Aim: to introduce the Harder-Narasimhan formalism HN and to show how it applies to vector bundles on X.

Program:

§1: Brief overview of the situation after FF1–FF7

§2: $\mathrm{Pic}(X) \simeq \mathbb{Z}$

§3: Harder-Narasimhan formalism

§4: Applications

Recall:

FF1 Maremma: Let C be a perfectoid field of char. p

$$|X| = \{\text{"untilts of } C\text{"}\}/\varphi^{\mathbb{Z}} = \underbrace{\left\{ (K,\iota) \;\middle|\; \begin{array}{l} K/E \\ K/\mathbb{Q}_p \text{ alg. closed completely valued} \\ \iota: K^\flat \xrightarrow{\sim} C \end{array} \right\}}_{\sim}$$

FF2 Bambozzi: The classification of the possible kernels of $\theta: W(C) \twoheadrightarrow O_K$ provides the classification of the untilts of C.

$\mathrm{Ker}\,\theta = \langle \xi \rangle$ with ξ distinguished

$$\underbrace{\{\text{distinguished elements of } A_{\inf}\}}_{\text{units}} \xleftrightarrow{1:1} \frac{\{\text{untilts of } C\}}{\text{isomorphisms.}}$$

FF3 Baldassarri: the Fontaine-Fargues curve is $|Y| = \{[(K,\iota)]_\sim = y \mid (K,\iota)\}$

FF4 Mazzari: $Y = \mathrm{Spec}\left(A_{\inf}\left[\frac{1}{p[\varpi]}\right]\right) \subseteq \mathcal{Y} = \mathrm{Spa}(A_{\inf}) \setminus \{p[\varpi]\}$

$(\xi) = \mathrm{Ker}\,(\theta_y: A_{\inf}[\frac{1}{p}] \to K)$

$$B^+_{dR,y} = \hat{\mathcal{O}}_{Y,y} = \xi\text{-adic completion of } A_{\inf}[\tfrac{1}{p}] = \varprojlim_{n \geq 0} \left(\frac{A_{\inf}}{(\xi)^{n+1}}[\tfrac{1}{p}]\right)$$

Proposition FF4: $B^+_{dR,y}$ is a complete DVR with uniformizer ξ i.e.

a) ξ is not a zero divisor in $B^+_{dR,y}$

b) $B^+_{dR,y}$ is (ξ)-adically complete

c) $\dfrac{B^+_{dR,y}}{(\xi)} = K$ field

FF5 Bertapelle: $\mathrm{Ker}\,\theta = (\xi) = ([a] - \pi u)$ with $\xi \in \mathrm{Prim}^1$ i.e.
$\quad\;\; \neq 0$

$\xi = \sum_{i\geq 0} [a_i]\pi^i \quad a_0 \neq 0 \quad a_0 \in m_F \quad a_1 \in \mathcal{O}_F^\times \quad \Rightarrow \xi = [a] - u\pi \quad u \in A_{inf}^\times$

$\alpha = a_0 \qquad \neq 0$

$|Y| = \{ I \trianglelefteq A_{inf} := W_{\mathcal{O}_E}(\mathcal{O}_C) : I = (\xi) \}$

$\forall I \subset (0,\infty)$ interval $\quad \underset{\text{PID}}{\hat{B}_I} = \varprojlim_{J \subseteq I \text{ compact}} \underset{\text{UFD}}{B_J} = \hat{B}^b_{\{v(r), r \in I\}}$

FF6 Longo: $|Y| = \mathrm{Prim}^1 / A_{inf}^\times \qquad \dfrac{\text{Primitive elements degree } 1}{\text{invertible elements in } A_{inf}}$

$\xi \in |Y| \qquad \widehat{(B_I)}_\xi \underset{\text{canonicamente}}{\simeq} B^+_{dR,\xi}$

$$\boxed{X := \mathrm{Proj}\left(\bigoplus_{d\geq 0} P_d\right)} \qquad P_d := B^{\varphi = \pi^d} \qquad B = \hat{B}^b_{\{v_r, r \in (0,\infty)\}}$$

FF7 Vanni $\quad B = \hat{B}^b_{\{v_r, r\in(0,\infty)\}} = B_{]0,\infty[}$

$P := \bigoplus_{d\geq 0} P_d$ is a graded ring generated by elements in P_1.

