Newton polygons and factorisations

10 January 2025 14:01

State :

$$\begin{array}{c} C \ \text{alg closed non arch complete} \\ \uparrow \\ E \\ \uparrow \text{finite} \\ \mathbb{Q}_p \end{array}$$

$$\begin{tikzcd}
W_{\mathcal{O}_E}(\mathcal{O}_F) \arrow[dr, "\theta"] & & & F \ \text{alg closed complete non arch.} \\
 & \mathcal{O}_C & \mathcal{O}_C^\flat = \mathcal{O}_F \ \text{valued ring} \arrow[l, dashed, "(\ )^\#"'] \arrow[ull, dashed, bend right, "{[\ ]}"'] \arrow[ur] & \\
W_{\mathcal{O}_E}(\mathbb{F}_q) = \mathcal{O}_E \arrow[uu] \arrow[ur] \arrow[rr, two heads] & & \frac{\mathcal{O}_E}{(\pi)} = \mathbb{F}_q \arrow[u] & \\
\mathbb{Z}_p \arrow[u] & & &
\end{tikzcd}$$

$\mathcal{O}_C^\flat = \varprojlim_{x \to x^q} \mathcal{O}_C/\pi\mathcal{O}_C$ $\qquad$ $\theta\left(\sum [a_i]\pi^i\right) = \sum a_i^\# \pi^i$

Question : how can we describe untilts of $\mathcal{O}_F$?

Let $|Y| := \{$ ideals in $\mathbb{A}_{\inf} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ generated by elements of the type $[a] - u\pi$ with $\underset{\neq 0}{a} \in m_F$, $u \in \mathbb{A}_{\inf}^\times$ $\}$

primitive of rk 1 $= \text{Prim}^1 / \mathbb{A}_{\inf}^\times$

thm 5.4 ($\overset{\text{cor}}{2.2.22}$ in FF) says that there is a bijection

$$\{ (C, \iota),\ C/E \text{ alg. } \overset{\text{complete}}{\text{closed}} \text{ N. arch},\ \iota : \mathcal{O}_C^\flat \overset{\text{as valued rings}}{\simeq} \mathcal{O}_F \}/_{\sim} \longrightarrow |Y|$$

$$(C, \iota) \longmapsto \mathfrak{p}_y = \ker\left(\mathbb{A}_{\inf} \xrightarrow[\sim]{W(\iota^{-1})} W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \xrightarrow{\theta} \mathcal{O}_C \right)$$

$$\mathfrak{p}_y = (\underbrace{[a] - \pi u_y}_{\xi_y})$$

In order to understand $|Y|$ we have to understand $\operatorname{Spec} \mathbb{A}_{\inf}\left[\frac{1}{\pi[\varpi]}\right] \ni ([a] + \pi u)$

and how elements in $\mathbb{A}_{\inf}\left[\frac{1}{\pi[\varpi]}\right]$ factors $\qquad \varpi \in m_F \setminus \{0\}$

Aim: describe the untilts of $\mathcal{O}_F$. Up to iso they are of the type $\dfrac{W_{\mathcal{O}_E}(\mathcal{O}_F)}{(d)}$ with $d = [a_0] + \pi u$ invertible, $a_0 \in m_F \setminus \{0\}$

NB $\operatorname{Spec} W_{\mathcal{O}_E}(\mathcal{O}_F) \ni (0),\ (\pi, [\varpi]),\ (\pi),\ W(m_F),\ \overset{\text{closed}}{(d)}$

§ Newton polygon in different contexts

If K is a non-arch. field, $v: K \to \mathbb{R} \cup \{\infty\}$ valuation

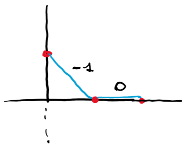
$f = a_0 + \dots + a_n T^n \in K[T]$

$\text{Newt}_{\text{poly}}(f) :=$ largest convex polygon below the points $(i, v(a_i))$

