Part 1: P is graded factorial and generated in degree 1

1 Notation and recollections

Let p be a prime number, $E/\mathbb{Q}_p$ a finite extension, $\pi \in \mathcal{O}_E$ a uniformiser, $q := \#(\mathcal{O}_E/(\pi))$.

Fix an algebraically closed complete non-archimedean extension of $\mathbb{F}_q$ whose topology is defined by a valuation $\nu\colon F \to \mathbb{R} \cup \{\infty\}$.

One is interested in classifying the isomorphism classes of untilts of F (over E) modulo the Frobenius. Recall that these are automatically algebraically closed, since F is. We will not consider the trivial untilt of F (so all the untilts will have characteristic 0).

Recall that there is a natural identification

$$|Y| := \{\text{primitive elements of degree 1}\}/\mathbb{A}_{\text{inf}}^{\times} \to \{\text{(non-trivial) untilts of } F\}/\cong \tag{1}$$

taking (ξ_y) to a pair (C_y, ι), with $C_y = \frac{\mathbb{A}_{\text{inf}}}{(\xi_y)}\left[\frac{1}{\pi}\right]$ and ι suitably defined. The map $\mathbb{A}_{\text{inf}} \to C_y$ will be denoted by θ_y.

The inverse of the previous bijection takes $(C, \iota\colon \mathcal{O}_C^{\flat} \xrightarrow{\sim} \mathcal{O}_F)$ to the class of any primitive element of degree 1 generating

$$\ker\left(\mathbb{A}_{\text{inf}} \xrightarrow[\sim]{W(\iota^{-1})} W_{\mathcal{O}_E}(\mathcal{O}_C^{\flat}) \xrightarrow{\sharp} \mathcal{O}_C\right),$$

where $W_{\mathcal{O}_E}(\mathcal{O}_C^{\flat}) \xrightarrow{\sharp} \mathcal{O}_C$ maps $\sum [a_i]\pi^i$ to $\sum a_i^{\sharp}\pi^i$.

Since, given ξ_y and $\xi_{y'}$ in $\mathbb{A}_{\text{inf}}$ primitive of degree 1, $\xi_{y'} \in \xi_y\mathbb{A}_{\text{inf}}$ if and only if $\xi_{y'}\mathbb{A}_{\text{inf}} = \xi_y\mathbb{A}_{\text{inf}}$, we shall often tacitly identify $|Y|$ with the set of ideals of $\mathbb{A}_{\text{inf}}$ generated by primitive elements of degree 1. We shall also sometimes write (a bit sloppily) ξ_y for $(\xi_y) \in |Y|$, or also $y \in |Y|$ for (ξ_y).

Fix a pseudo-uniformiser $\varpi \in \mathcal{O}_F$ and let $B^b = \mathbb{A}_{\text{inf}}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$; for any interval $I \subseteq]0, \infty[$, let B_I be the completion of B^b with respect to the family of Gauss norms $\{\nu_r\}_{r \in I}$. Let $d(0, -)\colon |Y| \to \mathbb{R}_{\geq 0}$ be the "distance from zero", given by $d(0, y) = \nu(a)$ if y is the untilt corresponding to the primitive element $[a] - u\pi$; set further $|Y_I| = \{y \in |Y| \mid d(0, y) \in I\}$.

Theorem 0.0.1. *If $I \subseteq]0, \infty[$ is a compact interval, B_I is a PID. Moreover, the map*

$$|Y_I| \to \operatorname{Specm} B_I,$$

$\xi_y \mapsto \xi_y B_I$ is a bijection.

