THE FARGUES-FONTAINE CURVE

Lecture 1: BACKGROUND ON PERFECTOID FIELDS

All the subject is local. So we fix a prime number p from the beginning.

Let's start with a definition.

Definition 1

A *perfectoid field* K is a topological field whose topology can be defined by a non-archimedean valuation $|\cdot|: K \longrightarrow \mathbb{R}_{\geq 0}$ such that

1. the residue field $k = \frac{\mathcal{O}_K}{m_K}$ has characteristic p.
2. K is complete.
3. the Frobenius map $\varphi: \frac{\mathcal{O}_K}{p\mathcal{O}_K} \xrightarrow{x \mapsto x^p} \frac{\mathcal{O}_K}{p\mathcal{O}_K}$ is surjective
4. $\exists\, x \in K$ such that $|p|_K < |x|_K < 1$

Remark 2

Conditions 3. + 4. imply that m_K is not finitely generated.

Conditions 3. + 4. imply that m_K [illegible]

Because for $|p|_K < |x|_K < 1$ we have $x = y^p + pz$ for some $y, z \in \mathcal{O}_K$. But $|pz|_K \leq |p|_K < |x|_K \Rightarrow |y^p|_K = |x|_K \Rightarrow |y|_K^p = |x_K|$

$\Rightarrow |y|_K > |x|_K$.

<u>Remark 3</u>

If K has characteristic p, then $p=0 \Rightarrow (3. \Rightarrow K$ is perfect$)$.

So perfectoid fields of char p are perfect fields complete with respect to a <u>non-trivial</u> valuation.

<u>Examples 4</u>

- $\mathbb{C}_p$ is clearly perfectoid of char 0.
- the completion of the algebraic closure of $\mathbb{F}_p((t))$ is perfectoid of char p.
- Consider $\mathbb{Q}_p\left(p^{\frac{1}{p^\infty}}\right) = \bigcup_{n \in \mathbb{N}} \mathbb{Q}_p\left(p^{\frac{1}{p^n}}\right) = C$, has a canonical norm and we can consider $K = \hat{C}$.

For sure the norm on K is not discrete.

We should check that Frobenius on $\mathcal{O}_K/p$ is surjective.

$$\mathcal{O}_K/p = \frac{\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}}{p} \underset{\substack{p\text{-adic}\\ \text{Topology}}}{\cong} \frac{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}{p} \cong \frac{\frac{\mathbb{Z}_p[t^{\frac{1}{p^\infty}}]}{(t-p)}}{p} \cong \frac{\mathbb{F}_p[t^{\frac{1}{p^\infty}}]}{(t-p)} \cong$$

p-adic Topology

$$\frac{\mathbb{F}_p[t^{\frac{1}{p^\infty}}]}{t}$$ ⇝ here Frobenius is surjective.

- $\mathbb{Z}_p^{cyc} = \widehat{\bigcup_{n\in\mathbb{N}} \mathbb{Z}_p[\zeta_{p^n}]}$ and $\mathbb{Q}_p^{cyc} = \mathbb{Z}_p^{cyc}[\frac{1}{p}]$.

The valuation is not discrete (use cyclotomic polynomials to check that $|\zeta_p - 1| = p^{\frac{1}{p-1}}$)

Again

$$\frac{\mathbb{Z}_p^{cyc}}{p} = \bigcup_{n\in\mathbb{N}} \mathbb{F}_p[\zeta_{p^n}] = \frac{\mathbb{F}_p[t^{\frac{1}{p^\infty}}]}{(t)}$$

showing that the Frobenius is surjective.

<u>Definition 5</u>

Let K be any field. We denote

$$K^\flat = \varprojlim \left(\cdots \to K \xrightarrow{x\mapsto x^p} K \xrightarrow{x\mapsto x^p} K \to \cdots \right).$$

$K^\flat$ is always a multiplicative monoid, because $x \mapsto x^p$ is a multiplicative map and we can just define

$$\{x_n\} \cdot \{y_n\} = \{x_n y_n\}$$

$\forall \{x_n\}, \{y_n\} \in K^\flat$.

Since $x \mapsto x^p$ is not additive there is no way to define an addition on $K^\flat$ in general. But, if K is complete with respect to a non-archimedean norm and $|p|_K < 1$, then it is possible to give the structure of a field to $K^\flat$.