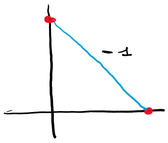
Not confuse it with $\varphi_f : \mathbb{R} \to \mathbb{R}$ piece-wise linear function connecting $(i, v(a_i))$

e.g. $K = \mathbb{Q}_p$

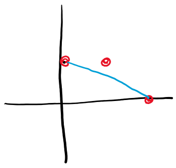
$f = p + (p+1)T + T^2 = (p+T)(1+T)$



$f = (T-p)(T+p) = T^2 - p^2$



$f = p + pT + T^2$



Prop: If $\alpha_1, \dots, \alpha_n \in \overline{K}$ are the zeros of f, then $-v(\alpha_1), \dots, -v(\alpha_n)$ are the slopes of the $NP(f)$ with correct multiplicities

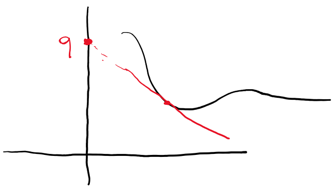
A description of $NP(f)$ in terms of Legendre transforms (Inverse Leg. transf.) will help the study of $f \in K[\![x]\!]$ and $f \in \mathcal{A}_{\text{inf}}$.

Given $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\mp\infty\}$ $\quad \operatorname{Im} \varphi \neq \{+\infty\}$

$\mathcal{L}(\varphi) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$
$\quad \lambda \longmapsto \inf_{x \in \mathbb{R}} (\varphi(x) + \lambda x)$

$\tilde{\mathcal{L}}(\varphi) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$
$\quad \lambda \longmapsto \sup_{x \in \mathbb{R}} (\varphi(x) - \lambda x)$

$\mathcal{F} := \{\mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}\}$ $\quad \mathcal{L}, \tilde{\mathcal{L}} : \mathcal{F} \longmapsto \mathcal{F}$

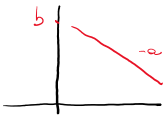


Consider the pencil of straight lines of slope $-\lambda$ (resp. of slope λ)

$y \pm \lambda x = q$

Take those passing through a point $(x, \varphi(x))$ and take the inf (resp. sup) of the set of possible q's.

Example



$\varphi(x) = ax + b$ $\quad \mathcal{L}(\varphi) : \lambda \longmapsto \begin{cases} b & \text{if } \lambda = -a \\ -\infty & \lambda \neq -a \end{cases}$

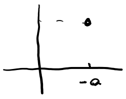
$\tilde{\mathcal{L}}(\varphi) : \lambda \longmapsto \begin{cases} b & \text{if } \lambda = a \\ +\infty & \text{if } \lambda \neq a \end{cases}$

$\tilde{\mathcal{L}}\mathcal{L}(\varphi) = \varphi$

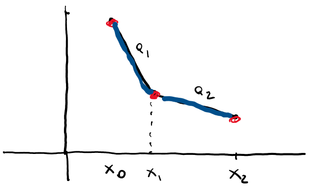
$\lambda \longmapsto \sup(b - \lambda(-a)) = a\lambda + b$

$\cdots$ $\begin{cases} & \\ +\infty & \text{if } d\neq a \end{cases}$

$\tilde{\mathcal{L}}\mathcal{L}(\varphi) = \varphi$

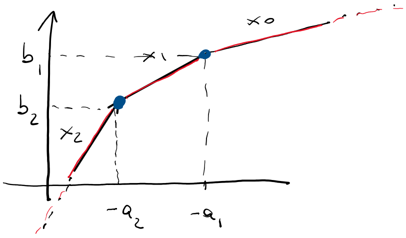


$d \longmapsto \sup\big(b - d(-a)\big) = ad + b$



$\varphi(x) = \begin{cases} a_1 x + b_1 & x_0 \le x \le x_1 \\ a_2 x + b_2 & x_1 \le x \le x_2 \end{cases}$ interc

with $a_1 x_1 + b_1 = a_2 x_1 + b_2$



$$\mathcal{L}(\varphi) : d \longmapsto \begin{cases} \cdots & d \ge -a_1 \\ b_1 & \text{if } d = -a_1 \\ x_1 d + (a_1 x_1 + b_1) & \text{if } -a_2 \le d \le -a_1 \\ b_2 & \text{if } d = -a_2 \\ \cdots & d \le -a_2 \end{cases}$$

Lemma $\mathcal{L} : \mathcal{F} \longrightarrow \mathcal{F}$ sends a convex piecewise linear function φ to a concave piecewise linear function and conversely using $\tilde{\mathcal{L}}$. (with the assumption non-extendable)

Note Convex says that slopes increase

Concave - - - - - - - decrease.