Recall that by definition

$$B_{\text{dR},y}^{+} := \varprojlim_n \frac{\mathbb{A}_{\text{inf}}}{(\xi_y^n)}\left[\frac{1}{\pi}\right];$$

this is an equal-characteristic discrete valuation ring with uniformiser ξ_y and residue field C_y. Its valuation will be denoted by

$$\mathrm{ord}_y\colon B^+_{\mathrm{dR},y} \to \mathbb{N} \cup \{\infty\}.$$

Theorem 0.0.2. *For any interval $I \subseteq]0,\infty[$ and any ξ_y, the ξ_y-adic completion of B_I is canonically identified with $B^+_{\mathrm{dR},y}$.*

Finally, set $B = B_{]0,\infty[}$. Recall that the Frobenius $\varphi\colon \mathbb{A}_{\mathrm{inf}} \to \mathbb{A}_{\mathrm{inf}}$ extends to $\varphi\colon B \to B$ by continuity. One sets

$$P = \bigoplus_{d \geq 0} B^{\varphi = \pi^d},$$

$X = \mathrm{Proj}\, P$.

We shall use the following fact:

Proposition 0.0.3. *One has $P_0 = E$, $P_n = 0$ if $n < 0$.*

The thrust of the first part of this talk will be to prove that P is graded factorial and is generated in degree 1. This makes use of the theory of divisors.

2 Divisors

Definition 0.0.4. Let $I \subseteq]0,\infty[$ be an interval. We let $\mathrm{Div}^+(|Y_I|)$ be the monoid of formal sums $\sum_{y \in |Y_I|} n_y[y]$ with non-negative integral coefficients $n_y \in \mathbb{N}$ such that for any compact interval $J \subseteq I$, $n_y = 0$ for almost all $y \in |Y_J|$.

Notice that $\mathrm{Div}^+(|Y_I|)$ is naturally a partially ordered monoid.

For any interval $I \subseteq]0,\infty[$, one can associate a divisor $\mathrm{div}(f) \in \mathrm{Div}^+(|Y_I|)$ to any $f \in B_I \setminus \{0\}$:

Lemma 0.0.5. *For any interval $I \subseteq]0,\infty[$, taking $f \in B_I \setminus \{0\}$ to $\sum_{y \in |Y_I|} \mathrm{ord}_y(f)[y]$ defines a morphism of monoids $\mathrm{div}\colon B_I \setminus \{0\} \to \mathrm{Div}^+(|Y_I|)$. Here we regard f as an element of $B^+_{\mathrm{dR},y}$ via the canonical map $B_I \to B^+_{\mathrm{dR},y}$.*

Proof. One has to prove that the map is well-defined, i.e. that for any compact interval $J \subseteq I$ and any $f \in B_I \setminus \{0\}$, $\mathrm{ord}_y(f) = 0$ for almost all $y \in |Y_J|$. The canonical map $B_I \to B^+_{\mathrm{dR},y}$ factors through $B_J \to B^+_{\mathrm{dR},y}$, so the assertion follows from the next remark. □

Remark 0.0.6. Let $J \subseteq]0, \infty[$ be a compact interval. By Theorem 0.0.1 and since every non-zero non-invertible element in a PID is a product of irreducible elements, for any $f \in B_J$ one can write $f = u\xi_{y_1}^{e_1} \times \cdots \times \xi_{y_n}^{e_n}$ for some $u \in B_J^\times$, $\xi_{y_i} \in |Y_J|$ with $\xi_{y_i} \notin \mathbb{A}_{\inf}^\times \xi_{y_j}$ if $i \neq j$ and positive integers $e_1, \ldots, e_n$. Then for any $y \in |Y_J|$, $\mathrm{ord}_y(f) = e_i$ if $y \in \xi_{y_i}\mathbb{A}_{\inf}^\times$ and 0 otherwise: indeed, by Theorem 0.0.2 $B_J \to B_{\mathrm{dR},y}^+$ is the adic comlpetion, which induces, for each non-negative integer n, an isomorphism

$$\frac{B_J}{(\xi_y^n)} \xrightarrow{\sim} \frac{B_{\mathrm{dR},y}^+}{\xi_y^n B_{\mathrm{dR},y}^+};$$

since the exponent of ξ_y in the factorisation of f in B_J is the highest n for which $f \in (\xi_y^n)$ and, similarly, $\mathrm{ord}_y(f)$ is the highest m for which $f \in \xi_y^m B_{\mathrm{dR},y}^+$, the assertion follows.

