$ with remainder in integral form. Sobolev's integral representation formula. Consequences: pointwise estimates for functions and intermediate derivatives.

6. Embedding theorems. The notion of embedding. Continuous embeddings between Sobolev spaces are equivalent to the corresponding inclusions. Sobolev's embedding Theorem for the space $W^{l,p}(\Omega)$ into $C_b(\Omega)$ (the case $lp > N$).

[1]For this chapter, we refer also to [4]
[2]For this example, we refer to [2]
[3]For this example, we refer to [5]
[4]For this part, we refer to [3]
[5]For this and the rest of Chapter 4, we have followed the approach presented in [5]

Sobolev's embedding Theorem for the space $W^{l,p}(\Omega)$ into $L^{q^*}(\Omega)$ where q^* is the Sobolev's exponent (the case $lp < N$): the argument via Young's inequality and the argument via Hardy-Littlewood-Sobolev inequality. The exponent q^* cannot be improved. Sobolev's embedding Theorem for the the space $W^{l,p}(\Omega)$ into $L^q(\Omega)$ for any $q \in [p, \infty[$ (the case $lp = N$). Gagliardo's inequality (the case $p = 1$ without proof). Example of an open set for which the Sobolev's embedding doesn't hold (outer cusps). The Poincaré's inequality. Applications: existence, uniqueness and stability of weak solutions to the Poisson problem for the Laplace operator.

7. Estimates for intermediate derivatives. Classes of open sets: open sets with resolved and quasi-resolved boundaries, open sets with continuous and quasi-continuous boundaries, open sets with Lipschitz boundaries, open sets with C^l-boundaries. Relation between open sets with Lipschitz boundaries and open sets satisfying the cone condition. Estimates for intermediate derivatives (without proof). Sobolev's embedding Theorem in the general case $W^{l,p}(\Omega) \subset W^{m,q}(\Omega)$.

8. Compact embeddings, traces, and applications. The notion of compact operator. The Rellich-Kondrakov Theorem (without proof). Application of the Rellich-Kondrakov Theorem to the Helmholtz equation: existence of the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. The Trace Theorem (without proof) and the Besov-Nikolskii spaces B^l_p. Application of the Trace Theorem to the Dirichlet problem for the Laplace operator: existence of weak solutions.

This lecture course has been integrated by a "Corso integrativo" delivered by Prof. V.I. Burenkov in the form of a cycle of seminars (10 hours). These seminars were devoted to the problem of extension for functions in the Sobolev spaces with the following program (which is not part of the exam[6]): 1. Extensions in the one-dimensional case. Estimates for the minimal norm of an extension operator. 2. Extensions in a multi-dimensional case with preservation of smoothness for domains with sufficiently smooth boundaries. 3. Extensions in a multi-dimensional case with preservation of smoothness for domains with Lipschitz boundaries. 4. Extensions in a multi-dimensional case with minimal deterioration of smoothness for domains with Holder boundaries. 5. Extensions in a multi-dimensional case with preservation of at least some smoothness.

References

[1] V.I. Burenkov, *Sobolev spaces on domains*, B.G. Teubner, Stuttgart, 1998.

[2] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics Vol. 19, American Mathematical Society, Providence, Rhode Island, 1998.

[3] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, London, 1992.

[6]Interested students can find this material in [1, Chapter 7].

[4] G.B. Folland, *Real Analysis. Modern techniques and their applications*, John Wiley & Sons, Inc., New York, 1999.

[5] V.G. Maz'ya and S.V. Poborchii, *Differentiable functions on bad domains*, World Scientific Publishing, 1997.