$|Y_J| = \mathrm{Specm}\, B_J = \{ y \in |Y| \mid v(a) \in J \} \qquad y = (\xi) \quad \xi = [a] - \pi u$

$$\mathrm{Div}^+(|Y_I|) = \left\{ \sum_{y \in |Y_I|} n_y [y] \;\middle|\; \begin{array}{l} n_y \in \mathbb{N}, \ \forall J \subseteq I \text{ int. compatto} \\ n_y = 0 \text{ per quasi tutti gli } y \text{ in } J \end{array} \right\}$$

$$B_I \setminus \{0\} \xrightarrow{\mathrm{div}} \mathrm{Div}^+(|Y_I|)$$
$$f \longmapsto \mathrm{div} f$$

$$\deg : \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}) \longrightarrow \mathbb{N}$$
$$D \longmapsto \sum n_y$$

Theorem FF7: $\quad \mathrm{div} : \underbrace{\bigcup_{d\geq 0} P_d \setminus \{0\} / E^\times}_{}\ \overset{\text{monoide}}{\xrightarrow{\ \simeq\ }} \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$

and P is a UFD with irreducible elements in P_1.

Def: $\mathcal{O}_X(n)$ is the invertible sheaf $\overset{eq}{\Leftrightarrow}$ line bundle associated to $P(n)$

Remark: $|X| = \dfrac{P_1 \setminus \{0\}}{E^\times} \simeq \dfrac{|Y|}{\varphi^{\mathbb{Z}}}$

Theorem: $\forall t \in P_1 = B^{\varphi=\pi}$ $t \neq 0$ $B_t := P[\frac{1}{t}]_0$ is a PID and

$\mathrm{Proj}(P) \simeq \mathrm{Spec}(B_t) \cup \{\infty_t\}$

$\Rightarrow$ X is noetherian, regular, of Krull dimension 1.

§2. $\mathrm{Pic}(X) \simeq \mathbb{Z}$

Definition: a Dedekind scheme is a Noetherian integral (connected) scheme of dim 1 all of whose local rings are regular. Let denote by η the generic point

① $\mathcal{O}_{X,\eta}$ is a field $K(X)$

② $\forall x \in |X|$ $\mathcal{O}_{X,x}$ is a DVR with fraction field $\mathcal{O}_{X,\eta}$

③ $\forall$ $U \subseteq X$ open affine $U \neq \emptyset$ $\mathcal{O}_X(U)$ is a Dedekind domain.

Definition: let $X = \mathrm{Proj}(P)$ and $x \in |X|$; $B^+_{dR,x} := \widehat{\mathcal{O}}_{X,x}$ with field of fractions $B_{dR,x}$.

For any Dedekind scheme X

Definition: $\mathrm{Div}(X)$ free abelian group on $|X|$;

Degree: $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ surjective group homomorphism.

$$D = \sum_{x \in |X|} n_x x \longmapsto \deg(D) := \sum_{x \in |X|} n_x$$

$\forall$ $f \in K(X)$ function field of the FF curve $\mathrm{div}(f) := \sum_{x \in |X|} \mathrm{ord}_x(f) x$

$\mathrm{div} : K(X) \setminus \{0\} \to \mathrm{Div}(X)$

where $\mathrm{ord}_x : \mathcal{O}_{X,x} \longrightarrow \mathbb{Z} \cup \{\infty\}$ is the valuation on $\mathcal{O}_{X,x}$ DVR.

Definition: $\mathrm{Pic}(X) = \{ [\mathcal{L}]_\sim \mid \substack{[\mathcal{L}]_\sim \text{ isomorphism class} \\ \text{an invertible sheaf}} \text{ with } \mathcal{L} \}$, it is a group with the operation $\otimes$.

Theorem: given X a Dedekind scheme:

① $\mathrm{PrDiv}(X) = \{ \mathrm{div}(f) \mid f \in K(X)^\times \} \leq \mathrm{Div}(X)$

② $\underset{H^1(X, \mathcal{O}_X^*)}{\underset{\parallel}{\mathrm{Pic}(X)}} \overset{\varphi}{\simeq} \dfrac{\mathrm{Div}(X)}{\mathrm{PrDiv}(X)} =: \mathrm{Cl}(X)$ with $\varphi(\mathcal{O}_X(D)) = [D]$ and if $D = [x]$ $x \in |X|$
$\mathcal{O}_X(x)^\vee =: \mathcal{O}_X(-x) =$

Example: $X = \mathbb{P}^1_k$ $K(X)^\times = K(t)$ $\mathrm{div} : K(X)^\times \to \mathrm{Div}(X)$

$$\mathrm{PrDiv}(X) = \{ D \in \mathrm{Div}(X) \mid \deg(D) = 0 \} = \mathrm{Ker}(\deg) \Rightarrow \mathrm{Pic}(X) \simeq \frac{\mathrm{Div}(X)}{\mathrm{PrDiv}(X)} \simeq$$

$$\simeq \frac{\mathrm{Div}(X)}{\mathrm{Ker}(\deg)} \simeq \mathbb{Z}$$

Let us prove the analog for $X = \mathrm{Proj}(P)$

Proposition [Ans 10.9]: $\forall f \in K(X)^\times$ we have $\deg(\mathrm{div}(f)) = 0$ and

$\deg : \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$ is an isomorphism.

