THE FARGUES–FONTAINE CURVE - STUDY GROUP 2024/2025

Abstract. The goal of this study group is to construct and study the basic geometric properties of the Fargues–Fontaine curve. Time permitting, we will also have talks about applications.

1. Overview

The aim of this study group is to define and study the basic properties of the Fargues–Fontaine curve, a geometric object central to many recent developments in arithmetic geometry. The program is structured to provide both foundational knowledge and insights into the more advanced aspects of the theory.

We will start with an introduction to (or refresher about) perfectoid fields. Then, we will define the schematic Fargues–Fontaine curve X, study the structure of its divisors, its Picard group and show that X is, in fact, a Dedekind scheme. This will require some technical ideas, such as, for example, the study of Newton polygons for $\mathbf{A}_{\text{inf}}$, which we will cover in the first few talks.

Later talks will address the Harder–Narasimhan formalism and the classification of vector bundles. This includes the relation of semistable vector bundles with the Dieudonné–Manin classification of isocrystals. We will also discuss the étale fundamental group of the Fargues–Fontaine curve, showing in particular that X is geometrically simply connected.

Key topics include:

- Introduction to perfectoid fields, tilts, untilts and relation to period rings.
- Schematic (and possibly adic) descriptions of the Fargues–Fontaine curve.
- Newton polygons, divisors, and factorization results on the curve.
- Harder–Narasimhan filtration for vector bundles and its applications.
- Connections to p-adic Hodge theory and vector bundles via isocrystals.

2. Organisation

- The talks will be 1.5 hours long (with a short break in the middle).
- The speakers are invited to provide at least a summary of their talk with precise references, preferably a full set of notes, written up in LaTeX or scanned.
- The last few talks are to be determined; we welcome suggestions (especially from people willing to give those talks).

Organisers:

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3. Program (DRAFT)

For the sake of speakers and organizers, we will mainly give references to the sets of lecture notes [Ans, Lur], but we would like to stress that the ultimate source for much of the material discussed in this study group is [FF], to which [Ans] gives itself more precise references. For an historical account of the development of the ideas discussed in this study group, we refer the reader to [Col18].

We invite speakers and attendees to communicate with organizers and to discuss with them the details of what should be covered in which talk, as well as to point out potential issues with the structure of the talks. There is only so much foresight we can put into devising a program in advance.

(1) **Overview**. This talk will give an overview of the study group. Some time at the end will be devoted to assigning talks to speakers (voluntarily or otherwise).

(2) **Background on perfectoid fields**. Recall the definition of perfectoid field, tilting, untilts, Witt vectors, $\mathbf{A}_{\mathrm{inf}}$, Fontaine's θ map, distinguished elements and their relation with untilts.

Cover [Lur, lectures 2-3]. From lecture 2 you may safely skip the proofs of Propositions 6 and 13, as well as Exercises 9 and 14; sketch the proof of Theorem 17 only if time allows, but mention Lemma 18. From lecture 3 you may skip page 1, except for the definition of untilt; at the end sketch as much as you can of the proof of Proposition 16 and its corollaries. When recalling the tilting functor for perfectoid fields, mention in addition (without proof) the fundamental theorem [Sch, Theorem 3.7]. When recalling the ring of Witt vectors, mention in addition the existence of an endomorphism lifting the Frobenius map and acting as p-power on Teichmüller lifts; also mention the existence of a ring of *ramified* Witt vectors (given a finite extension $E|\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, you may take [Ans, Lemma 3.8] as a definition and add details at your discretion from loc. cit. Proposition 3.7 until Remark 3.10). These notions apply in particular to $\mathbf{A}_{\mathrm{inf}}$, see the beginning of [Ans, lecture 4]. You may consult [Ans, lectures 3-5] for more material and additional references.

