Course program[1] of Teoria delle Funzioni

Academic Year 2014-2015

P.D. Lamberti

Almost all material presented during this lecture course can be found in [1].

1. Preliminaries and notation[2]. Basic facts from the theory of L^p-spaces in open subsets of $\mathbb{R}^N$. Minkowski's inequality for integrals. Convolution in $\mathbb{R}^N$. Young's inequality for convolution. Generalized Young's inequality for convolution (without proof). Mollifiers. Properties of mollifiers: pointwise convergence, uniform convergence, convergence in L^p. Density of $C^\infty_c(\Omega)$ in $L^p(\Omega)$ for $p \in [1, \infty[$. Fundamental Lemma of the Calculus of Variations.

2. Weak derivatives. Motivation: integration by parts in $\mathbb{R}^N$ and weak formulation of a differential problem (the case of the Poisson problem for the Laplace operator). Definition of weak derivatives via integration by parts. Equivalent definitions: definition of weak derivatives via approximation by means of regular functions, definition of weak derivatives in $\mathbb{R}$ via absolutely continuous functions. Weak differentiation under integral sign. Weak derivatives and convolution. Existence of intermediate weak derivatives. The chain rule.

3. Sobolev Spaces. Definition of the Sobolev Space $W^{l,p}(\Omega)$ and its variants $\widetilde{W}^{l,p}(\Omega)$, $w^{l,p}(\Omega)$. Completeness. Basic examples: an example of a function in the Sobolev space $W^{1,p}(\Omega)$ which is unbounded in any neighborhood of any point[3]. Equivalent norms. The Nikodym's domain[4]. The notion of differential dimension of a function space and the differential dimension of the Sobolev space $w^{l,p}(\mathbb{R}^N)$. Lipschitz continuous functions[5]: classical derivatives and weak derivatives, extensions of Lipschitz continuous functions, the Rademacher's Theorem.

4. Approximation theorems. Density of $C^\infty_c(\mathbb{R}^N)$ in $W^{l,p}(\mathbb{R}^N)$ for $p \in [1, \infty[$. The space $W^{l,p}_0(\Omega)$. Partitions of unity and the Meyers-Serrin Theorem[6] $H = W$ concerning the density of $C^\infty(\Omega) \cap W^{l,p}(\Omega)$ in $W^{l,p}(\Omega)$ for $p \in [1, \infty[$. Counterexamples to the density of $C^\infty(\bar{\Omega})$ in $W^{l,p}(\Omega)$. An important consequence: characterization of weak derivatives via functions which are absolutely continuous in almost all lines parallel to the coordinate axes. Removable singularities: $W^{l,p}(\Omega) = W^{l,p}(\Omega \setminus F)$ if F is a closed subset of Ω with zero $(N-1)$-dimensional Hausdorff measure[7].

5. Integral representations. Star-shaped domains with respect to a ball and domains satisfying the cone condition. Taylor's formula in $\mathbb{R}^N$ with remain-

[1] This lecture course replaces the course Teoria delle Funzioni 1 delivered in the previous academic years. The new program is an extension of the old program: the new parts are mainly contained in Chapters 9, 10 and in the Appendix.

[2] For this chapter, we refer also to [4]

[3] For this example, we refer to [2]

[4] For this example, we refer to [5]

[5] For this part, we refer to [3]

[6] For this and the rest of Chapter 4, we have followed the approach presented in [5]

[7] For this theorem, we refer to [5]

der in integral form. Sobolev's integral representation formula. Consequences: pointwise estimates for functions and intermediate derivatives.

6. Embedding theorems. The notion of embedding. Continuous embeddings between Sobolev spaces are equivalent to the corresponding inclusions. Sobolev's embedding Theorem for the space $W^{l,p}(\Omega)$ into $C_b(\Omega)$ (the case $lp > N$). Sobolev's embedding Theorem for the space $W^{l,p}(\Omega)$ into $L^{q^*}(\Omega)$ where q^* is the Sobolev's exponent (the case $lp < N$): the argument via Young's inequality and the argument via Hardy-Littlewood-Sobolev inequality. The exponent q^* cannot be improved. Sobolev's embedding Theorem for the the space $W^{l,p}(\Omega)$ into $L^q(\Omega)$ for any $q \in [p, \infty[$ (the case $lp = N$). Gagliardo's inequality (the case $p = 1$ without proof). Example of an open set for which the Sobolev's embedding doesn't hold (outer cusps). The Poincaré's inequality. Applications: existence, uniqueness and stability of weak solutions to the Poisson problem for the Laplace operator.

