Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam ('Quarto appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 12 July 2012.

SURNAME.................... NAME.................... MATRICOLA...................

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 2AB45 on 13 July 2012 at 14:00.

Duration: 150 minutes

Question 1.

(i) State the Fundamental Lemma of the Calculus of Variations.
(ii) Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. Let $u : \bar{\Omega} \to \mathbb{R}$ be a function of class $C^2(\bar{\Omega})$. Let $f \in L^2(\Omega)$. Assume that for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u \Delta \varphi dx = \int_\Omega f \varphi dx.$$

Prove that f is equivalent to a continuous function (Hint: integrate by parts and use (i)).

Answer:

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Question 2.

(i) Give at least two equivalent definitions of weak derivative.

(ii) Let $f_\alpha : (-1/2, 1/2) \to \mathbb{R}$ be the function defined by

$$f(x) = |\log |x||^\alpha ,$$

for all $x \in (-1/2, 1/2) \setminus \{0\}$ and $f(0) = 0$, where $\alpha < 0$. Find all values of $\alpha < 0$ such that the weak derivative $(f_\alpha)'_w$ of f_α exists in $(-1/2, 1/2)$.

(iii) For every value of α determined in (ii), find all values of $p \in [1, \infty]$ such that f_α belongs to $W^{1,p}(-1/2, 1/2)$.

Answer:

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Question 3.

(i) State the Sobolev's embedding theorem for the space $W^{l,p}(\Omega)$ into $L^{q^*}(\Omega)$, specifying all details.

(ii) Let $0 < \gamma < 1$ and Ω be the open set in $\mathbb{R}^N$, $N \geq 2$, defined by

$$\Omega = \{(\bar{x}, x_N) \in \mathbb{R}^N : \ |\bar{x}| < 1, \ \ |\bar{x}|^\gamma < x_N < 1\},$$

where $\bar{x} = (x_1, \dots, x_{N-1})$. Let $f : \Omega \to \mathbb{R}$ be the function defined by

$$f(\bar{x}, x_N) = x_N^\delta \,,$$

for all $(\bar{x}, x_N) \in \Omega$, where $\delta \in \mathbb{R} \setminus \mathbb{N}_0$. Let $l \in \mathbb{N}_0$ and $1 \leq p < \infty$. Find all values of $\delta \in \mathbb{R} \setminus \mathbb{N}_0$ such that $f \in W^{l,p}(\Omega)$.

(iii) Deduce by (ii) that in general the Sobolev's embedding does not hold for an arbitray bounded open set in $\mathbb{R}^N$.

Answer:

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Question 4.

(i) State the theorem concerning the estimates for intermediate derivatives of functions in the Sobolev space $W^{l,p}(\Omega)$.

(ii) Let Ω be an open set in $\mathbb{R}^N$ of class C^1. Let $f \in W_0^{2,p}(\Omega)$ with $1 \leq p < \infty$. Prove that the traces in $\partial\Omega$ of the first order derivatives of f are well-defined and satisfy $\mathrm{Tr}\frac{\partial f}{\partial x_i} \equiv 0$ for all $i = 1, \ldots, N$.