(3) **Definition of the curve** X. We would like the speaker to follow the exposition of [Lur, Lecture 4] while extending to the case of a generic base field E finite extension of $\mathbb{Q}_p$, which is the setting of [Ans]. Introduce the ring $B^b = \mathbf{A}_{\mathrm{inf}}[1/[\varpi], 1/\pi] = W_{\mathcal{O}_E}(\mathcal{O}_C^\flat)[1/[\varpi], 1/\pi]$, see [Ans, Def. 4.1] and the line before [Ans, Def. 8.1]. Reproduce the discussion on [Lur, p. 1-3], in particular: discuss how the absolute value on K untilt of $C^\flat$ is normalised, how that gives an invariant $0 \le |\pi|_K < 1$, explain the meaning of inverting $[\varpi]$ and π in $\mathbf{A}_{\mathrm{inf}}$ in analogy with complex power series, write down the table on [Lur, p. 3], briefly discuss the example of the p-adic logarithm. Present the definition of $B_{[a,b]}$ as a p-adic completion of $\mathbf{A}_{\mathrm{inf}}[\frac{[\varpi_a]}{\pi}, \frac{\pi}{[\varpi_b]}]$ and discuss how this is endowed with natural morphisms to certain untilts K (say which ones and what that means in the analogy with holomorphic functions on the unit disk), see [Lur, p. 4]. Also mention the description of $B_{[a,b]}$ as a completion with respect to certain Gauß norms, see [Lur, Def. 10, Cor. 19 of Lecture 5]. Describe the natural restriction maps $B_I \to B_{I'}$ for $I' \subseteq I \subseteq (0,1)$. Define the Fréchet algebra $B = \lim_I B_I$, stressing the analogy with holomorphic functions on the punctured unit disk which are not necessarily meromorphic. Explain how the Witt lift of Frobenius to B^b extends by continuity to an endomorphism φ of B, see [Ans, Def. 8.3].

Define the Frobenius eigenspaces $B^{\varphi=\pi^d}$ and the schematic curve as in [Ans, Def 8.4]. The details on the structure of these Frobenius eigenspaces will be covered in subsequent lectures.

(4) B_{dR} **and divisors**. Construct the period ring B_{dR}^+, prove its basic properties and show the existence of a map $B_I \to B_{\mathrm{dR}}^+$. Introduce the divisors of elements of B interpreted as functions on the set Y of characteristic 0 untilts, state their main properties without proof and discuss the crucial example based on logarithms. You are free to do everything in the setup of a finite extension E of $\mathbb{Q}_p$ or just mention that this is possible (mutatis mutandis).

This talk is based on [Lur, lectures 8-9]. Fix $C^\flat$, $\mathcal{O}_C^\flat$ and π as in the beginning of [Lur, lecture 8]; you may assume right away that $C^\flat$ is algebraically closed. Also introduce the sets Y and Y_I as in the beginning of [Lur, lecture 9], recalling the relation with distinguished elements of $\mathbf{A}_{\mathrm{inf}}$ (see Corollary 18 from [Lur, lecture 3] or the end of Talk 2). Skip the first part of [Lur, lecture 8] for now and discuss B_{dR}^+ from Construction 8 until the end. You will need some inputs from [Lur, lecture 3], Exercise 19 and beginning of the proof of Proposition 16: you may leave this as an exercise or explain as much as you like. Passing to [Lur, lecture 9], only cover until Corollary 4, plus the first assertion of Corollary 8. You may state Theorem 1 and Corollary 4 together; postpone their proof to the next talks. Regarding Example 3 and the first assertion of Corollary 8, make sure to state the results, then prove as much as you can in the remaining time, picking the necessary material from the first part of [Lur, lecture 8] (until Remark 7 included) and possibly from [Lur, lecture 7] (from Exercise 1 until Remark 3 included); in particular, it would be nice to phrase Corollary 4 of [Lur, lecture 8] and to get an idea of why we need $C^\flat$ algebraically closed (we already know that this implies that the untilts are algebraically closed by Talk 2). For more material, other points of view and additional references, you may consult [Ans, lecture 4] (from Definition 4.4 until before Definition 4.7), [Ans, lecture 9] and the beginning of [Ans, lecture 10].

(5) **Newton polygons and factorisations**. Discuss the elementary properties of Newton polygons of elements of $\mathbf{A}_{\mathrm{inf}}$ and B. Show that B_I is a PID; factorisation of functions on the curve in terms of $B^{\varphi=p}$ (that is, the underlying graded ring of X is *graded factorial*). Here we would like to follow primarily [Ans, lectures 6 and 9], but the speaker is invited to also look at [Lur, lecture 10]. The speaker should take their time in setting everything up, the discussion of some of the material in this lecture (including proofs) may carry over to the next one.