In the notes more general functions than those piecewise linear are treated

Back to NP $\quad f \in K[T] \quad f = a_0 + a_1 T + \cdots + a_n T^n = \sum_{i \in \mathbb{Z}} a_i T^i \quad a_i = 0$ if $i \notin \{0,\dots,n\}$

$Newt_{poly}(f)$ can be seen as the largest convex function $\mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ below the set $\{(i, v(a_i))\}$

Fix $r \in \mathbb{R}$ and let $v_r(f) := \inf_{i \in \mathbb{Z}} (v(a_i) + ir)$. Then its Legendre transform becomes

$\mathcal{F} \ni \mathcal{L}(Newt_{poly}(f)) : r \longmapsto v_r(f)$

NOTE: Since it is a polygon it is sufficient to take inf on the vertices

Geometric interpretation

If $r \in v(\overline{K}^*)$ v_r is a well-known Gauss valuation and

$v_r(f) = \inf_{\substack{x \in \overline{K} \\ v(x) \ge r}} v(f(x))$

$|x| \le \rho \Rightarrow q^{-v(x)} \le q^{-r} \Leftrightarrow v(x) \ge r$

Interesting $\mathcal{L}(Newt_{poly}(f)) = \mathcal{L}(\varphi_f)$ with $\varphi_f : \mathbb{R} \longrightarrow \mathbb{R}$ is

the piecewise linear function obtained by connecting the points $(i, v(a_i))$

What is the relation between $Newt_{pol}$ of f, g, and $f \cdot g$?

If v of zeros of f is $>$ v zeros of g (i.e. the slopes of $Newt_{pol}(f)$ $<$ all slopes of $Newt_{pol}(g)$) then $Newt_{poly}(fg)$ is simply obtained by concatenation of the two NP.

$Newt_{pol}(f)$ < all slopes of $Newt_{pol}(g)$) then $Newt_{poly}(fg)$ is simply obtained by concatenation of the two NP.

In general one uses the convolution construction

<u>Lemma</u> $v_r(f \cdot g) = v_r(f) + v_r(g) \quad \forall r \in \mathbb{R}, \quad f, g \in K[T]$

$$\Rightarrow \mathcal{L}(\varphi_{fg}) = \mathcal{L}(\varphi_f) + \mathcal{L}(\varphi_g)$$

<u>Def</u> Let $\varphi, \psi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$

Define the convolution $\varphi * \psi : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$

$$x \longmapsto \inf_{a+b=x} (\varphi(a) + \psi(b))$$

<u>Lemma</u> $\mathcal{L}$ maps $*$ to $+$ and.

$$\mathcal{L}(Newt_{poly}(f) * Newt_{poly}(g)) = \mathcal{L}(Newt_{poly}(fg))$$

$$\mathcal{L}(\varphi_f) + \mathcal{L}(\varphi_g) = \mathcal{L}(\varphi_{fg})$$

Hence $Newt_{poly}(fg) = Newt_{poly}(f) * Newt_{poly}(g)$ always

What we have done above can be generalized to power series!

Since we are interested in roots in the ring of integers, we truncate the Newton polygon to negative slopes. Hence we consider

$Newt(f)$ as the largest decreasing convex function below $(i, v(a_i))$ i.e the one with associated Legendre Transf.

$$r \longmapsto \begin{cases} v_r(f), & r \geq 0 \\ -\infty & r < 0 \end{cases} \qquad v_r(f) = \inf_{i \in \mathbb{Z}} (v(a_i) + ir)$$

this definition can be extended to power series

$f \in \mathcal{O}_K[[T]]$

$\sum_{i \geq 0} a_i T^i$

Theorem (Lazard) Let $f \in \mathcal{O}_K[[T]]$, $\lambda < 0$ a slope of $Newt(f)$ then there exists some $\alpha \in \hat{\overline{K}}$ with $f(\alpha) = 0$ and $v(\alpha) = -\lambda$

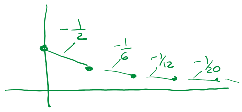
Note: here there is no assumption of convergence of f in a ball. If this is done, then the $Newt(f)$ has finitely many edges of finite length and negative slope $\Rightarrow$ finitely many zeros.