Theorem 0.0.7 (FF 2.7.4)**.** *Let $I \subseteq]0, \infty[$ be an interval.*

1. *For $f, g \in B_I \setminus \{0\}$, one has $\mathrm{div}(f) \geq \mathrm{div}(g)$ if and only if $f \in gB_I$.*

2. *The morphism $\mathrm{div}\colon (B_I \setminus \{0\})/B_I^\times \to \mathrm{Div}^+(|Y_I|)$ is injective.*

Proof. Notice that the first point implies the second one because $B_I = \varprojlim_{J \subseteq I \text{compact}} B_J$ is a domain.

The first point follows from the previous remark if I is compact; if not, write it as a union of compact intervals J_n and let $\rho_n\colon B_I \to B_{J_n}$ be canonical maps. It is clear that $\mathrm{div}(f) \geq \mathrm{div}(g)$ if $f \in gB_I$; conversely, if $\mathrm{div}(f) \geq \mathrm{div}(g)$, then for each n there exits a unique $h_n \in B_{J_n}$ with $\rho_n(f) = h_n\rho_n(g)$. By uniqueness, the h_n define a compatible sequence $h := (h_n)_n \in \varprojlim_{J \subseteq I \text{compact}} B_J = B_I$ satisfying $f = hg$. □

The next step is defining a Frobenius action on $\mathrm{Div}^+(|Y|)$ (i.e. a $(\mathbb{Z}, +)$-action via the Frobenius map). For an untilt y defined by a primitive element of degree 1 $\xi_y = [a] + u\pi$, define $\varphi^*(y)$ to be the untilt corresponding to $\varphi(\xi_y)$. First, a preliminary

Remark 0.0.8. For any $y = ([a] - u\pi) \in |Y|$ one has $d(0, \varphi^*(y)) = \nu(a^q) = q\nu(a) = qd(0, y)$; hence for any interval $I \subseteq]0, \infty[$ the operation φ^* just defined produces a bijection $|Y_I| \xrightarrow{\sim} |Y_{qI}|$.

Lemma 0.0.9. *The mapping $\varphi^*\colon |Y| \to |Y|$ $y \mapsto \varphi^*(y)$ on $|Y|$ extends to a $(\mathbb{Z}, +)$-action on $\mathrm{Div}^+(|Y|)$.*

Notation 0.0.10. For any integer n, we write

$$(\varphi^*)^n\colon \mathrm{Div}^+(|Y|) \to \mathrm{Div}^+(|Y|)$$

for the action by n on $\mathrm{Div}^+(|Y|)$; in particular $\varphi^*([(\xi_y)]) = [(\varphi(\xi_y))]$.

Proof. One has to prove that the condition on the support of divisors is respected. This follows from the previous remark: indeed, for any

$$D = \sum_{y \in |Y|} n_y[y] \in \mathrm{Div}^+(|Y|)$$

and compact interval $J \subseteq]0, \infty[$,

$$\mathrm{Supp}(\varphi^*(D)) \cap |Y_J| = \{\varphi^*(y) \in |Y_J| \mid n_y \neq 0\};$$

this set is in bijection with $\{y \in |Y_{\frac{1}{q}J}| \mid n_y \neq 0\} = \mathrm{Supp}(D) \cap |Y_{\frac{1}{q}J}|$, which is finite. □

Remark 0.0.11. Anschuetz adopts a different convention, defining φ^* to be what I have called $(\varphi^*)^{-1}$; it seems to me that, contrary to what he writes, the equation (2) below is not valid for his choice of notation, and one would have to write $(\varphi^*)^{-1}$ where I wrote φ^*. Fargues and Fontaine do not make their choice explicit, though their notation seems to suggest that they adopt the same convention as above. This is ultimately inconsequential, as one is interested in Frobenius-invariant elements.

Notation 0.0.12. We denote by $\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$ the set $\mathrm{Div}^+(|Y|)^{\varphi}$ of divisors D which are invariant under the action above (i.e. such that $\varphi^*(D) = D$).