PROPOSITION 6

Let K be a complete non-arch. field with residue char. p. Then, the canonical map $\mathcal{O}_K \to \mathcal{O}_K / p\mathcal{O}_K$ induces a bijection

$$\mathcal{O}_K^\flat \to \varprojlim \left(\cdots \to \mathcal{O}_K / p\mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p\mathcal{O}_p \to \cdots \right)$$

$$\| $$

$$\varprojlim \left(\cdots \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K \to \cdots \right)$$

inverse system of rings

inverse system of monoids

Proof

If $\mathrm{char}(K) = p$, then $p\mathcal{O}_K = 0 \Rightarrow$ nothing to prove.

In general, we can write a diagram

$$\varprojlim \mathbb{Z}(n) \quad \to \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K \xrightarrow{x \mapsto x^p} \varprojlim \mathcal{O}_K / p\mathcal{O}_K = \mathcal{O}_K$$

This is a commutative projective

$$
\begin{tikzcd}
{} \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
Z(n) \arrow[r, dashed] \arrow[d, "\pi_n"', "\wr"] & \mathcal{O}_K/p^n\mathcal{O}_K \arrow[r, "x\mapsto x^p"] \arrow[d] & \mathcal{O}_K/p^n\mathcal{O}_K \arrow[r, "x\mapsto x^p"] \arrow[d] & \mathcal{O}_K/p^n\mathcal{O}_K \arrow[d] \\
Z(n-1) \arrow[r, dashed] \arrow[d, "\wr"] & \mathcal{O}_K/p^{n-1}\mathcal{O}_K \arrow[r, "x\mapsto x^p"] \arrow[d] & \mathcal{O}_K/p^{n-1}\mathcal{O}_K \arrow[r, "x\mapsto x^p"] \arrow[d] & \mathcal{O}_K/p^{n-1}\mathcal{O}_K \arrow[d] \\
\vdots \arrow[d, "\wr"] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
Z(1) \arrow[r, dashed] & \mathcal{O}_K/p\mathcal{O}_K \arrow[r, "x\mapsto x^p"] & \mathcal{O}_K/p\mathcal{O}_K \arrow[r, "x\mapsto x^p"] & \mathcal{O}_K/p\mathcal{O}_K
\end{tikzcd}
$$

projective limit of monoids

projective limit of rings

The claim is that This projective system is Trivial.

So let's consider $\{x_i\} \in Z(n)$, then

$$\pi_n(\{x_i\}) = \{x_i \bmod p^{n-1}\}.$$

Now if I have $\{y_i\} \in Z(n-1)$, I can consider

$$\left\{\tilde{y}_i^{\,p}\right\}_{\underline{i>0}} \in Z(n), \quad \tilde{y}_i \equiv y_i$$

because

$$x \equiv y \bmod p^{n-1} \Rightarrow x^p \equiv y^p \bmod p^n.$$

Notice that

$$\pi_n(\{x_i\}) = \{x_i^p\}_{i\geq 0}$$

we just got back the original sequence. ◻

Corollary 7

$\mathcal{O}_K^\flat$ can be equipped with a ring structure using the isomorphism of the theorem. So, let $\{x_i\}, \{y_i\} \in \mathcal{O}_K^\flat$ and write

$$\{x_i\} + \{y_i\} = \{z_i\}.$$

By the theorem we have a canonical map

$$\mathcal{O}_K^\flat \xrightarrow{\cong} \mathbb{Z}(1) = \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p \xleftarrow{\cong} \varprojlim_{x\mapsto x^p} \mathcal{O}_K$$

So, we deduce that

$$x_i + y_i \equiv z_i \mod (p),$$

and if write

$$z_i = x_i + y_i + pw$$

for some $w \in \mathcal{O}_K$, then we use

$$\Rightarrow z_i^{p^i} = (x_i + y_i + pw)^{p^i} = \sum \binom{p^i}{j} (pw)^j (x_i + y_i)^{p^i - j}$$

$$z_0 = z_i^{p^i} = (x_i + y_i + pw)^{p^i} = \sum_{j=0}^{p^i} \binom{p^i}{j} (pw)^j (x_i + y_i)^{p^i - j}$$

$$\equiv (x_i + y_i)^{p^i} \mod (p^i)$$

So

$$z_0 = \lim_{i \to \infty} (x_i + y_i)^{p^i}.$$

More generally

$$z_n = \lim_{i \to \infty} (x_{i+n} + y_{i+n})^{p^i}.$$

<u>Examples 8</u>

In proposition 6 we did not assume that K is perfectoid.
If it is not then the result of tilting is not very interesting.
For example, if $K = \mathbb{Q}_p$ then $K^\flat = \mathbb{F}_p$.
For $K = \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$, $K^\flat \cong \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$,
for $K = \widehat{\mathbb{Q}_p^{cyc}}$, $K^\flat \cong \widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}$

Notice that

$$\widehat{\mathbb{Q}_p^{cyc}} \not\cong \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}.$$

<u>Definition 9</u>

Definition 9

Let K be a complete valued field of residue char p and $x = \{x_n\} \in K^\flat$.

