Part 2 : The geometry of X.

Recall that $X = \operatorname{Proj} \bigoplus_{d \geq 0} B^{\varphi = \pi^d}$.

We now want to study some of the main properties of X: in particular we shall prove that X is a Dedekind scheme with $\operatorname{Pic}(X) \simeq \mathbb{Z}$.

The main tool will be the following

Thm [Fundamental exact sequence of p-adic Hodge theory, FF 6.4.1].
Let $y \in |Y|$ and $t \in B^{\varphi=\pi}$ s.t $\operatorname{div}(t) = \sum_{n\in\mathbb{Z}} \varphi^{*n}[y] \in \operatorname{Div}^+(|Y|/\varphi^{\mathbb{Z}})$.
For $d \geq 0$, the natural sequence

$$0 \to t^d E \longrightarrow B^{\varphi=\pi^d} \longrightarrow \frac{B^+_{dR,y}}{\xi_y^d B^+_{dR,y}} \longrightarrow 0$$

is exact [Here $t^dE \to B^{\varphi=\pi^d}$ is the inclusion and
$B^{\varphi=\pi^d} \to \frac{B^+_{dR,y}}{\xi_y^d B^+_{dR,y}}$ the composition $B^{\varphi=\pi^d} \hookrightarrow B \xrightarrow[\xi_y\text{-adic completion}]{\alpha} B^+_{dR,y} \to \frac{B^+_{dR,y}}{\xi_y^d B^+_{dR,y}}$]

Pf Let $x \in B^{\varphi=\pi^d}$ map to 0 in the quotient. Then
$\operatorname{div}(x) \geq d[y] = d\cdot \operatorname{div}(t) = \operatorname{div}(t^d)$.
as both $\operatorname{div}(x)$ and $\operatorname{div}(t^d)$ are in
$\operatorname{Div}^+(|Y|/\varphi^{\mathbb{Z}})$ of degree d, by ~~Thm~~ 0.0.7 they differ
by an element of $E^\times$.
It remains to show that $B^{\varphi=\pi^d} \longrightarrow \frac{B^+_{dR,y}}{\xi_y^d B^+_{dR,y}}$ is surjective.
We shall reduce it to the case $d=1$, which we have already seen for $E=\mathbb{Q}_p$, but we will not prove that

$\cancel{B}$ $B^{\varphi=\pi} \longrightarrow \dfrac{B^+_{dR,y}}{\xi_y B^+_{dR,y}} = C_y$ is surjective in general.

So let $d \geq 2$ and assume the statement holds ~~for~~ in degrees $1, \dots, d-1$. For $x \in B^+_{dR,y}$, let $w \in B^+_{dR,y}$ be s.t. $w^d \equiv x \bmod \xi_y B^+_{dR}$ (recall that $\dfrac{B^+_{dR,y}}{\xi_y B^+_{dR,y}} = C_y$ is algebraically closed) and let $a \in B^{\varphi=\pi}$ be s.t. $\alpha(a) \equiv w \bmod \xi_y B^+_{dR}$. Then $\alpha(a^d) - x \equiv 0 \bmod \xi_y B^+_{dR}$, so that we can write $\alpha(a^d) - x = \alpha(t) x'$ for some $x' \in B^+_{dR,y}$ (recall that, by the very choice of t, $\mathrm{ord}_y(t) = 1$, so $\alpha(t) = \xi_y u$ for some $u \in (B^+_{dR,y})^\times$). By the inductive assumption $\exists b \in B^{\varphi=\pi^{d-1}}$ s.t. $x' \equiv \alpha(b) \bmod \xi_y^{d-1} B^+_{dR,y}$ whereby $a^d - tb \in B^{\varphi=\pi^d}$ and $\alpha(a^d - tb) \equiv x \bmod \xi_y^d B^+_{dR,y}$. $\square$

Denote again by $\theta_y : B \to C_y$ the canonical map.