Remark 0.0.8 also implies that for any $y \in |Y|$, the formal sum

$$\sum_{n \in \mathbb{Z}} \varphi^{*n}[y]$$

defines an element of $\mathrm{Div}^+(|Y|)$, which clearly belongs to $\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$. This immediately implies the following important

Lemma 0.0.13. *The monoid* $\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$ *is the free abelian monoid over* $|Y|/\varphi^{\mathbb{Z}}$ *via the injection*

$$\begin{aligned} |Y|/\varphi^{\mathbb{Z}} &\hookrightarrow \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}), \\ y \mod \varphi &\mapsto \sum_{n \in \mathbb{Z}} \varphi^{*n}[y]. \end{aligned}$$

In particular, any $D \in \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$ can be written uniquely as

$$D = \sum_{y \mod \varphi \in |Y|/\varphi^{\mathbb{Z}}} n_y \sum_{l \in \mathbb{Z}} (\varphi^*)^l[y],$$

with $n_y \in \mathbb{N}$ for any $y \mod \varphi$. Let us stress that the sum on the left in the equation above runs over the *equivalence classes* of points of $|Y|$ under the Frobenius action.

With this at hand, we can at last define a degree function on the Frobenius-invariant divisors:

Definition 0.0.14. Let $\deg\colon \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}) \to \mathbb{N}$ the morphism of (additive) monoids such that, for any D as in the previous paragraph,

$$\deg(D) = \sum_{y \mod \varphi} n_y.$$

Notice that the previous sum is finite because $n_y = 0$ for almost all $y \in |Y_{[a,aq]}|$, where a is any positive real number.

3 The main theorem

Since, for each $d \geq 0$ and $x \in P_d$,

$$\varphi^*(\mathrm{div}(x)) = \mathrm{div}(\varphi(x)) = \mathrm{div}(\pi^d) + \mathrm{div}(\varphi(x)) = \mathrm{div}(\varphi(x)) \tag{2}$$

because π is invertible in $B^+_{\mathrm{dR},y}$, $\mathrm{div}\colon B_I \setminus \{0\} \to \mathrm{Div}^+(|Y|)$ induces a morphism of monoids

$$\mathrm{div}\colon \bigcup_{d\geq 0}(P_d \setminus \{0\})/E^\times \to \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}). \tag{3}$$

Theorem 0.0.15 (FF 6.2.7)**.** *The morphism* (3) *is an isomorphism of monoids.*

Notice that we are not (yet) claiming that div is a *graded* morphism.

Proof (injectivity): Let $x \in P_d$, $x' \in P_{d'}$ with $d' \geq d$ and $\mathrm{div}(x) = \mathrm{div}(x')$. By Theorem 0.0.7, this implies that $x = ux'$ for some $u \in B^\times$. Moreover, as

$$\pi^d ux' = \pi^d x = \varphi(x) = \varphi(u)\varphi(x') = \pi^{d'}\varphi(u)x',$$

$u \in P_{d-d'}$, which, since $d' \geq d$, is 0 unless $d = d'$. It ensues that $d = d'$ and $u \in P_0^\times = E^\times$.

We shall prove surjectivity only in the case when $E = \mathbb{Q}_p$, which we assume for the rest of the section.

We shall need some auxiliary results, which are interesting in their own right.

Construction 0.0.16. Let $(C, \iota\colon \mathcal{O}_C^\flat \to \mathcal{O}_F)$ be a (non-trivial) untilt of F (recall that $C = \bar{C}$). Fix an element $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_{C}^\flat,$, where ζ_p is a primitive p-th root of unity (and thus ζ_{p^n} is a primitive p^n-th root of unity). Associate to (C, ι) the element $\iota(\varepsilon) \in (1 + \mathfrak{m}_F) \setminus \{1\}$.

