Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam ('secondo appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 23 February 2012.

SURNAME.................... NAME.................... MATRICOLA...................

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 2AB40 on 27 February 2012 at 14:30.

Duration: 150 minutes

Question 1.

(i) Let Ω be an open set in $\mathbb{R}$ and $f \in L^1_{loc}(\Omega)$.

(a) Give the definition of weak derivative $f_w^{(l)}$ of order $l \in \mathbb{N}$ in terms of absolute continuous functions.

(b) Give the definition of weak derivative $f_w^{(l)}$ of order $l \in \mathbb{N}$ via integration by parts.

(ii) Let $g : (-1, 1) \to \mathbb{R}$ be the function defined by $g(x) = x + 1$ if $-1 < x < 0$ and $g(x) = \alpha + \beta x + \gamma x^2$ if $0 \leq x < 1$, where $\alpha, \beta, \gamma \in \mathbb{R}$.

(c) Use the definition in (a) in order to find all values of $\alpha, \beta, \gamma \in \mathbb{R}$ such that $g_w^{(2)}$ exists in $(-1, 1)$.

(d) Fix α, β, γ as in (c). Use the definition in (b) to prove that if $\gamma \neq 0$ then $g_w^{(3)}$ does not exist in $(-1, 1)$.

Answer:

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Question 2.

Let Ω be an open set in $\mathbb{R}^N$ and $l \in \mathbb{N}$, $p \in [1, \infty[$.
(i) Give the definition of the space $W_0^{l,p}(\Omega)$.
(ii) Let $f \in W_0^{l,p}(\Omega)$ and $g : \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \in \Omega\,, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

Prove in detail that $g \in W^{l,p}(\mathbb{R}^N)$.
(iii) Is it true that $g \in W_0^{l,p}(\mathbb{R}^N)$?

Answer:

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Question 3.

(i) State Taylor's formula with remainder in integral form for a function defined on an open set in $\mathbb{R}^N$.

(ii) Give the definition of open set star-shaped with respect to a ball and open set satisfying the cone condition.

(iii) Let $\gamma > 0$ and Ω be the open subset of $\mathbb{R}^2$ defined by

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \ -1 < x < 1, \ -2 < y < |x|^\gamma\}.$$

Find all values of $\gamma > 0$ such that Ω is star-shaped with respect to a ball (give a clear motivation, at least drawing some pictures that clarify the main ideas).

Answer:

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Question 4.

(i) State the Sobolev's embedding Theorem for the space $W^{l,p}(\Omega)$ into $W^{m,q}(\Omega)$ (parts 2 and 3) specifying all appropriate values of l, p, m, q.

(ii) State the Trace Theorem for the Sobolev Space $W^{l,p}(\Omega)$ in terms of the appropriate Besov spaces.

(iii) Assume that Ω is an open set for which the Trace Theorem holds. Prove that if $f \in W_0^{l,p}(\Omega)$ then $\mathrm{Tr} f = 0$.