The Fargues-Fontaine curve : stage 4

<u>Plan</u> : • define B_{dR}^{+}

• extend $\theta : A_{inf}[1/p] \to K$

$$\begin{array}{ccc} A_{inf}[1/p] & \to & K \\ \downarrow & & \uparrow \\ B_{[a,b]} & \to & B_{dR}^{+} \end{array}$$

<u>Setting</u> p = prime , C = perf'oid field of char p

ϖ = pseudo-unif of C (e.g. $C = \mathbb{C}_p^{\flat}$)

$0 < |\varpi|_C < 1$

$$Y = \mathrm{Spec}\left(A_{inf}\left[\frac{1}{p[\varpi]}\right]\right) \subseteq \mathcal{Y} = \mathrm{Spa}(A_{inf}) \setminus \{p[\varpi]\}$$

cl. pts "$\overset{!}{=}$" $\{ y = (K, \iota)$ untilt of $C \}$

char $K = 0$, $\iota : C \xrightarrow{\sim} K^{\flat}$

$$y \leftrightarrow \ker\left(\theta_y : A_{inf}[1/p[\varpi]] \to K\right) \xrightarrow{|\cdot|_K} \mathbb{R}$$

$$\sum [c_i] p^i \mapsto \sum (\iota c_i)^{\#} p^i$$

Rmk 1) Normalization of $|\cdot|_K$: $\forall x \in C,\ |x|_C = |(\iota x)^\#|_K$,

2) have a function $\begin{array}{ccc} Y & \xrightarrow{r} & \mathbb{R} \\ y & \longmapsto & r(y) = |p|_K \end{array}$

3) $\mathcal{Y} \neq \mathrm{Spa}(A_{inf}[\frac{1}{p[\varpi]}])$, $\mathcal{Y} = \mathrm{Spa}(B) = \bigcup_{0<a\leq b<1} \mathrm{Spa}(B_{[a,b]})$

Picture

$$\begin{array}{ccc} \Gamma(Y, \mathcal{O}_Y) = A_{inf}[\frac{1}{p}] & \xrightarrow{\theta_y} & \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} = K(y) = K \text{ untilt} \\ \downarrow & & \uparrow \\ \Gamma(Y_{[a,b]}, \mathcal{O}^{an}) = B_{[a,b]} & \longrightarrow & \widehat{\mathcal{O}}_{Y,y} =: B^+_{dR,y} \\ \text{\scriptsize $\in$} & & \\ y & & \end{array}$$

where $Y_{[a,b]} = Y \cap \mathrm{Spa}(B_{[a,b]}) = \{ y \in Y : a \leq \underset{\substack{\| \\ |p|_K}}{r(y)} \leq b \}$

Definition of B^+_{dR} fix $y = (K, \iota)$

let $(\xi) = \mathrm{Ker}(\theta_y : A_{inf}[1/p] \to K)$

$\xi = \xi_y$ depends on y is a distinguished element
i.e. of the form $[\varpi] + up$, $u \in A_{inf}^\times$

$A_{inf} = W(\mathcal{O}_C)$ (e.g. $W(\mathcal{O}_{\mathbb{C}_p}^\flat)$)

$B^+_{dR,y} =$ "$\widehat{\mathcal{O}}_{Y,y}$" $=$ "(ξ)-adic completion of $A_{inf}[1/p]$"
$\stackrel{!}{=} \boxed{\lim_{n\geq 0} \left(A_{inf}/(\xi)^{n+1} [1/p] \right)}$

Prop. $B^+_{dR,y}$ is a complete dvr with unif. ξ — i.e.

a) ξ is not a zero-divisor in $B^+_{dR,y}$

b) $B^+_{dR,y}$ = (ξ)-adically complete

c) $B^+_{dR,y}/(\xi)$ = field = K

Proof $\quad$ y fixed, drop it from notation.

a) $x \in B^+_{dR}$, $\quad x = (x_n)_{n\geq 0}$, $\quad x_n \in A_{inf}/(\xi)^{n+1}\left[\frac{1}{p}\right]$

ξ is not zero-div. in A_{inf} (because dist.)

p " " " in $A_{inf}/(\xi) \subseteq K$ is char 0 untilt

$\Rightarrow$ (ex) $A_{inf}/(\xi)^{n+1} \subseteq A_{inf}/(\xi)^{n+1}\left[\frac{1}{p}\right]$

