Question 1	Question 2	Question 3	Question 4	Sum	Final score

'Second compitino' and written exam ('primo appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 2 February 2012.

SURNAME.................... NAME.................... MATRICOLA....................

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 2AB40 on 6 February 2012 at 14:30.

Duration: 150 minutes

Please tick one of the two options below:

□ Second 'compitino': questions 3 and 4. Time 80 Minutes.

□ Written exam: all questions. Time 150 Minutes.

Question 1.

(i) Let $f : (-\pi/2, \pi/2) \to \mathbb{R}$ be the function defined by $f(x) = \cos x$ if $0 \le x < \pi/2$ and $f(x) = \sin x$ if $-\pi/2 < x < 0$. Let $g : (-\pi/2, \pi/2) \setminus \{0\} \to \mathbb{R}$ be the restriction of f to $(-\pi/2, \pi/2) \setminus \{0\}$.

(a) Prove that f does not admit a weak derivative f'_w in $(-\pi/2, \pi/2)$, without using the characterization of weak derivatives via absolute continuous functions.

(b) Does the function g belong to the Sobolev space $W^{1,p}((-\pi/2, \pi/2) \setminus \{0\})$ for some $p \in [1, \infty]$? If yes, for which values of $p \in [1, \infty]$?

(ii) Let Ω be an open set in $\mathbb{R}^N$ and $u \in L^1_{loc}(\Omega)$. Let $\alpha, \beta \in \mathbb{N}_0^N$, $\beta < \alpha$. Assume that there exist $D_w^\alpha u$ and $D_w^\beta u$. Prove that $D_w^{\alpha-\beta}(D_w^\beta u)$ exists and $D_w^{\alpha-\beta}(D_w^\beta u) = D_w^\alpha u$.

(iii) Is it true that the spaces $w^{l,p}(\Omega)$, $W^{l,p}(\Omega)$ coincide if Ω is a bounded open set in $\mathbb{R}^N$? If it is true, prove it. If it is false, give a counterexample.

Answer:

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Question 2.

(i) Give an example of an open set Ω in $\mathbb{R}^N$ such that $C_c^\infty(\Omega)$ is dense in $W^{l,p}(\Omega)$ for $p \in [1, \infty[$. Is it possible to find an example like that also for $p = \infty$?

(ii) Give the definition of the space $W_0^{l,p}(\Omega)$.

(iii) Give an example of a bounded domain Ω in $\mathbb{R}^2$ such that $C^\infty(\bar{\Omega})$ is not dense in $W^{1,p}(\Omega)$ for $p \in [1, \infty[$ and provide a detailed motivation.

Answer:

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Question 3.

(i) State the Sobolev's integral representation formula (all details are required).

(ii) Let Ω be an arbitrary open set in $\mathbb{R}^N$ star-shaped with respect to a ball $B \subset \Omega$ and $p \in [1, \infty[$. Is it true that there exists a constant $C > 0$ such that

$$\|f\|_{L^p(\Omega)} \leq C(\|f\|_{L^p(B)} + \|\nabla f\|_{L^p(\Omega)})$$

for all $f \in W^{1,p}(\Omega)$?

(iii) Give the precise definition of cone (specifying the parameters 'height', 'radius' and 'aperture') and the definition of open set satisfying the cone condition.

Answer:

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Question 4.

(i) State the Sobolev's embedding Theorem part 1, i.e., embedding of the space $W^{l,p}(\Omega)$ into the space of continuous functions (all details are required).

(ii) Let $k \in \mathbb{N}$. Assume that the space $w^{3k,p}(\mathbb{R}^N)$ is continuously embedded into the space $w^{k,q}(\mathbb{R}^N)$ for some $q \in]p, \infty[$. Prove that if $N > 2kp$ then

$$q \leq \frac{pN}{N - 2kp}$$

(Hint: use the notion of differential dimension.)

(iii) Give the definition of the Besov-Nikolskii space $B^l_p(\mathbb{R}^N)$ for $l > 0, 1 \leq p \leq \infty$.