7. Estimates for intermediate derivatives. Classes of open sets: open sets with resolved and quasi-resolved boundaries, open sets with continuous and quasi-continuous boundaries, open sets with Lipschitz boundaries, open sets with C^l-boundaries. Relation between open sets with Lipschitz boundaries and open sets satisfying the cone condition. Estimates for intermediate derivatives (without proof). Sobolev's embedding Theorem in the general case $W^{l,p}(\Omega) \subset W^{m,q}(\Omega)$.

8. Compact embeddings. The notion of compact operator (including a general discussion on the fundamental Enflo-Grothendieck approximation problem). The Kolmogorov criterion for compactness in $L^p(\mathbb{R}^N)$ (without proof). The Rellich-Kondrakov Theorem for the embedding of $W^{l,p}(\Omega)$ into $W^{m,q}(\Omega)$ with $q < q^*$ (proof in the case of the embedding of $W_0^{l,p}(\Omega)$ into $L^p(\Omega)$ when Ω has finite measure). Non-compactness of the embedding of $W^{l,p}(\Omega)$ into $W^{m,q^*}(\Omega)$. Application of the Rellich-Kondrakov Theorem to the Helmholtz equation: existence of the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. Glimpses of nonlinear theory: the p-Laplacian and its first eigenvalue.

9. Trace theorems and Besov-Nikolskii spaces B_p^l with $l \in]0, \infty[$. Motivation: the Dirichlet problems for the Laplace operator. Definition of trace on a linear subspace of $\mathbb{R}^N$ of dimension $m < N$ for functions in $L^1_{loc}(\mathbb{R}^N)$. Existence of traces for functions in the Sobolev space $W^{l,p}(\mathbb{R}^N)$. Example of a function without trace. Besov-Nikolskii spaces $B_p^l(\mathbb{R}^N)$: definition via differences $\Delta_h^\sigma f$ of order σ and step h of a function f. Hölder-Zygmund spaces $B_\infty^l(\mathbb{R}^N)$, in particular the Zygmund space $B_\infty^1(\mathbb{R}^N)$ and the strict inclusion $\mathrm{Lip}(\mathbb{R}^N) \subsetneq B_\infty^1(\mathbb{R}^N)$. Estimates and representation formulas for $\Delta_h^\sigma f$ (proof not required at the oral exam). One-dimensional Hardy's inequality (without proof). The Trace Theorem for linear subspaces of $\mathbb{R}^N$: $\mathrm{Tr}_{\mathbb{R}^m} W^{l,p}(\mathbb{R}^N) = B_p^{l-(N-m)/p}(\mathbb{R}^m)$ (only the proof of the continous inclusion). Definition of traces on $\partial\Omega$ for functions in $W^{l,p}(\Omega)$ and definition of the Besov-Nikolskii space $B_p^l(\partial\Omega)$. The Trace Theorem for smooth boundaries: $\mathrm{Tr}_\Omega W^{l,p}(\Omega) = B_p^{l-1/p}(\partial\Omega)$ (without proof). Total trace: definition and corresponding Total Trace Theorem (without proof). Application of the Trace Theorem to the existence of weak solutions to the Dirichlet problem for the Laplace operator.

10. Extension theorems. Existence of a bounded linear extension operator from $W^{l,p}(a,b)$ to $W^{l,p}(\mathbb{R})$. Existence of a bounded linear extension operator from $W^{l,p}(\Omega)$ to $W^{l,p}(\mathbb{R}^N)$ for Lipschitz open sets Ω (proof only in the case of an elementary open set of class C^l).

Appendix. Sobolev spaces via Fourier transform. Definition of Fourier Transform for functions in $L^1(\mathbb{R}^N)$. The Plancherel Theoreom (without proof). Equivalent definition of the Sobolev spaces $W^{l,2}(\mathbb{R}^N)$ and of the Besov-Nikolskii spaces $B_2^l(\mathbb{R}^N)$ via Fourier transform.

References

[1] V.I. Burenkov, *Sobolev spaces on domains*, B.G. Teubner, Stuttgart, 1998.

[2] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics Vol. 19, American Mathematical Society, Providence, Rhode Island, 1998.

[3] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, London, 1992.

[4] G.B. Folland, *Real Analysis. Modern techniques and their applications*, John Wiley & Sons, Inc., New York, 1999.

[5] V.G. Maz'ya and S.V. Poborchii, *Differentiable functions on bad domains*, World Scientific Publishing, 1997.