If $\sum a_i x^i \quad v(a_i) > 0$ but $v(a_i) \longrightarrow 0$

slope $\Rightarrow$ finitely many zeros.

If $\sum a_i x^i$ $\quad v(a_i)>0$ but $v(a_i)\longrightarrow 0$

Infinitely many zeros.



(Think about this!) think about $\mathcal{O}_{\mathbb{C}_p}[[T]]$

$p + \sqrt[2]{p}\,T + \sqrt[3]{p}\,T^2 + \cdots\cdots$ $\quad v(p)=1 \quad v(\sqrt[2]{p})=\frac{1}{2} \quad v(\sqrt[m]{p})=\frac{1}{m}$

Generalization to $\mathbb{A}_{\inf} = W_{\mathcal{O}_E}(\overline{\mathcal{O}_F}) \Rightarrow f = \sum_{i=0}^{\infty} [a_i]\pi^i$ $\quad a_i \in \mathcal{O}_F$, v its valuation

$0 \leq v_r(f) := \inf_{i\in\mathbb{N}} (v(a_i) + ir)$

$Newt(f) :=$ convex decreasing piecewise linear function with

Legendre transform $\quad r \longmapsto \begin{cases} v_r(f) & \text{if } r \geq 0 \\ -\infty & \text{if } r<0 \end{cases}$

this is increasing, concave.

Again $Newt(fg) = Newt(f) * Newt(g)$

deduced from $v_r(fg) = v_r(f) + v_r(g)$

① $f = [\overset{\neq 0}{a}] \Rightarrow v_r(f) = v(a)$

② $v_r(f+g) \geq \inf\big(v_r(f), v_r(g)\big)$

$f = \sum [a_i]\pi^i \qquad g = \sum [b_i]\pi^i$

Fix k such that $kr > v_r(f), v_r(g)$ $\quad (a) = (a_0, \dots, a_k) \quad (b) = (b_0, \dots b_k)$ in $\mathcal{O}_F \quad (c) = (a, b)$ in $\mathcal{O}_F$

$f+g = [a_0+b_0] + [c_1]\pi + \cdots [c_k]\pi^k + \cdots \qquad c_i \in (c)$

$\in [c]\mathbb{A}_{\inf} + \pi^{k+1}\mathbb{A}_{\inf} \qquad v(c) = \inf\big(v(a), v(b)\big)$

$v(f+g) \geq v(c) = \inf(v(a_i), v(b_j)) = \inf(v_r(f), v_r(g))$ ✓ $\qquad = \inf(v(a_i), v(b_j))$

③ $v_r(fg) \geq v_r(f) + v_r(g)$

$f\cdot g = \sum_{n+m\leq k} [a_n][b_m]\pi^{n+m} + \pi^{k+1}(\cdots)$

$v_r(f\cdot g) \geq \inf\Big(v_r\big(\sum [a_n b_m]\pi^{n+m}\big), \boxed{v_r(e) + r\cdot(k+1)}\Big)$

$\inf\big(v(a_n) + v(b_m) + r(n+m)\big)$

$\vee$

$A = \underbrace{\inf(v(f), v(g))}$ $\quad \swarrow$ greater for $k \gg 0$

$\exists\, k \geq 0$ s.t. $r(k+1) > A$

④ It remains to check $=$ (p. 36)

In conclusion: $v_r : \mathbb{A}_{\inf} \longmapsto \mathbb{R} \cup \{\infty\}$

$r>0 \qquad \sum_{i=0}^{\infty} [a_i]\pi^i \longmapsto \inf_i (v(a_i) + ri)$

$0 \longmapsto \infty$

is a valuation: $v_r(fg) = v_r(f) + v_r(g)$

$v_r(f+g) \geq \min\big(v_r(f), v_r(g)\big)$

is a valuation : $v_r(fg) = v_r(f) + v_r(g)$
$v_r(f+g) \geq \min(v_r(f), v_r(g))$