We now set to relate $\iota(\varepsilon)$ to $\iota(\varepsilon')$ for a different choice of roots of unity: more precisely, consider $\varepsilon' = (1, \zeta'_p, \ldots)$ with ζ'_p a primitive p-th root of 1; for each n there then exists

$$a_n \in \frac{\mathbb{Z}}{(p^n)}$$

with $\zeta'_{p^n} = \zeta_{p^n}^{a_n}$, and the a_n yield an element $(1, a_1, \ldots) \in \mathbb{Z}_p^\times$. This defines a $\mathbb{Z}_p^\times$-action on the subset of $\mathcal{O}_C^\flat$ $\{(1, \zeta_p, \ldots) \in \mathcal{O}_C^\flat \mid \zeta_p \text{ is a primitive } p\text{-th root of unity}\}$ with

$$a.\varepsilon = (1, \zeta_p^{a_1}, \zeta_{p^2}^{a_2}, \ldots).$$

Such action corresponds via ι to the $\mathbb{Z}_p^\times$-action on $(1 + \mathfrak{m}_F) \setminus \{1\}$ given by

$$a.x = \sum_{n=0}^{\infty} \binom{a}{n} (x-1)^n.$$

We omit any further details, and just observe that mapping (C, ι) to the class of $\iota(\varepsilon)$ modulo $\mathbb{Z}_p^\times$ then defines a map of sets

$$\frac{\{\text{untilts of F}\}}{\cong} \to \frac{(1 + \mathfrak{m}_F) \setminus \{1\}}{\mathbb{Z}_p^\times} : \tag{4}$$

indeed, by the definition of isomorphism of untilts, changing untilt in the isomorphism class of (C, ι) produces the same result as taking a different $\varepsilon \in \mathcal{O}_C^\flat$.

Remark 0.0.17. Let us observe that, given $x \in 1 + \mathfrak{m}_F \setminus \{1\}$, in order to check that (the class of) x is the image of some untilt $(C, \iota \colon \mathcal{O}_C^\flat \to \mathcal{O}_F)$ under the previous map, it is sufficient (though perhaps not necessary) to show that $(\iota^{-1}(x)^{1/p})^\sharp$ is a primitive p-th root of 1, since, unwinding, this amounts precisely to the fact that one can take $\varepsilon = \iota^{-1}(x)$ (notation as above).

Proposition 0.0.18. *1. For any $x \in (1 + \mathfrak{m}_F) \setminus \{1\}$, the element*

$$\xi_x := 1 + [x^{1/p}] + \cdots + [x^{(p-1)/p}] \in \mathbb{A}_{\text{inf}}$$

is primitive of degree 1*.*

2. *The map* (4) *is bijective; its inverse takes the class of $x \in (1 + \mathfrak{m}_F) \setminus \{1\}$ to the untilt defined by ξ_x.*

Proof sketch: 1: One has to prove that $\xi_x = [a] + up$ with $a \in \mathfrak{m} \setminus \{0\}$ and u invertible. The image of ξ_x under $\mathbb{A}_{\text{inf}} = W(\mathcal{O}_F) \to W(\mathcal{O}_F/\mathfrak{m}_F) = W(k)$ is p, whereby one can write $\xi_x = [a] + up$ with u invertible and $a \in \mathfrak{m}$. If a were 0, then reducing modulo p one would get $\xi_x = 0 \mod p$ and thus $0 = 1 + x^{1/p} + \cdots x^{(p-1)/p}$; as F has characteristic p, this implies that $1 + x + \cdots + x^{p-1} = 0$, whence $x = 1$, a contradiction.

2: Letting $(C_{\xi_x}, \iota \colon \mathcal{O}_{C_{\xi_x}}^\flat \to \mathcal{O}_F)$ be the untilt associated to ξ_x, it suffices to prove that $\varepsilon := \iota^{-1}(x) = (1, \zeta_p, \ldots)$ for some primitive p-th root of unity ζ_p (see the previous remark).