Proof: $f \in K(X)^\times$ is a quotient $f = \frac{g}{h}$ $\quad g, h \in P_d \setminus \{0\}$ of the same degree. since g, h are product of irreducible elements in P_1 $\quad g = \prod_{i=1}^{\ell} g_i^{m_i}$ $\quad h = \prod_{j=1}^{k} h_j^{n_j}$ with $g_i, h_j \in P_1 \setminus \{0\}$. Hence $\deg\left(\frac{g}{h}\right) = \sum_{i=1}^{\ell} m_i \deg(g_i) - \sum_{j=1}^{k} n_j \deg(h_j) = 0$ because $\sum m_i = \sum n_j = d$ and for any $t \in P_1 \setminus \{0\}$ $\quad \mathrm{div}(t) = \infty_t \Rightarrow$ $\deg(\mathrm{div}(t)) = 1$.

$$\begin{tikzcd}
P_1 = B^{\varphi = \pi} \arrow[r] \arrow[d] & B^+_{dR,y} \\
B_t \arrow[r] & \mathcal{O}_{X,x} \arrow[u]
\end{tikzcd}$$

This proves that $\mathrm{PrDiv}(X) \subseteq \mathrm{Ker}(\deg)$. Let recall the s.e.s

$$0 \longrightarrow \mathbb{Z} \cdot [\mathcal{O}_X(1)] \xrightarrow{\simeq} \mathrm{Pic}(X) \longrightarrow \underset{\substack{\parallel \\ 0}}{\mathrm{Pic}(\mathrm{Spec}(B_t))} \longrightarrow 0$$

If $D = \sum_{x \in |X|} n_x x$ has $\deg D = 0 \rightarrow \sum n_x = 0$ $\qquad X_+ := \{ x \in |X| \mid n_x > 0 \}$, $\; X_- := \{ x \in |X| \mid n_x < 0 \}$

$$= \sum_{x \in X_+} n_x x - \sum_{x \in X_-} (-n_x) x =$$

$$= \mathrm{div}\left(\frac{\prod_{x \in X_+} t_x^{n_x}}{\prod_{x \in X_-} t_x^{-n_x}} \right)$$

Proposition [Ans 10.10]: $H^i(X, \mathcal{O}_X(n)) = \begin{cases} B^{\varphi = \pi^n} = P_n & \text{if } i = 0 \\ 0 & \text{if } i \geq 2 \text{ or } i = 1 \text{ and } n \geq 0 \\ B^+_{dR,x} \big/ \mathrm{Fil}^{-n} B^+_{dR,x} + E & \text{if } i = 1 \text{ and } n \leq -1 \end{cases}$

$X = \mathrm{Spec}\, B_t \cup \mathrm{Spec}\, B_{t'}$

$$0 \to \mathcal{O}_X(n-1) \xrightarrow{t} \mathcal{O}_X(n) \to K(\infty_t) \to 0$$

$$\text{r. } H^1(X, \mathcal{O}_X) = 0$$

$$\mathrm{Fil}^j B_{dR} := \xi^j B^+_{dR}$$

with $x \in |X|$. Remark that $H^1(X, \mathcal{O}_X(-1)) \simeq K(\infty_t)/E \neq 0$

Remark: recall that $H^i(\mathbb{P}^1_E, \mathcal{O}_{\mathbb{P}^1_E}(n)) = \begin{cases} (E[x_0, x_1])_n & \text{if } i = 0,\ n \geq 0 \\ \left(\frac{1}{x_0 x_1} E[\frac{1}{x_0}, \frac{1}{x_1}]\right)_n & \text{if } i = 1,\ n \leq -1 \\ 0 & \text{otherwise} \end{cases}$

§3 Harder-Narasimhan formalism.

- $\mathcal{C}$ quasi abelian category: additive $\mathrm{Hom}(A,B) \in \mathbb{Z}$-mod, $\circ$ bilinear

  Object $\bigoplus_{i=1}^{n} \simeq \prod_{i=1}^{n}$

  Exact sequences $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$

  strict mono $i = \ker p$ strict epi $p = \mathrm{coker}\, i$: Stable by pushouts, pullbacks, compositions.

- Two functions

  deg: $Sk(\mathcal{C}) := \underset{\text{iso}}{Ob(\mathcal{C})} \longrightarrow \mathbb{Z}$ rk: $Sk(\mathcal{C}) \longrightarrow \mathbb{Z}$

  additive on s.e.s and $rk(A) = 0 \Leftrightarrow A = 0$.