<u>Corollary</u> [6.4.3] In the notation above, the morphism of graded E-algebras

$$\frac{P}{tP} \longrightarrow \{ f \in C_y[T] \mid f(0) \in E \} =: S \quad \text{is an isomorphism.}$$

$$\sum_{d \geq 0} x_d + tP \longmapsto \sum_{d \geq 0} \theta_y(x_d) T^d$$

(with $\deg(x_d) = d$)

In particular, $\mathrm{Proj}(P/tP) = \{(0)\}$

~~Ex:~~ This tells us that for $t \in B^{\varphi=\pi}$, $V_+(t)$ consists of a single point ("a function of degree 1 vanishes at a single point").

Pf: Denote by β the map in the statement.

$\left(\dfrac{P}{tP}\right)_0 = \dfrac{P_0}{(tP)_0} = \dfrac{P_0}{tP \cap P_0} = P_0 \simeq E$, whereby β is an isomorphism in degree 0.

Surjectivity is clear because for each d one has a commutative triangle

$$\begin{tikzcd}
B^{\varphi=\pi^d} \arrow[r, two heads] \arrow[d] & \frac{B^+_{dR,C_y}}{\xi_y^d B^+_{dR,C_y}} \arrow[dl, two heads] \\
C_y = \frac{B^+_{dR,C_y}}{\xi_y B^+_{dR,C_y}} &
\end{tikzcd}$$

, the horizontal map being surjective by the previous theorem.

If $x \in P_d$, $d \geq 1$, is mapped to 0 by θ_y, $x \equiv 0 \mod (\xi_y) \Rightarrow$ one can write $x_d \equiv t x_{d-1}$ mod $\xi_y^d B^+_{dR,y}$; thus, $x_d - t x_{d-1} \in t^d E \subseteq tP$ by the previous theorem.

In order to show that $\mathrm{Proj}\, S$ consists of a single point, pick a graded prime ideal $\mathfrak{p} \subseteq S$. Being a graded ideal, $\mathfrak{p}$ is generated by homogeneous elements, so if $\mathfrak{p} \neq (0)$, $cT^d \in \mathfrak{p}$ for some $c \in C_y^\times$ and $d \geq 1$. Then $T^{d+1} = (cT^d)(c^{-1}T) \in \mathfrak{p}$, whereby $\mathfrak{p}$ contains the irrelevant ideal (T), and so $\mathfrak{p} \notin \mathrm{Proj}\, S$. $\square$

As P is a graded ring generated in degree 1 (i.e. P is generated by P_1 as a P_0-algebra), for each $n \in \mathbb{Z}$ one has an invertible sheaf $\mathcal{O}(n)$, the sheaf on X associated to the graded P-module $P(n)$, where $P(n)_d = P_{d+n}$ $\forall d \in \mathbb{Z}$.

Thm [FF 6.5.2, 5.2.7). Let $t \in P_1 \setminus \{0\}$, ∞_t the unique point of $V_+(t) \subseteq \mathrm{Proj}\, P = X$, $B_t = P\left[\frac{1}{t}\right]_0$.

~~Then~~ The ring B_t is a PID of Krull dim 1 and $X = \mathrm{Spec}\, B_t \cup \{\infty_t\}$.

In particular, X is a Dedekind scheme.

Pf. In order to show that B_t is a PID, we shall use that a factorial domain whose irreducible elements generate a maximal ideal is a PID.

Let $x \in B_t \neq 0$. One can write $x = \frac{t'}{t^d}$ for some $d \geq 0$, $t' \in P_d$.