We define $x^\# = x_0 \in K$. Moreover we define the map

$$|\cdot|_\flat : K^\flat \longrightarrow \mathbb{R}_{\geq 0}$$
$$x \longmapsto |x^\#|_K = |x_0|_K .$$

The map

$$K^\flat \longrightarrow K$$
$$x \longmapsto x^\# = x_0$$

is a morphism of commutative monoids.

Remark 10

If K is perfect of char p, then $(-)^\#$ is a bijection.

Proposition 11

Let K be perfectoid.

① $\forall x \in \mathcal{O}_K$, $\exists y \in \mathcal{O}_K^\flat$ satisfying $x \equiv y^\# \bmod p$,

② $\forall x \in K$, $\exists y \in K^\flat$ such that $|x|_K = |y|_K$.

Proof

① Proposition 6 + 3 of definition of perfectoid.

② Similar. ∎

... $|K^\flat| = |K|$. So choose $\pi \in K^\flat$ such that

(2) Similar.

So by proposition 11, $|K^\flat|_\flat = |K|_K$. So choose $\pi \in K^\flat$ such that $0 < |\pi|_\flat < 1$ and consider the subsets

$$\pi^{-n} \mathcal{O}_K^\flat = \left\{ x \in K^\flat \mid |x|_\flat \leq |\pi|_\flat^{-n} \right\}$$

for $n \in \mathbb{Z}$.

We have

$$K^\flat = \varinjlim_{n \in \mathbb{N}} \pi^{-n} \mathcal{O}_K^\flat = \left(\cdots \to \mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \xrightarrow{\pi} \cdots \right)$$

as multiplicative monoids.

<u>Proposition 12</u>

There is a bijection of multiplicative monoids

$$K^\flat \cong \mathcal{O}_K^\flat [\pi^{-1}]$$

and this bijection induces a ring structure on $K^\flat$.

The addition law is given again by

$$\{x_n\} + \{y_n\} = \left\{ \lim_{i \to \infty} (x_{n+i} + y_{n+i})^{p^i} \right\}.$$

With this structure and the map $|\cdot|_\flat$, the field $K^\flat$ becomes a perfectoid field of char p.

becomes a perfectoid field of char p.

<u>Proof</u>

$K^\flat$ is a field because if $\{x_n\} \neq 0$ then $\{x_n^{-1}\}$ is the inverse.

The isomorphism of proposition 6 shows that $\mathrm{char}(K^\flat) = p$.

$|0|_\flat = 0$, $|1|_\flat = 1$ and $|xy|_\flat = |x|_\flat |y|_\flat$ are obvious

while

$$|x+y|_\flat \leq \max\{|x|_\flat, |y|_\flat\}$$

is deduced from the fact that

$$(x+y)^\# = (x+y)_0 = \lim_{i \to \infty} (x_i + y_i)^{p^i}$$

So

$$|(x+y)_0|_K = \left|\lim_{i\to\infty} (x_i+y_i)^{p^i}\right|_K = \lim_{i\to\infty} |x_i+y_i|_K^{p^i} \leq$$

$$\lim_{i\to\infty} \max\left\{|x_i|_K^{p^i}, |y_i|_K^{p^i}\right\} = \max\{|x_0|_K, |y_0|_K\} =$$

$$\max\{|x|_\flat, |y|_\flat\}.$$

It remains to show that $K^\flat$ is complete.

Let us assume $|\pi|_\flat = |p|_K$.

For this it is easy to show that

$$
\begin{tikzcd}
\varprojlim \frac{\mathcal{O}^\flat}{\pi^{p^n}} \cong \varprojlim \Big(\arrow[d] & \arrow[r] & \mathcal{O}^\flat/\pi^p \arrow[r] \arrow[d, "\wr"] & \mathcal{O}^\flat_K/\pi \Big) \arrow[d, "\wr"] \\
\mathcal{O}_K^\flat = \varprojlim \Big(\cdots \arrow[rr] & & \mathcal{O}_K/p \arrow[r, "\varphi"] & \mathcal{O}_K/p\mathcal{O}_K \Big)
\end{tikzcd}
$$

Where the vertical isomorphisms are induced by $\#$.

