STUDY GROUP ON THE FARGUES-FONTAINE CURVE: OVERVIEW

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§. Introduction

p prime number, $\mathbb{Q}_p$, $C := \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ (alg. closed completely valued field of char 0, res. field $\overline{\mathbb{F}_p}$).

$\rightsquigarrow$ "tilt" $C^\flat$: an alg. closed completely valued field of char. p, res. field $\overline{\mathbb{F}_p}$; Frobenius: $\varphi \circlearrowright C^\flat$.

$\rightsquigarrow$ the Fargues-Fontaine curve $X = X_{\varphi, C^\flat}$ (associated to $\mathbb{Q}_p$, $C^\flat$) is a:

1-dimensional, geometrically simply connected, regular, quasi-compact Dedekind scheme over $\operatorname{Spec}\mathbb{Q}_p$, not of finite type. (★)

[Why are we interested in X? There are several points of view...]

- As a set: $|X| = \{\text{"untilts" of } C^\flat\}/\varphi^{\mathbb{Z}} = \left\{(K,\iota) \,\middle|\, \begin{array}{l} K|\mathbb{Q}_p \text{ alg. closed completely valued field} \\ \iota : K^\flat \xrightarrow{\sim} C^\flat \end{array}\right\} \Big/ \begin{array}{c}(K,\iota)\sim(K,\iota\circ\varphi^r),\\ r\in\mathbb{Z}\end{array}$ (closed points)

IN FACT SHOULD TAKE ISO CLASSES OF THESE (IN APPROPRIATE SENSE)

[Idea: one can go back from $C \mapsto C^\flat$ but there are many ways and $|X|$ classifies them (up to Frobenius); we would like to give geometric structure to this set...]

- As a diamond: $X^\diamond = \operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spa} C^\flat/\varphi^{\mathbb{Z}}$ [one can recover $|X|$ from the $C^\flat$-points of $X^\diamond$].

RMK: In function field case, one can consider the adic space $\operatorname{Spa}\mathbb{F}_p((t)) \times_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa} C^\flat \approx$ (punctured open unit disc with coefficients in $C^\flat$).

We would like to replace $\mathbb{F}_p((t))$ by $\mathbb{Q}_p$, but this only makes sense as diamonds.

Ideally, the adic space associated to X should be the "punctured open unit disc in the variable p and with coefficients in $C^\flat$" $/\varphi^{\mathbb{Z}}$.

[To make sense of this...]

$\mathcal{O}_{C^\flat} \subseteq C^\flat$ valuation ring, $A_{\inf} := W(\mathcal{O}_{C^\flat})$ ring of Witt vectors of $\mathcal{O}_{C^\flat}$: A RING S.TH. $W(\mathcal{O}_{C^\flat})_{/(p)} = \mathcal{O}_{C^\flat}$, WITH a multiplicative section $[\cdot] : \mathcal{O}_{C^\flat} \longrightarrow W(\mathcal{O}_{C^\flat})$ [called "Teichmüller lift"].

$\rightsquigarrow A_{\inf} = \left\{\sum_{k\in\mathbb{N}} [x_k]p^k \,\middle|\, x_k \in \mathcal{O}_{C^\flat}\right\} \circlearrowleft \varphi : \sum_k [x_k]p^k \longmapsto \sum_k [x_k^p]p^k$ lift of Frobenius,

$\vartheta : A_{\inf} \twoheadrightarrow \mathcal{O}_C$ ($\subseteq C$ valuation ring), $\operatorname{Ker}\vartheta = (p - [\varpi])$ where $\varpi \in \mathcal{O}_{C^\flat}$, $0 < |\varpi| < 1$.

- As an adic space: $X^{ad} = \left(\operatorname{Spa} A_{\inf} \setminus V(p\cdot[\varpi])\right)/\varphi^{\mathbb{Z}}$ [$X^\diamond$ = diamond associated to X^{ad}].

To finally get X as a scheme, we need to consider "holomorphic functions in the variable p":

$B^b := W(\mathcal{O}_{C^\flat})\left[\frac{1}{p}, \frac{1}{[\varpi]}\right] \subseteq B := \varprojlim_{I \subseteq (0,1) \text{ interval}} B_I$, where B_I is the completion of B^b wrt to a set of norms indexed by I.

$\rightsquigarrow B$ a "Fréchet algebra", φ extends to B, $P := \bigoplus_{d\geq 0} B^{\varphi = p^d} \subseteq B$ graded algebra.

- As a scheme: $X = \operatorname{Proj}(P)$ [X^{ad} = adic space associated to X; $X^\diamond$ = diamond associated to X; $|X|$ = closed pts of X].