Recall the classical definition of Newton polygon, following [Ans, Def. 6.1], as well as [Ans, Prop. 6.2]. Present the definition of Legendre transform, following [Ans, Def. 6.3-4]. State [Ans, Lemma 6.12] and discuss the case of Newton polygons of polynomials, defining the valuations ν_r, giving their “geometric interpretation” and stating [Ans, Lemma 6.13]. Discuss how the definition of Newton polygon is extended from polynomials to power series, following [Ans, Def. 6.18] (should also mention Weierstrass preparation?). State [Ans, Thm. 6.19]. Redefine ν_r and Newton polygons for $\mathbf{A}_{\mathrm{inf}}$ and $B^b = \mathbf{A}_{\mathrm{inf}}[1/p, 1/[\varpi]]$. State [Ans, Theorem 7.1]. Re-define the B_I in terms of ν_r, see [Ans, Def. 8.1]. State and sketch the proof of [Ans, Thm. 9.3].

(6) **Structure of the curve**. In this lecture we aim to complete the proof of [Ans, Thm. 9.3], stated in the previous lecture, and begin the study of divisors, which will carry on in the next lecture. We will mainly follow [Ans], but the speaker might also want to have a look at [Lur, Lect. 7, Ex. 1 to Rmk. 3] and [Lur, Lect. 9].

Begin by following [Ans, Lect. 7]. Recall the statement of [Ans, Thm. 7.1]. Define the metric on $|Y|$ following [Ans, Def. 7.2]. State without proof [Ans, Lemma 7.4]. State without proof [Ans, Prop. 7.5]. Briefly sketch the proof of [Ans, Thm. 7.1], possibly looking at [FF, Sec. 2.4, Thm. 2.4.5]. Next, we move to the material left over from [Ans, Lect. 8]. Recall the definition of Newton polygons for B^b, extending the definition for $\mathbf{A}_{\mathrm{inf}}$ seen in the previous lecture. Further extend the definition of Newton polygon to the B_I's, see [Ans, Def. 8.9-11]. Now, let us move on to material from [Ans, Lect. 9]. State and prove [Ans, Thm. 9.7] (we have already proved [Ans, Lemma 9.5] in a previous lecture). State without proof [Ans, Lemma 9.6]. Complete the proof of [Ans, Thm. 9.3], see [Ans, p. 48]. Cover as much as you can of the material in [Ans, 9.9 to 9.13] + definition of Frobenius for B, see end of [Ans, p. 42].

(7) **Line bundles on the curve**. In this talk we aim to finally prove that $X = \mathrm{Proj}(P)$ is a Dedekind scheme and to study the structure of $\mathrm{Pic}(X)$. The main source will be [Ans], but the speaker might also look at [Lur, Lect. 10, 19].

Recall the definition of the graded algebra P [Ans, Def. 9.1]. State and prove [Ans, Thm. 9.14] and show how it implies [Ans, Thm. 9.2], stating the latter. State [Ans, Lemma 9.15], leave the proof as an exercise. State and prove [Ans, Thm. 10.1, Lemma 10.2]. State and prove [Ans, Cor. 10.3]. Cover the material from [Ans, 10.4] through [Ans, 10.10], possibly skipping the proof of [Ans, 10.10], if you are short on time.

(8) **Harder–Narasimhan filtration**. The aim of this talk is to introduce the the Harder–Narasimhan (HN) formalism and show that it applies to vector bundles on X. This talk should follow [Lur, lecture 20] and [Ans, lecture 11], with some inspiration from [FF, 5.5].

Cover the following material left over from the the previous talk: give [Ans, Def. 10.8], state and prove [Ans, Prop. 10.9] and state without proof [Ans, Prop. 10.10].

Discuss the notions of degree, rank and slope in the context of an exact category $\mathcal{C}$ over an abelian category $\mathcal{A}$, see beginning of [Ans, lecture 11] until [Ans, Def. 11.1] and also [FF, 5.5.1] and references therein. State [Ans, 11.2]. State and prove [Ans, 11.3]. State [Ans, 11.4]. Discuss the example of the projective line over a field, see for instance [Ans, p. 62], [FF, 5.6.1]. Discuss the example of isocrystals, from [Ans, 11.5] to the statement (not the proof) of [Ans, 11.7]; see also [FF, 5.5.2.3], [Lur, Lect. 26, Def. 1-Thm. 6]. Show that the HN formalism can be applied to vector bundles on the Fargues–Fontaine curve. If there is time left at the end of the talk, cover the rest of the material in [Ans, Lect. 11], from [Ans, 11.9] onwards; alternatively, follow [Lur, Lect. 26], from [Lur, Lect. 26, Constr. 7] until the end.