If $\rho \in (0,1)$ $\quad |f|_\rho = \sup_{i\in\mathbb{N}} |a_i|\rho^i \qquad |\pi|_\rho := \rho = q^{-r} \qquad |a_i| = q^{-v(a_i)}$

$\Rightarrow |\ \ |_\rho = q^{-v_r(\ )}$

This can be extended uniquely to $B^b = \mathbb{A}_{\inf}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$

$$= \left\{ \sum_{i\gg-\infty} [a_i]\pi^i,\ a_i \in \mathcal{O}_F,\ \inf_i^{\flat} v(a_i) > -\infty \right\}$$

Recall that ϖ is a pseudo uniformizer of $\mathcal{O}_F$: hence $\varpi^i \to 0$
$\left(\Rightarrow v(\varpi^i) \to \infty \Rightarrow v\left(\frac{1}{\varpi^i}\right) \to -\infty\right)$

$$\sum_{i\geq 0} \frac{1}{[\varpi^i]}\pi^i \notin B^b$$

We are interested in closed prime ideals of $\mathbb{A}_{\inf}$ of the type (d) with $d = [a_0] + \pi u$ with $a_0 \in m_F$ and $u \in \mathbb{A}_{\inf}^*$

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{A}_{\inf} & \overset{\text{open}}{\hookleftarrow} & \operatorname{Spec} B^b \\ \cup\!\!| & & \cup\!\!| \\ \mathcal{P} = \underset{\text{ideals}}{\text{closed prime}} & \longleftrightarrow & \mathcal{P} \cap \operatorname{Spec} B^b = \{(0), (d)\} \end{array}$$

NB Up to a translation, also Newton polygons of el. of B^b are well understood!

$f \in B^b \qquad f = \frac{1}{\pi^n [\varpi]^m} \tilde{f} \qquad \tilde{f} \in \mathbb{A}_{\inf}$

Theorem (F.F) Let $f \in \mathbb{A}_{\inf}$ and let $\lambda < 0$ be a slope of $\mathrm{Newt}(f)$ then there exists an el. $a \in \mathcal{O}_F$ s.t. $v(a) = -\lambda$ and $f = (x - [a])g$ for ~~some~~ $g \in \mathbb{A}_{\inf}$.

Idea: $f = \sum_{n\geq 0} [a_n]\pi^n \quad , \quad a_0 \neq 0 \quad$ WLOG

$$f = \lim \underbrace{\sum_{n=0}^{d} [a_n]\pi^n}_{f_d}$$

Consider $X_d = \{ y \in |Y| \text{ s.t. } f_d(y) = 0 \quad v_y(\pi) = -\lambda \}$ and construct a sequence of Cauchy $(y_i) \in \prod X_d \rightsquigarrow y_i \rightsquigarrow y$ zero of f.

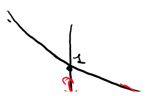
Def Let $B^b = \mathbb{A}_{\inf}\left[\frac{1}{\pi[\varpi]}\right]$ and $I \subseteq (0,\infty)$ an interval.

B_I := completion on B^b for the family of valuations v_r, $r \in I$.

$v_r : B^b \longrightarrow \mathbb{R} \cup \{\infty\} \qquad v_r\left(\sum [a_i]\pi^i\right) = \inf_i (v(a_i) + ir)$

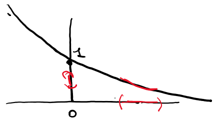
If $\rho = q^{-r} = \left(\frac{1}{q}\right)^r \quad |x|_\rho = q^{-v_r(x)} \quad \left[\ |[1]| = q^0 = 1 \ , \ |\pi| = q^{-r} = \frac{1}{q^r} < 1 \right]$

$r \in (0,\infty) \Rightarrow 0 < \rho < 1$



$r \in (0,\infty) \Rightarrow 0 < \rho < 1$

$r \in I \iff \rho \in q^{-I} = I'$



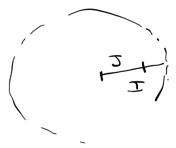
B_I = completion of B^b w.r.t. v_r, $r \in I$
$\quad |\ |_\rho \quad \rho \in I'$

- $B_I = \varprojlim_{U \in \mathcal{F}} B^b/U \qquad \mathcal{F} = \left\{ \bigcap_{i=1}^{n} v_{r_i}^{-1}([m_i, \infty)),\ n, r_i, m_i \right\}$

NB If $x, y \in v_{r_i}^{-1}([m_i,\infty)) \Rightarrow x+y \in v_{r_i}^{-1}([m_i,\infty))$
It is a subgroup. Hence the topology is linear
Hence the above subgroup is open.