In other words, one has to check that $(\varepsilon^{1/p})^\sharp$ is a root of the p-th cyclotomic polynomial. By the definitions of ξ_x and θ_{ξ_x}, this amounts precisely to the canonical map

$$\theta_{\xi_x} : \mathbb{A}_{\mathrm{inf}} \to C_{\xi_x}$$

annihilating ξ_x, which holds by construction. □

Interlude: logarithms

Recall that for a complete non-archimedean field C of mixed characteristic $(0, p)$, the logarithm is the morphism of monoids

$$\log\colon (1 + m_C, \cdot) \to (C, +)$$

with

$$\log(z) = \sum_{i=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}.$$

One can similarly define a morphism of monoids

$$\log[-]\colon (1 + \mathfrak{m}_F, \cdot) \to (B, +)$$

with

$$\log[x] = \sum_{i=1}^{\infty} (-1)^{n-1} \frac{([x]-1)^n}{n}.$$

It is important to notice that the image of $\log[-]$ lies in $B^{\varphi=p}$, because

$$\varphi(\log[x]) = \log[x^p] = p \log[x].$$

Lemma 0.0.19. *For any untilt (C, ι) of F, the square*

$$\begin{array}{ccc} 1 + \mathfrak{m}_F & \xrightarrow{\log[-]} & B^{\varphi=p} \\ \downarrow{\scriptstyle (-)^\sharp \circ \iota^{-1}} & & \downarrow{\scriptstyle \theta} \\ 1 + \mathfrak{m}_C & \xrightarrow{\log} & C \end{array}$$

commutes.

Proof. Notice first that the left vertical map is well-defined since if $z \in \mathfrak{m}_{C^\flat}$, $(1+z)^\sharp = \lim_{n\to\infty} (1 + z_n)^{p^n}$.

The commutativity follows at once from the equalities

$$\theta(\log[x]) = \theta\left(\sum_{n\geq 1} \frac{([x]-1)^n}{n}\right) = \sum_{n\geq 1} \frac{(\varepsilon^\sharp - 1)^n}{n} = \log(\varepsilon^\sharp)$$

for $x \in 1 + \mathfrak{m}_F$ and $\varepsilon = \iota^{-1}(x)$. □

We shall also need the following lemma, which we do not prove.

Lemma 0.0.20. *For any complete non-archimedean field C of mixed characteristic $(0,p)$, the logarithm induces a bijection*

$$\{z \in C \mid |z-1|_C < |p|_C^{1/p}\} \xrightarrow{\sim} \{z \in C \mid |z|_C < |p|_C^{1/p}\}.$$

Corollary 0.0.21. *If in addition C is algebraically closed,*[1] *then* $\log\colon 1+\mathfrak{m}_C \to C$ *is surjective.*

Proof. Given $z \in C$, one has $|p^n z|_C < |p|_C^{1/p}$ for $n \gg 0$; hence by the previous lemma there exists $x \in 1+\mathfrak{m}_C$ with $\log x = p^n z$. Letting $w \in 1+\mathfrak{m}_C$ be a p^n-th root of x, one then has

$$p^n \log w = \log x = p^n z,$$

whence $\log w = z$. $\square$

For any untilt (C,ι), the map $(-)^\sharp$ is surjective; hence the previous lemma and Lemma 0.0.19 imply the following corollary, which will be used in the second part of the talk:

Corollary 0.0.22. *For any untilt (C,ι), the canonical map $\theta\colon B^{\varphi=p} \to C$ is surjective.*

We can finally come back to the proof of the main theorem.

Proof that (3) *is surjective ($E = \mathbb{Q}_p$):* It clearly suffices to show that for each $(\bar{C},\bar{\iota}) = \bar{y} \in |Y|$, there exists $t \in B^{\varphi=p}$ with

$$\mathrm{div}(t) = \sum_{n\in\mathbb{Z}} (\varphi^*)^n[\bar{y}].$$

Let $\bar{x} = \bar{\iota}(\bar{\varepsilon})$ (notation as above) correspond to $\bar{y}$ under the map (4). We claim that one can take $t = \log[\bar{x}]$.

Step 1. *For each $y \in |Y|$ in the same Frobenius orbit as $\bar{y}$,* $\mathrm{ord}_y(\log[\bar{x}]) = 1$.