RMK: $(B^b)^{\varphi = p^d} = 0$ for all $d \neq 0$ [that is why we pass to the completion].

GOAL: Introduce X (as a scheme) and prove all properties (★) + study vector bundles on X.

§. Motivation (p-adic Hodge theory)

1) Functions on X are closely related to "p-adic period rings".

E.g.: (C, id) is an untilt of $C^\flat \rightsquigarrow$ chosen point $\infty \in |X|$.

$$\widehat{\mathcal{O}}_{X,\infty} = B_{dR}^+$$ [this is a ring of periods in the following sense...]

EXPLAINED ORALLY

[$\mathcal{Y}$ smooth proper variety$/\mathbb{Q}_p \rightsquigarrow H^n_{ét}(\mathcal{Y}_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^n_{dR}(\mathcal{Y}) \otimes_{\mathbb{Q}_p} B_{dR}$ "p-adic étale-de Rham comparison" ($\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$-equivariant)

in analogy with:

$\mathcal{Y}$ smooth projective variety$/\mathbb{Q} \rightsquigarrow H^n_{sing}(\mathcal{Y}(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^n_{dR}(\mathcal{Y}) \otimes_{\mathbb{Q}} \mathbb{C}$.

Here, the role of $2\pi i \in \mathbb{C}$ is played by an element $t \in B_{dR}^+$, which is also a local coordinate at $\infty \in X$ [wrt $\widehat{\mathcal{O}}_{X,\infty} = B_{dR}^+$]

and $B_{dR} = B_{dR}^+[\frac{1}{t}]$ [these are the "periods": coefficients needed to write down the above iso's].]

On the other hand: $\mathcal{O}_X(X \setminus \{\infty\}) = B_{cris}^{\varphi=1} \subseteq B_{cris} \circlearrowleft \varphi$

[another ring of periods, endowed with a Frobenius φ; this one is needed for the "étale-cristalline comparison"].

We have a short exact sequence: $0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(X\setminus\{\infty\}) \longrightarrow \widehat{\mathcal{O}}_{X,\infty}[\frac{1}{t}]/\widehat{\mathcal{O}}_{X,\infty} \longrightarrow 0$ [similar to Beauville-Laszlo]

$\Leftrightarrow 0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{cris}^{\varphi=1} \longrightarrow B_{dR}/B_{dR}^+ \longrightarrow 0$ "fundamental exact sequence of p-adic Hodge theory".

2) Vector bundles on X are closely related to p-adic Galois representations (and other p-adic Hodge-theoretic objects).

$$\begin{tikzcd}
(\varphi\text{-modules}) = (\text{isocrystals}) \arrow[r, leftrightarrow, "\substack{\text{same objects, but}\\ \text{not an equivalence}}"] & \mathrm{Bun}_X := (\text{vector bundles on } X) \overset{(\mathrm{GAGA})}{=} (\text{vector bundles on } X^{ad}) \\
(\text{Filtered } \varphi\text{-modules}) \arrow[u] \arrow[r, hook] & \mathrm{Bun}_X^{G_{\mathbb{Q}_p}} \arrow[u] \\
\left(\substack{\text{weakly admissible}\\ \text{Filtered } \varphi\text{-modules}}\right) \arrow[u, hook] \arrow[r, hook] \arrow[d, shift right, "D_{cris}"] & (\mathrm{Bun}_X^{G_{\mathbb{Q}_p}})^{ss\text{-}0} \arrow[u, hook] \\
\mathrm{Rep}_{\mathbb{Q}_p}^{cris}(G_{\mathbb{Q}_p}) \arrow[u, shift right, "V_{cris}"] \arrow[r, hook] & \mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \arrow[u, "\wr"]
\end{tikzcd}$$

(objects endowed with action of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$)

affords a "Harder-Narasimhan" formalism analogous to vector bundles on smooth projective complex curve.

$\Downarrow$

classification of vector bundles on X similar to Dieudonné-Manin classification of isocrystals

both categories afford an HN-formalism [on the right because HN-filtration is $G_{\mathbb{Q}_p}$-equivariant]

$\Downarrow$

restrict to semi-stable objects of slope $\emptyset$

the fact that (V_{cris}, D_{cris}) define an equivalence of categories is the theorem "weakly admissible $\Leftrightarrow$ admissible" and it can be proved via the curve working in $(\mathrm{Bun}_X^{G_{\mathbb{Q}_p}})^{ss\text{-}\emptyset}$.

(p-adic Galois representations)

("cristalline" representations, defined using B_{cris})