- $B := B_{(0,\infty)} \qquad I = (0,\infty)$

- $J \subseteq I \qquad B_J \hookleftarrow B_I \qquad B_I = \varprojlim_{\substack{J \subseteq I \\ \text{compact}}} B_J$



⚠ FF usano la notazione con le norme
$I \subseteq (0,1) \qquad B_I$ = completamento rispetto alle norme $|\ |_\rho$
al variare di $\rho \in I$
$|f|_\rho = \sup_n \{ |a_n|_{\mathcal{O}_F} \rho^n \}$

$I = [\overset{\rho_1 \text{ in FF}}{a}, \overset{\rho_2 \text{ in FF}}{b}] \subseteq (0,1)$

$B_I = \widehat{\mathbb{A}_{\inf}\left[\frac{[t_a]}{\pi}, \frac{\pi}{[t_b]}\right]}\left[\frac{1}{\pi}\right] = B_{[a,b]}$

$|t_a| = a = q^{-v(t_a)}$

$|t_b| = b = q^{-v(t_b)}$

b in FF

Theorem (9.3) Let $I \subseteq (0,\infty)$ compact (multipl. $I \subseteq (0,1)$ compact)

then B_I is a PID

Proof By general C.A. it suffices to prove that:

- B_I is an integral domain. Suffices to prove that B_J is integral with J compact.
$B^b \in \mathrm{Frac}(\mathbb{A}_{\inf})$ is an integral domain $\qquad f \cdot g = 0 \qquad f, g \in B_{[a,b]}$
Passing to completion one can loose integrity $\qquad$ can find a, b

$\mathbb{A}_{\inf}\left[\frac{[t_a]}{\pi}, \frac{\pi}{[t_b]}\right]$ standard calculations.

- B_I is UFD
- each irreducible el of B_I generates a maximal ideal

} hard part

Assume we have proved that:

$f \in B_I$ $f \neq 0$, $f \notin B_I^\times$ then $f = u\,\xi_{y_1} \cdots \xi_{y_n}$ $u \in B_I^\times$

$y_1, \dots, y_n \in |Y_I| = \{ y \in |Y| \mid v(\pi(y)) \in I \}$ ↑ multiple

$$A_{inf} \longrightarrow \frac{A_{inf}}{p_y} \underset{\mathcal{O}_{C_y}}{\simeq} \text{untilt di } \mathcal{O}_F$$

$\pi \longmapsto \cdot$ avrà una valutazione

Ora ξ_y sono i generatori dell' id. max.
$= [a_y] - \pi u_y$

If f irreducible $\Rightarrow$ $f = u\,\xi_y$ $\exists y$ $\Rightarrow$ $(f) = (\xi_y) = M_y$ maximal

because Lemma 9.5 Ans. says that $\theta_y : B^b \longrightarrow C_y$ extends to $\theta_y : B_I \to C_y$ and $\ker \theta_y$ is still generated by ξ_y (the one that generates the kernel of $A_{inf} \twoheadrightarrow C_y$

then: for proving that $f = u\,\xi_{y_1} \cdots \xi_{y_n} \in B_I$

By approximation argument for the analogous result on B^b and A_{inf} if $\lambda < 0$ is a slope of $Newt(f)$ then $\exists$ $a \in m_F$ with $v(a) = -\lambda$ and $f = ([a] - \pi u)g$ for some $g \in B_I$.

If I compact $Newt(f)$ finitely many slopes. Iterating.

$$f = ([a_1] - \pi u_1)\underbrace{(\ \cdots\ )}_{\text{one slope less}}(\ \cdots\ )\,u$$

↑ no slopes $\Rightarrow$ invertible !