(9) **Covers and semistable vector bundles**. Define the functor from isocrystals (that is, φ-modules) to vector bundles on the curve, describe the main properties of this construction and state the classification of vector bundles on the

curve. Obtain as a corollary a description of p-adic Galois representations in terms of Galois-equivariant vector bundles on the curve.

Cover [Ans, lecture 11] from Definition 11.9 until the end, minus what was possibly already covered in the previous talk; you may compare this part with [Lur, lecture 26] from Construction 7 until the end (although the functor here is dual), also see [FF, §8.2.4] for the full classification statement. Next, introduce the rings $\mathbf{A}_{\mathrm{crys}}$ and B^+_{crys} following [Ans, lecture 4], Definition 4.11 and Lemma 4.12 (you may directly give the concrete description of $\mathbf{A}_{\mathrm{crys}}$), then state [Ans, Proposition 10.15] as a fact and discuss the comments after the proof of loc. cit. (note the typo $x = \varepsilon$). Finally, move to [Ans, lecture 14] and cover until Corollary 14.8, skipping the part from Theorem 14.1 to Lemma 14.3 included. Time permitting, continue until Definition 14.10 included, state Lemma 14.15 giving the construction of the functor for granted and mention that its restriction to semistable objects of slope zero induces an equivalence as in Theorem 14.1.

(10) **The étale fundamental group of X.** Show that X is geometrically simply connected and $\pi_1^{\acute{e}t}(X) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ assuming the classification theorem for vector bundles on the curve; see the beginning of [Lur, lecture 27] (and note that Lemma 2 is usually a non trivial statement in Harder–Narasimhan theories) or [FF, §8.6]. In the remaining time, sketch the proof of the classification theorem for vector bundles on the curve [FF, Théorème 8.2.10] following [Ans, lecture 13] (cf. [FF, §8.1, §8.3]); note that [Ans, Theorem 13.1] implies the final result by [FF, Théorème 5.6.29] and Anschütz misuses the word "assumption" to mean "statement"). The proof uses the background material in [Ans, lecture 12] until Definition 12.19 (although Theorem 12.20 and Corollary 12.24 are also very interesting facts). For your exposition, feel free to choose what to explain more in detail and what to leave out.

(11) **Adic and relative curves**. Out of necessity, this talk will be less formal and detailed than the earlier ones and we will leave some wiggle room to the speaker regarding the choice of topics to discuss. That being said, we would like to see the definition of the adic curve, together with the GAGA-like results which link it to the schematic curve. We would also like to have an overview of the relative curve, describing its functor of points and introducing the point of view of diamonds more generally. See [Far23, lectures 4 and 7] and references therein.

(12) **What else?** Here are some (partially overlapping) options:
   (i) Prove the relation between isocrystals and vector bundles on the curve via Scholze–Weinstein, Lubin–Tate theory and period maps.
   (ii) Weakly admissible implies admissible via equivariant vector bundles on the curve.

   These two topics could be covered together, following the last 2/3 lectures of [Ans], and be made in 2 or 3 talks.
   (iii) Banach–Colmez spaces and their relation with the curve.

   This talk needs background on the relative Fargues–Fontaine curve, which should then be introduced in the 11th talk.

References

[Ans] Johannes Anschütz, *Lectures on the Fargues–Fontaine Curve*, `https://janschuetz.perso.math.cnrs.fr/skripte/vorlesung_the_curve.pdf`.

[Col18] Pierre Colmez, *Préface par Pierre Colmez: la courbe de Fargues et Fontaine*, Astérisque (2018), no. 406, 1–50. MR 3948097

[Far23] Laurent Fargues, *Some new geometric structures in the Langlands program*, `https://webusers.imj-prg.fr/~laurent.fargues/Eilenberg.pdf`, 2023, Eilenberg Lectures.

[FF] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique, with a preface by Pierre Colmez*, Astérisque 406 : xiii-382 (2018). `https://hal.science/hal-01510045v1/file/Courbe_fichier_principal.pdf`.

[Lur] Jacob Lurie, *The Fargues–Fontaine Curve - Lecture notes*, `https://www.math.ias.edu/~lurie/205.html`.

[Sch] Peter Scholze, *Perfectoid spaces*, Publications Mathématiques de l'IHES 116 : 245-313 (2012). `http://www.numdam.org/item/10.1007/s10240-012-0042-x.pdf`.