Observe first that since $[\bar{x}^{1/p}]$ is mapped to a primitive p-th root of unity by

$$\theta_{\bar{y}}\colon \mathbb{A}_{\mathrm{inf}} \to C_{\bar{y}} \hookrightarrow B^+_{\mathrm{dR},\bar{y}},$$

$[\bar{x}^{1/p}] - 1$ is invertible in $B^+_{\mathrm{dR},\bar{y}}$; writing

$$\xi_{\bar{y}} = 1 + [\bar{x}^{1/p}] + \cdots + [\bar{x}^{(p-1)/p}] = \frac{[\bar{x}]-1}{[\bar{x}^{1/p}]-1},$$

[1] As is always the case for untilts of F.

this implies that $[\bar{x}]-1$ is (mapped to) a uniformiser in $B^+_{\mathrm{dR},\bar{y}}$, and since

$$\log[\bar{x}] \equiv [\bar{x}]-1 \mod ([\bar{x}]-1)^2$$

by definition, one has $\mathrm{ord}_{\bar{y}}(\log[\bar{x}])=1$.

If $y \in |Y|$ corresponds to $\varphi^n(\xi_{\bar{y}})$ for some $n \in \mathbb{Z}$, then

$$p^n \log[\bar{x}] = \log[\varphi^n(\bar{x})] \equiv [\varphi^n(\bar{x})]-1 \mod ([\varphi^n(\bar{x})]-1)^2;$$

since $[\varphi^n(\bar{x})]-1=\varphi^n([\bar{x}]-1)$ is (mapped to) a uniformiser in $B^+_{\mathrm{dR},y}$ and p is invertible in $B^+_{\mathrm{dR},y}$, one gets $\mathrm{ord}_y(\log[\bar{x}])=1$ as well.

Step 2. *If $y=(C,\iota)$ is an untilt for which $\mathrm{ord}_y(\log[\bar{x}]) \geq 1$, then $(C,\iota) \cong (\bar{C}, \varphi^n \circ \bar{\iota})$ for some $n \in \mathbb{Z}$.*

Let $\varepsilon = \iota^{-1}(\bar{x}) \in \mathcal{O}_C^\flat$. If $\mathrm{ord}_y(\log[\bar{x}]) \geq 1$, then for any $m \in \mathbb{Z}$ one has

$$\log\big((\varepsilon^{p^m})^\sharp\big) = \theta_y(\log[\bar{x}^{p^m}]) = p^m \theta_y(\log[\bar{x}]) = 0,$$

where the first equality holds by Lemma 0.0.19. Therefore, by the injectivity part in Lemma 0.0.20, if $l \in \mathbb{Z}$ is such that $|(\varepsilon^{p^l})^\sharp - 1|_C < |p|_C^{1/p}$, then $(\varepsilon^{p^l})^\sharp = 1$.

Since $\bar{x} \in 1+\mathfrak{m}_F \setminus \{1\}$, the set of such integers l is non-empty and has a minimum n. The equality $(\varepsilon^{p^n})^\sharp = 1$ implies that ε is of the form $\varepsilon^{p^n} = (1, a_1, \cdots)$, and by the minimality of n, $a_1 = (\varepsilon^{p^{n-1}})^\sharp \neq 1$, whereby a_1 is a primitive p-th root of unity in C: consequently, calling F the map in (4), one has $\bar{x}^{p^n} = \iota(\varepsilon^{p^n}) = F((C,\iota))$ (see Remark 0.0.17). On the other hand, by definition one also has $F((\bar{C}, \varphi^n \circ \bar{\iota})) = \varphi^n\big(F((\bar{C},\bar{\iota}))\big) = \bar{x}^{p^n}$; as F is injective (Proposition 0.0.18), one obtains that $(\bar{C}, \varphi^n \circ \bar{\iota})$ is isomorphic to (C,ι), which concludes the proof (up to relating the Frobenius action on the left-hand side of (1) defined above with the one on the right-hand side determined by $(C,\iota) \mapsto (C, \varphi \circ \iota)$, which we will not do). □

