Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam (‘quarto appello’) of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 27 August 2013.

SURNAME.................... NAME.................... MATRICOLA...................

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 2AB45 on 28 August 2013 at 09.00.

Duration: 150 minutes

Question 1.

Let $\alpha > 0$, $\beta \geq 0$ and $f : (0, 2) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^{\alpha} \log x, & \text{if } \ 0 < x \leq 1, \\ (x-1)^{\beta}, & \text{if } \ 1 < x < 2 \end{cases}$$

(i) Find all values of $\alpha > 0$, $\beta \geq 0$ such that f has the weak derivative f'_w in $(0, 2)$ (give a detailed motivation).

(ii) Find all values of $\alpha > 0$, $\beta \geq 0$ and $p \in [1, \infty[$ such that $f \in W^{1,p}(0, 2)$.

Answer:

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Question 2.

(i) Let Ω be an open set in $\mathbb{R}^N$. Let $f, g, h \in L^1_{loc}(\Omega)$ and $\alpha, \beta \in \mathbb{N}^N$ multi-indeces. Assume that $D^\alpha_w f$, $D^{\alpha+\beta}_w f$ exist and that $g = D^\alpha_w f$ and $h = D^{\alpha+\beta}_w f$. Is it true that $D^\beta_w g$ exists and that $h = D^\beta_w g$ (almost everywhere)? If yes, prove it. If not, give a counterexample.

(ii) Let $F \in L^1_{loc}(\mathbb{R}^2)$. Give an example showing that the existence of the weak derivative $\left(\frac{\partial^2 F}{\partial x_1 \partial x_2}\right)_w$ does not imply the existence of the weak derivatives $\left(\frac{\partial F}{\partial x_1}\right)_w$, $\left(\frac{\partial F}{\partial x_2}\right)_w$.

(iii) With reference to question (ii), state the well-known sufficient condition ensuring the existence of so-called intermediate weak derivatives.

Answer:

Question 3.

(i) State the Taylor Formula in $\mathbb{R}^N$ with remainder in integral form.

(ii) State the Sobolev Embedding Theorem part 2.

(iii) State the Hardy-Littlewood-Sobolev inequality and, with reference to question (ii), explain its importance in a few words.

Answer:

Question 4.

(i) State the Poincaré inequality for functions in the Sobolev space $W_0^{1,p}(\Omega)$.

(ii) Is it true that the Poincaré inequality holds for all functions in the Sobolev space $W^{1,p}(\Omega)$? If yes, prove it. If not, give a counterexample.

(iii) Give the definition of Besov-Nikolskii spaces and state the Trace Theorem.

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