By Thm 0.0.23, one can factor t' as a product of elements $t_1, \ldots, t_d \in B^{\varphi = \pi}$, so that $x = \frac{t_1}{t} \cdots \frac{t_d}{t}$: it ensures that the irreducible elements of B_t are those of the form

$\frac{t'}{t}$ for $t' \in P_1 \setminus tE$. Denoting by a bar the reduction

mod $t'P$, one has an isomorphism

$$\frac{B_t}{\left(\frac{t'}{t}\right) B_t} \simeq \left(\frac{P}{t'P}\right)\left[\frac{1}{\bar{t}}\right]_0 .$$

We have previously seen that $\frac{P}{t'P} \simeq \{ f \in G[T] \mid f(0) \in E \} =: S$,

so it is enough to observe that for each $f \in S_1$, $S\left[\frac{1}{f}\right]_0 \simeq G$, in

particular a field. This concludes the proof that B_t is a PID of Krull dimension 1.

One has $X = D_+(t) \cup V_+(t) = \operatorname{Spec} B_t \cup \{\infty_t\}$ by the previous corollary.

Taking $0 \neq t' \in P_1$ s.t. $\infty_{t'} \in \cancel{} D_+(t')$ (this exists as

P is generated in degree 1, so that $X = \bigcup_{t' \in P_1} D_+(t')$ as a topological

space), one gets

$X = \operatorname{Spec} B_t \cup \operatorname{Spec} B_{t'}$, so that X has a finite cover by

spectra of Dedekind domains. Finally, X is irreducible

as it has (0) as its unique generic point

$\square$

Let us denote by $|X|$ the set of closed points of X.

Since $X = \mathrm{Spec}\, B_t \cup \{\infty_t\}$ for $t \in P_1 \setminus \{0\}$, and ∞_t is closed and the maximal ideals of B_t are exactly those generated by elements of the form $\frac{t'}{t}$ for $t' \in P_1 \setminus Et$ (see previous proof), we have canonical identifications $|X| = (P_1 \setminus \{0\})/E^\times \xrightarrow{\sim} |Y|/\varphi^{\mathbb{Z}}$ (comp. with Thm 0.0.15 and Corollary 0.0.24).

Thus the closed points of X do indeed classify the isomorphism classes of untilts of F modulo the Frobenius action.

Lemma If $y \in |Y|$, and x is its image under the canonical map $|Y| \longrightarrow |Y|/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|$, there is a canonical isomorphism

$$\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} B^+_{dR,y}.$$

Pf Let x and y be as in the statement and let $t \in P_1 \setminus \{0\}$ be such that $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} \varphi^{*n}[y]$, i.e. s.t. $V_+(t) = \{x\} \subseteq X$.

Let $t' \in B^{\varphi = \pi} \setminus Et$, so that $x \in \mathrm{Spec}\, B_{t'}$. One has a factorisation

$$\begin{tikzcd} B^{\varphi=\pi} \arrow[r, hook] \arrow[d] & B \arrow[r] & B^+_{dR,y} \\ B_{t'} = B\left[\frac{1}{t'}\right]_0 \arrow[urr] & & \end{tikzcd}$$

(recall that the map $B \to B^+_{dR,y}$ is the ξ_y-adic completion).

The local ring $\mathcal{O}_{X,x}$ is the localisation of $B_{t'}$ at the prime $\left(\frac{t}{t'}\right)$; since $B \to B^+_{dR,y}$ maps t to a uniformiser, the induced map

$\widehat{\mathcal{O}}_{X,x} \longrightarrow B^+_{dR,y}$ maps the uniformiser $\frac{t}{t'}$ to a uniformiser.

Moreover, the fundamental exact sequence of p-adic Hodge theory for $d=1$ reads

$0 \to Et \to B^{\varphi=\pi} \to C_y$, whereby $\frac{\widehat{\mathcal{O}}_{X,x}}{\frac{t}{t'}\widehat{\mathcal{O}}_{X,x}} \simeq \frac{B_{t'}}{\frac{t}{t'}B_{t'}} \simeq C_y$ and the map $\widehat{\mathcal{O}}_{X,x} \to B^+_{dR,y}$ is

an isomorphism since it is a map of dvr's taking a uniformiser to a uniformiser and inducing an isomorphism of residue fields. $\boxtimes$

Definition For $x \in |X|$, let $B^+_{dR,x} := \widehat{\mathcal{O}}_{X,x}$ and $B_{dR,x} = \mathrm{Frac}\, B^+_{dR,x}$.

