Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam ('secondo appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 20 February 2013.

SURNAME.................... NAME.................... MATRICOLA...................

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 1AD100 on 25 February 2013 at 14:30.

Duration: 150 minutes

Question 1.

Let $\alpha > 0$ and $f : (-1/2, 1/2) \to \mathbb{R}$ be the function defined almost everywhere by the formula

$$f(x) = \frac{1}{|\log |x||^{\alpha}}$$

(i) Find all values of $\alpha > 0$ such that f has the weak derivative f'_w in $(-1/2, 1/2)$ (give a detailed motivation).

(ii) Find all values of $\alpha > 0$ and $p \in [1, \infty]$ such that $f \in W^{1,p}(-1/2, 1/2)$.

(iii) State the Minkowski's inequality for integrals.

Answer:

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Question 2.

(i) Give the definitions of the Sobolev spaces $W^{l,p}(\Omega)$, $W_0^{l,p}(\Omega)$, $\widetilde{W}^{l,p}(\Omega)$ and $\omega^{l,p}(\Omega)$.
(ii) Prove that $\widetilde{W}^{l,p}(\Omega)$ is complete.
(iii) Prove that if $f \in W^{l,p}(\Omega)$ and has compact support then $f \in W_0^{l,p}(\Omega)$.

Answer:

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Question 3.

(i) State the Sobolev's Embedding Theorem in the general case $W^{l,p}(\Omega) \subset W^{m,q}(\Omega)$.

(ii) Let Ω be a bounded domain in $\mathbb{R}^N$, $p \in [1, \infty[$ and $l \in \mathbb{N}$ with $pl < N$. Prove that for any $\epsilon > 0$ there exists $f_\epsilon \in W^{l,p}(\Omega) \setminus L^{p^*+\epsilon}(\Omega)$. (Hint: you may assume directly that $0 \in \Omega$ but explain why.)

Answer:

Question 4.

(i) State the Trace Theorem for Sobolev spaces.

(ii) Give the definition of the Besov-Nikolskii spaces B_p^l and explain their role in the description of traces of functions in the Sobolev spaces.

(iii) Give the weak formulations of the following classical problems in a bounded open set Ω in $\mathbb{R}^N$

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

specifying the appropriate assumptions on g.