Course program of Spectral Theory

Academic Year 2015-2016
Scuola Galileiana di Studi Superiori
Pier Domenico Lamberti
30+2 hours[1], 4 ECTS

Motivations: Helmholtz and Poisson equations: Derivation of the Helmholtz equation from the membrane equation. Eigenvalues of the Dirichlet Laplacian on the rectangle. The Poisson equation, classical and weak formulations.

Weak derivatives and Sobolev Spaces: Integration by parts in $\mathbb{R}^N$ and weak derivatives. Sobolev spaces $W^{l,p}(\Omega)$ on domains Ω, main properties and approximation theorems. Characterisation of the Sobolev spaces $W^{l,2}(\mathbb{R}^N)$ via Fourier Transform, H^l spaces.

Unbounded operators in Hilbert space: Densely defined, closed, symmetric and self-adjoint operators, main properties and definitions. Examples, the Laplace operator in $L^2(\mathbb{R}^N)$.

Quadratic forms and Friedrichs Extensions: Lax-Milgram theorem in the context of closed (Hermitian) quadratic forms and the corresponding (self-adjoint) operators. Semi-bounded operators, Friedrichs Extension Theorem.

Important realisations: Rigorous definitions of most relevant self-adjoint operators via Friedrichs extensions: the Dirichlet and the Neumann Laplacians on smooth and non-smooth domains (hints of conormal derivatives for the Neumann case), Schrödinger operators, the harmonic oscillator, Hardy's inequality in $\mathbb{R}^N$, Schrödinger operator with Coulomb potential.

Spectral theory for bounded operators: Spectrum and resolvent of a bounded operator in Hilbert space, spectrum of a compact operator, spectrum of a bounded self-adjoint operator, Hilbert-Schmidt theorem.

Compact and non-compact resolvents: Compact embeddings and compact resolvents, Kolmogorov and Rellich-Kondrakov theorems (without proof), canonical form of the Dirichlet Laplacian on domains with finite measure, Schrödinger operators with or without compact resolvents.

Spectral resolutions of the identity and spectral integration: Orthogonal projectors in Hilbert space, spectral resolutions of the identity and associated spectral measures and integrals. Unbounded self-adjoint operators defined via spectral integrals. Spectral resolution of the Laplace operator in $\mathbb{R}^N$ via Fourier transform.

[1] Two extra lectures were devoted to discussing the last notions in the program, namely: functional calculus for Borel functions, norm of the resolvent and distance to the spectrum, discrete spectrum, essential spectrum, point spectrum, continuous spectrum. Spectrum of the Laplace operator in $\mathbb{R}^N$.

Spectral theorem and functional calculus: The Stone-von Neumann Spectral Theorem for the representation of a self-adjoint operator as a spectral integral of the identity, functional calculus for Borel functions, norm of the resolvent and distance to the spectrum, discrete spectrum, essential spectrum, point spectrum, continuous spectrum. Spectrum of the Laplace operator in $\mathbb{R}^N$.

References

[1] E.B. Davies, *Spectral theory and differential operators* . Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[2] B. Helffer, *Spectral Theory and its Applications*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2013.