It is enough to show, $\forall \pi \in K^\flat$ with $|p|_K \leq |\pi|_\flat < 1$, $\#$ induces an isomorphism $\mathcal{O}_K^\flat/(\pi) \xrightarrow{\cong} \mathcal{O}_K/(\pi^\#)$.

It is not clear that

$$x \longmapsto x^\#$$

is injective. Suppose that

$$x^\# \equiv 0 \mod \pi^\#$$

This means

$$|x|_\flat = |x^\#|_K \leq |\pi^\#|_K = |\pi|_\flat$$

$$\Rightarrow \quad x \in \pi \mathcal{O}_K^\flat \Rightarrow x = 0 \mod \pi. \quad \blacksquare$$

So we have defined an operation

$$\{\text{Perfectoid fields}\} \xmapsto{(-)^\flat} \{\text{Perfectoid fields char } p\}$$

called Tilt, that is a functor.

Moreover some properties of K are preserved by $(-)^\flat$, for example if K is algebraically closed then $K^\flat$ is algebraically closed. More is true

Proposition 12.1

Let K be a perfectoid field.

① Let L/K be a finite extension, then L is a perfectoid field.

② The Tilt functor $(-)^\flat$ induces an equivalence of categories between finite extensions of K and those of $K^\flat$, that preserves the degree of the extensions.

We have also seen that the functor $(-)^\flat$ is not faithful: it maps non-isomorphic perfectoid fields to the same Tilt.

So the natural question is: if we fix a perfectoid field of char p, how many char 0 perfectoid fields have this field as a Tilt?

Definition 13

Let C be a perfectoid field of char p. An untilt of C is a pair (K, ι) of a perfectoid field and an

C is a pair (K, ι) of a perfectoid field and an isomorphism $K^\flat \cong C$ of valued fields.

The goal of this seminar is to classify untilts up to Frobenius action.

<u>Remark 14</u>

$$C \underset{\substack{\uparrow \\ \text{valued} \\ \text{fields}}}{\cong} K^\flat \iff \mathcal{O}_C \cong \mathcal{O}_K^\flat = \varprojlim \left(\cdots \to \mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K \to \cdots \right)$$

And we have seen that $\exists\, \pi \in \mathcal{O}_C$ such that

$$\mathcal{O}_C/\pi \cong \mathcal{O}_K/p \iff C \cong K^\flat.$$

Thus we need to classify perfectoid fields with an isomorphism

$$\mathcal{O}_K/p \cong \mathcal{O}_C/(\pi).$$

Now purely algebraic isomorphism.

Let us use $\#$ also for the composition $C \xrightarrow[\cong]{\iota} K^\flat \xrightarrow{\#} K$.

This map is not surjective in general, but we have seen in proposition 11

that $\forall\, x \in \mathcal{O}_K$, $\exists\, c_0 \in \mathcal{O}_C$ such that $x - c_0^\# \in p\mathcal{O}_K$.

So we can write

$$x = c_0^{\#} + x' p$$

with $x' \in \mathcal{O}_K$. And we can keep going with

$$x' = c_1^{\#} + x'' p \Rightarrow x = c_0^{\#} + c_1^{\#} p + x'' p^2$$

etc..

$$x = c_0^{\#} + c_1^{\#} p + \dots + c_n^{\#} p^n + \dots = \sum_{i=0}^{\infty} c_i^{\#} p^i$$

that is a p-adically convergent series.

The representation of x in a series like this is not unique.

To make more sense of this observation we introduce the following concept.

<u>Definition 15</u>

Let R be a perfect ring of characteristic p. The <u>ring of Witt vectors</u> $W(R)$ of R is characterized by the following properties

1. $W(R)/pW(R) \cong R$,
2. p is not a zero divisor in $W(R)$
3. $W(R)$ is p-adically complete

Another characterization of $W(R)$ is by the universal property

[illegible] $\cong$ [illegible]

$$\mathrm{Hom}(W(R), A) \xrightarrow{\cong} \mathrm{Hom}(R, A/pA)$$

$\forall$ p-adically complete rings A.

<u>Example</u> 16

$W(\mathbb{F}_p) \cong \mathbb{Z}_p$

<u>Remark</u> 17

The quotient map $W(R) \to R \cong W(R)/p$ has a canonical multiplicative section

$$[-] : R \longrightarrow W(R)$$

called <u>Teichmüller lift</u>. One can compute

$$[x] = \lim_{n\to\infty} \left(\widetilde{x^{1/p^n}}\right)^{p^n}$$

where $\widetilde{x^{\frac{1}{p^n}}}$ is any preimage of $x^{\frac{1}{p^n}}$.