Denote by $k(X)$ the function field of X, by $\mathrm{Div}(X)$ the group of (Weil) divisors on X, by $\mathrm{ord}_x : \mathcal{O}_{X,x} \to \mathbb{Z} \cup \{\infty\}$ (for $x \in |X|$) the valuation of $\mathcal{O}_{X,x}$ and by $\mathrm{div} : k(X)^\times \longrightarrow \mathrm{Div}(X)$ the usual map $f \longmapsto \sum_{x \in |X|} \mathrm{ord}_x(f)\,[x]$, whose image is denoted by $\mathrm{Prin}(X)$.

There is a degree homomorphism $\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$ taking a divisor $\sum_{x\in|X|} n_x\,[x]$ to $\sum_{x \in |X|} n_x$.

Proposition The degree morphism $\mathrm{Div}(X) \longrightarrow \mathbb{Z}$ factors through $\frac{\mathrm{Div}(X)}{\mathrm{Prin}(X)} =: \mathrm{Cl}(X)$ and the resulting map $\mathrm{Cl}(X) \longrightarrow \mathbb{Z}$ is an isomorphism.

Pf Let us first show that ~~deg(f) = 0~~ $\deg(\mathrm{div}(f)) = 0$ $\forall f \in k(X)^\times$.

By the description of $|X|$ given above (page 7) and since ~~deg is~~ deg and div are homomorphisms, we can assume that

$f = \frac{t'}{t}$ for $t, t' \in B^{\varphi=\pi}$; it is easy to see that

$\mathrm{div}(t'/t) = [\infty_{t'}] - [\infty_t]$ (use the commutative diagram,

$$\begin{tikzcd} B^{\varphi=\pi} \arrow[r] \arrow[d] & B^+_{dR,y} \\ B_{t'} \arrow[r] & \mathcal{O}_{X,x} \arrow[u] \end{tikzcd}$$

the previous lemma and the computation of $\mathrm{ord}_y(t)$, $\mathrm{ord}_y(t')$), which has degree 0.

The induced map $\mathrm{Cl}(X) \to \mathbb{Z}$ is clearly surjective.

In order to prove injectivity, recall that for a Dedekind scheme Z (or more generally, mutatis mutandis, for a noetherian locally factorial integral scheme) morphism of abelian groups from $\mathrm{Div}(Z)$ to the group $\mathrm{Pic}(Z)$ of isomorphism classes of line bundles on Z which ~~sends~~ takes a closed point z to the line bundle $\mathcal{O}_Z(z) := \mathcal{I}_z^\vee$ (the dual of the invertible ideal sheaf cutting out z)

induces an isomorphism $Cl(X) \longrightarrow Pic(X)$)

We now claim that the diagram

$$\begin{tikzcd} Div(X) \arrow[r] \arrow[d] & Pic(X) & \mathcal{O}_X(n) \\ Cl(X) \arrow[r, "\deg"] & \mathbb{Z} \arrow[u] & n \arrow[u, mapsto] \end{tikzcd}$$

commutes, which implies that $\deg : Cl(X) \to \mathbb{Z}$ is injective, (as $Cl(X) \longrightarrow Pic(X)$ is).

It is enough to show that $\forall x \in |X|$, $\mathcal{O}_X(x) \cong \mathcal{O}_X(1)$; but this follows immediately since $\mathcal{O}_X(x)$ and $\mathcal{O}_X(1)$ have a (regular) global section cutting out the effective Cartier divisor x (see [Stack Project, 01X0]), namely the canonical section of $\mathcal{O}_X(x)$ and $t \in B^{\varphi=\pi} = \Gamma(X, \mathcal{O}_X(1))$, where t corresponds to x under the bijection $|X| \xrightarrow{\sim} P_1 \setminus \{0\} / E^\times$.

$\square$