4 P is graded factorial and generated in degree 1

As an almost immediate consequence of the main theorem, we get the result we were after:

Theorem 0.0.23. *The E-algebra P is a graded factorial domain generated in degree 1, i.e. for any $d \geq 0$ and $x \in P_d \setminus \{0\}$ there exist unique - up to multiplication by $E^\times$ and reordering - $t_1, \ldots, t_d \in P_1$ such that $x = t_1, \ldots, t_d$.*

More concisely,

$$\bigcup_{d \geq 1} (P_d \setminus \{0\})/E^\times$$

is the free abelian monoids over $P_1 \setminus \{0\}$. Notice that the statement makes sense when $d = 0$: it just means that $x \in E^\times$.

Proof. The case $d = 0$ is part of Proposition 0.0.3. For $d \geq 1$ and $x \in P_d \setminus \{0\}$, let $y \in \mathrm{Supp}(\mathrm{div}(x))$. By the main theorem and its proof, there exists $t \in P_1$ such that

$$\mathrm{div}(t) = \sum_{n\in\mathbb{Z}} (\varphi^*)^n [y],$$

unique up to multiplication by $E^\times$;[2] by Theorem 0.0.7, $x = x't$ for some $x' \in P_{d-1}$. We conclude by induction. $\square$

The same proof also yields the following

Corollary 0.0.24. *The map* div *in* (3) *is graded.*

As P is a graded ring generated in degree 1, for each integer n the sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(n)$ associated to the graded P-module $P(n)$ with $P(n)_d := P_{n+d}$ is a line bundle. We shall prove in the second part that $\mathrm{Pic}(X) \cong \mathbb{Z}$ and any line bundle on X us isomorphic to a (unique) $\mathcal{O}_X(n)$. For the moment, let us record a consequence of the previous result.

Lemma 0.0.25. *For each $n \in \mathbb{Z}$, the natural map $P_n \to \Gamma(X, \mathcal{O}_X(n))$ is an isomoprhism of E-modules.*

More generally, one has the following:

Lemma 0.0.26. *Let R be a graded factorial generated in degree* 1*, $X = \mathrm{Proj}\, R$. Then for any $n \in \mathbb{Z}$, the natural map $\alpha_n \colon R_n \to \Gamma(X, \mathcal{O}_X(n))$ is an isomorphism of R_0-modules.*

The proof is the same as for R being a polynomial ring.

Proof. By definition $\alpha_n(r) = r \in \Gamma(X, \mathcal{O}_X(n))$; more precisely, for any $f \in R_1$ one has a commutative diagram

$$\begin{tikzcd} R_n \arrow[r, "\alpha_n"] \arrow[dr] & \Gamma(X, \mathcal{O}_X(n)) \arrow[d] \\ & \Gamma(D_+(f), \mathcal{O}_X(n)) \end{tikzcd}$$

and $\Gamma(D_+(f), \mathcal{O}_X(n)) = R(n)\left[\frac{1}{f}\right]_0 = \left\{\frac{r}{f^a} \mid \deg(r) = n + a\right\}$. Since R is a domain, α_n is then injective.

[2] Actually, we only showed this for $E = \mathbb{Q}_p$, but this is true in general.

For surjectivity, take two $f, g \in R_1$ such that $(f) \neq (g)$ (if no two such elements exist, the statement is trivial) and let $x \in \Gamma(X, \mathcal{O}_X(n))$. Write $x_{|_{D_+(f)}}$ as $\frac{r}{f^a}$ and $x_{|_{D_+(g)}}$ as $\frac{s}{g^b}$ with, without loss of generality, $f \not| \, r$ and $g \not| \, s$. As R is a factorial domain, the equality

$$\left(\frac{r}{f^a}\right)_{|_{D_+(fg)}} = \left(\frac{s}{g^b}\right)_{|_{D_+(fg)}}$$

implies $rg^b = sf^a$, whence $a = b = 0$, $r = s$ and $\deg(r) = n$. Therefore, $x \in \mathrm{Im}(\alpha_n)$. $\square$