Let now $x \in W(R)$, then using $W(R) \underset{[-]}{\rightleftarrows} W(R)/p$

$$x = [c_0] + x' p$$

and so on

$$x = \sum_{n=0}^{\infty} [c^n] p^n$$

This is called Teichmüller <u>expansion</u> of x and is similar to

This is called Teichmüller expansion of x and ~

what we have done before with the map $\#$.

The difference is that now the expansion is unique!

This is quite different than before with $\#$.

Since W is a functor, to any morphism of perfect $\mathbb{F}_p$-algebras $R_1 \to R_2$, associates a morphism $W(R_1) \to W(R_2)$. In particular this is true for the Frobenius automorphism $R \circlearrowleft \varphi$ that gets lifted to an automorphism $W(R) \circlearrowleft \varphi$ called <u>lift of Frobenius</u>. In formulas, if

$$\sum_{i=0}^{\infty} [a_n] p^n \in W(R) \text{, then } \varphi\left(\sum_{n=0}^{\infty} [a_n] p^n\right) = \sum_{n=0}^{\infty} [\underbrace{\varphi(a_n)}_{a_n^p}] p^n .$$

<u>Definition 18</u>

Let C be a perfectoid field of char p. Then $\mathcal{O}_C$ is a perfect ring of char p. We denote $A_{\mathrm{inf}}(C) = \mathbb{A}_{\mathrm{inf}} = W(\mathcal{O}_C)$, often called <u>Fontaine's period ring</u> of C.

The map $\# : \mathcal{O}_C \to \mathcal{O}_K$ is not a ring homomorphism but when composed with $\mathcal{O}_K / p\mathcal{O}_K$ it becomes. Therefore by the universal property of the Witt vectors and the fact that $\mathcal{O}_K$ is p-adically complete we have unique morphism of rings

$$\theta : \mathbb{A}_{inf} \to \mathcal{O}_K$$

lifting #. It is given by the formula

$$\theta\left([c_0] + [c_1]p + [c_2]p^2 + \cdots\right) = c_0^\# + c_1^\# p + c_2^\# p^2 + \cdots$$

universal version of

In terms of commutative diagrams we have seen that

$$\begin{tikzcd}
& \mathbb{A}_{inf} \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_K \arrow[d] \\
\mathcal{O}_K^\flat \cong & \mathcal{O}_C \arrow[r, "\#"] & \mathcal{O}_K / p\mathcal{O}_K
\end{tikzcd}$$

is commutative.

Remark 19

We can find $\pi \in \mathcal{O}_C$ such that $|\pi|_\flat = |p|_K$, so

$$|\pi^\#|_K = |p|_K \Rightarrow \pi^\# = V p$$

for a $v \in \mathcal{O}_C^\times$. Let us write $v = \theta(u)$, for some $u \in \mathbb{A}_{inf}$.

Also u is a unit (because $u = \sum [c_i] p^i$ with $|c_0| = 1$) and therefore

$$[\pi] - up \in \operatorname{Ker} \theta.$$

Definition 20

An element $\xi \in \mathbb{A}_{inf}$ is called <u>distinguished</u> if it is of the form $\xi = [\pi] - up$, $|\pi|_b < 1$, $u \in \mathbb{A}_{inf}^\times$

$$\Updownarrow$$

$$\xi = [c_0] + [c_1]p + \cdots$$

$|c_0|_b < 1$, $|c_1|_b = 1$.

Remark 19 shows that $\operatorname{Ker}(\theta)$ always contains a distinguished element.

Proposition 21

Let $\xi \in \mathbb{A}_{inf}$ be a distinguished element. The quotient $\mathbb{A}_{inf}/(\xi)$ is the valuation ring of a perfectoid field K. The canonical morphism

$$\mathcal{O}_C = A_{\inf}/(p) \longrightarrow A_{\inf}/(\xi, p) \cong \mathcal{O}_K/(p)$$

exhibits K as an untilt of C.

Proof

Roughly, one checks that $A_{\inf}$ is (p, ξ)-adically complete and write down explicitly the isomorphisms claimed in the statement. $\blacksquare$

Corollary 22

$\forall$ untilts K of C the map $\vartheta : A_{\inf} \to \mathcal{O}_K$ is surjective with $\ker(\vartheta)$ principal and generated by a distinguished element.

Therefore, the construction

$$\xi \longmapsto \mathrm{Frac}\left(A_{\inf}/(\xi)\right)$$

induces a bijection

$$\frac{\{\text{distinguished el. of } A_{\inf}\}}{\text{multiplication by units}} \cong \frac{\{\text{untilts of } C\}}{\text{isomorphisms}}.$$

Last construction.