| Question 1 | | | Question 2 | | | | Sum | Final score |
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First compitino of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica 16/11/11.

SURNAME.................... NAME.................... MATRICOLA...................

**PLEASE NOTE.** During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered.

Duration: 70 minutes

**Question 1.**

(a) State the theorem concerning the Minkowski's inequality for integrals.

(b) Prove that convolution is associative (this means that if $f, g, h \in L^1(\mathbb{R}^N)$ then $(f*g)*h = f*(g*h)$).

(c) Let $f \in \mathrm{Lip}(\mathbb{R}^N)$, $w$ a non-negative kernel of mollification and $A_\delta f$ the associated mollifier with step $\delta > 0$. Prove that $A_\delta f \in \mathrm{Lip}(\mathbb{R}^N)$ and

$$\mathrm{Lip} A_\delta f \leq \mathrm{Lip} f.$$

**Answer:**

**Question 2.**

(i) Let $f : (-1, 1) \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$ if $0 \le x < 1$ and $f(x) = x$ if $-1 < x < 0$. Let $g : (-1, 1) \setminus \{0\} \to \mathbb{R}$ be the restriction of $f$ to $(-1, 1) \setminus \{0\}$.

(a) Prove that $f$ does not admit a weak derivative $f'_w$ in $(-1, 1)$, without using the characterization of weak derivatives via absolute continuous functions.

(b) Does the function $g$ belong to the Sobolev space $W^{1,p}((-1, 1) \setminus \{0\})$ for some $p \in [1, \infty]$? If yes, for which values of $p \in [1, \infty]$?

(ii) Let $\Omega$ be an open set in $\mathbb{R}^N$ and $u \in L^1_{loc}(\Omega)$. Let $\alpha, \beta \in \mathbb{N}_0^N$. Assume that there exist $D^\alpha_w u$ and $D^\beta_w(D^\alpha_w u)$. Prove that $D^{\alpha+\beta}_w u$ exists and $D^{\alpha+\beta}_w u = D^\beta_w(D^\alpha_w u)$.

(iii) Which is the main property enjoyed by the Nikodym's domain?

**Answer:**

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