There exists $k >> 0$ (dep. on n) s.t. $p^k \cdot x_n \in A_{inf}/(\xi)^{n+1}$

if now $\xi \cdot x = 0 \Rightarrow p^k x_n$ is killed by ξ in

$\Rightarrow p^k x_n = \xi^n \cdot y_n$ for some $y_n \in A_{inf}/(\xi)^{n+1}$

reducing modulo $(\xi)^n$ we get

$x_n \equiv x_{n-1} + (\xi)^n$ by definition

so that $p^k x_{n-1} \equiv 0 \mod (\xi)^n \Rightarrow x = 0$ $\square$

b)
$$\begin{tikzcd}
B^+_{dR} = \lim\left(A_{inf}/(\xi)^{n+1}\left[\frac{1}{p}\right]\right) \arrow[rr, two heads, "\rho"] \arrow[dr] & & A_{inf}/(\xi)^{m+1}\left[\frac{1}{p}\right] \\
& B^+_{dR}/(\xi)^{m+1} \arrow[ur, two heads, "\bar\rho"] &
\end{tikzcd}$$

enough to show that $\bar\rho$ iso for all m —

<u>Enough to show $\bar\rho$ injective</u>

take $x = (x_n)_{n\geq 0}$, $k_n >> 0$ as before $p^{k_n} \cdot x_n \in A_{inf}/(\xi)^{n+1}$

If $\rho(x) = 0 \Rightarrow p^{k_n}\cdot x_n \in (\xi)^{m+1}$ for $n \geq m$

$\Rightarrow p^{k_n}\cdot x_n = \xi^{m+1}\cdot y_n$ for some $y_n \in A_{inf}/(\xi)^{n-m}$

$\Rightarrow x = \xi^{m+1}\cdot\left(\frac{y_n}{p^{k_n}}\right)_{n\geq m}$ hence $x \bmod (\xi)^{m+1}$ is trivial

$\lim A_{inf}/(\xi)^{m+1}\cdot\left[\frac{1}{p}\right]$ $\rightarrow$ is a sequence

hence B^+_{dR} is ξ-adically complete and for $m=0$ get point (c) $\square$

<u>Prop.</u> Fix a, b such that $\underline{y} \in Y$ is in $Y_{[a,b]}$: i.e. $a \leq |p|_K \leq b$ – Consider

$$\begin{tikzcd}
A_{inf}\left[\frac{1}{p[\varpi]}\right] \arrow[r, hook, "\text{can. map}"] \arrow[d, bend right] & B^+_{dR,y} & \hat{f}_y \\
\frac{1}{p^m}\cdot\frac{1}{[\varpi]^k}\sum [c_n]p^n \ni f \arrow[r, mapsto] & \hat{f}_y & \exists \\
B_{[a,b]} = \widehat{\left(A_{inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]\right)}\left[\frac{1}{p}\right] \ni f \arrow[u, dotted] \arrow[uur, bend right]
\end{tikzcd}$$

(where $|\pi_a|_C = a$, $|\pi_b|_C = b$)

<u>Rmk</u> $A_{inf} \xrightarrow{\theta_y} \mathcal{O}_K$ $\qquad$ $\varpi \in C \xrightarrow{\iota} K^\flat$

$[\varpi] \longmapsto \iota\varpi^\#$ $\qquad$ $0 < |\varpi|_C = |\iota\varpi^\#|_K < 1$

$\Rightarrow (\iota\varpi)^\# \in K^\times$

K = residue fields of $B^+_{dR,y}$

Proof Lurie say **WLOG** $a = b = |p|_K$, because

$$B_{[|p|_K, |p|_K]} \longleftrightarrow B_{[a,b]}$$

Consider $e_n : A_{inf}\left[\frac{[\varpi]}{p}, \frac{p}{[\varpi]}\right] \longrightarrow B^+_{dR}/(\xi)^n$

induced by the canonical map as before

it is sufficient to show that $\boxed{e_n(\ \ //\ \ ) \subseteq \frac{1}{p^{k_n}} \cdot A_{inf}/(\xi)^n}$

exercise

let $\alpha = e_n\left(\frac{[\varpi]}{p}\right)$, $\beta = \alpha^{-1} = e_n\left(\frac{p}{[\varpi]}\right)$

and $|\alpha|_K = |\beta|_K = 1$ (can choose ϖ in such a way)

there exist α', β' in $A_{inf}/(\xi)^n$ such that

$$\begin{cases} \alpha \equiv \alpha' \\ \beta = \beta' \end{cases} \bmod (\xi) \Rightarrow \begin{cases} \alpha = \alpha' + \frac{\xi}{p^c}\alpha'' \\ \beta = \beta' + \frac{\xi}{p^c}\beta'' \end{cases}$$

$$\Rightarrow \alpha^m = \left(e_n\left(\frac{[\varpi]}{p}\right)\right)^m = \left(\alpha' + \frac{\xi\alpha''}{p^c}\right)^m = \text{binomial formula}$$

at worst you get $\frac{1}{p^{nc}} \cdot A_{inf}/(\xi)^n$

$\square$