- Generic fiber: $F: \mathcal{C} \to \mathbb{A}$ functor with $\mathbb{A}$ abelian

  F is faithfull and exact and $\forall\, N \in \mathcal{C}$

  $\{\text{strict subobjects of } N\} \simeq \{\text{subobjects of } F(\mathcal{E})\}$

  If $u: A \to B$ in $\mathcal{C}$ such that $F(u)$ iso $\Rightarrow \deg A \leq \deg B$

  and if $\deg A = \deg B \Rightarrow u$ isomorphism.

  $rk: \mathbb{A} \to \mathbb{N}$ $rk_{|\mathcal{C}}: \mathcal{C} \to \mathbb{N}$

Remark: if $\mathcal{C}$ is abelian we take $\mathbb{A} = \mathcal{C}$

Definition: $\mu: \mathcal{C} \to \mathbb{Q} \cup \{\infty\}$ $\mu: \mathcal{C} \setminus \{0\} \to \mathbb{Q}$ is called

$$X \longmapsto \frac{\deg(X)}{rk(X)}$$

the slope function (Y. André $\forall\, A \xrightarrow{u} B$ mono-epic $\Rightarrow \mu(A) \leq \mu(B)$).

Example 1:

X connected smooth proper curve $/K$

$\mathcal{C} = Bun_X$ $\mathcal{O}_X$-mod locally free of finite rank.

$\deg: Bun_X \to \mathbb{Z}$, $\mathcal{E} \longmapsto \deg(\Lambda^r \mathcal{E})$ where $r = rk(\mathcal{E})$ $rk: Bun_X \to \mathbb{N}$, $\mathcal{E} \longmapsto \dim_{K(X)} \mathcal{E}_{X,\eta}$

$\mathbb{A} = K(X)$-vect $F: Bun_X \to \mathbb{A}$, $\mathcal{E} \longmapsto \mathcal{E}_\eta$

$u: \mathcal{E} \to \mathcal{E}'$: $\mathcal{E}_\eta \xrightarrow{\simeq} \mathcal{E}'_\eta$ $\quad \deg(\mathcal{E}') = \deg(\mathcal{E}) + \deg(\mathcal{E}'/u(\mathcal{E}))$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vee\!\!\!\!0$

where $\mathcal{F} := \frac{\mathcal{E}'}{u(\mathcal{E})}$ $\quad \deg(\mathcal{F}) := \sum_{x \in |X|} \deg(x)\, \mathrm{lenght}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$.

$\deg(\mathcal{E}'/u(\mathcal{E})) = 0 \iff \mathrm{len}_{\mathcal{O}_{X,x}} \mathcal{F}_x = 0 \ \forall x \in X \iff \mathcal{F}_x = 0 \ \forall x \in X \iff \mathcal{F} = 0$
$\iff u$ iso.

Example 2: isocrystals: $\breve{E} := W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)[1/\pi]$

Let A be a ring $\varphi \in \mathrm{End}(A)$, a φ-module over A is a finite projective A-module M with an isomorphism $\varphi_M : \varphi^* M \to M$

$\mathcal{C} = \varphi\text{-}\mathrm{Mod}_A$ $\qquad\qquad \varphi_M : A \otimes_{\varphi, A} M \longrightarrow M$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad M \otimes$

If M is free with basis $e_1, \dots, e_n$ $\exists!$ associated matrix to φ_M
$a := (a_{ij}) \in Gl_n(A)$ such that if we change base accordiy to $g \in GL_n(A)$ the matrix associated to φ_M with respect to the new basis is $b = g\, a\, \varphi(g)^{-1}$.

$$\{\text{isom. classes } M \mid M \text{ free of rank } n\} \overset{1:1}{\longleftrightarrow} \{\varphi\text{-conjugacy classes } [a]_\varphi\}_{a \in GL_n(A)}$$

Let $A = \breve{E}$ $\qquad \mathcal{A} = \mathcal{C} = \varphi\text{-}\mathrm{Mod}_{\breve{E}}$ is abelian. r

$rk : sk(\mathcal{A}) \longrightarrow \mathbb{N}$
$\quad M \longmapsto \dim_{\breve{E}} M$

$$\deg : \underbrace{\{\varphi\text{-mod of rank } 1\}}_{\sim} \longrightarrow \{[a]_\varphi \mid a \in \breve{E}^\times \ a \sim \tfrac{a\, b}{\varphi(b)} \ b \in \breve{E}^\times\} \xrightarrow{v_{\breve{E}}} \mathbb{Z}$$

$\deg : sk(\mathcal{A}) \longrightarrow \mathbb{Z}$
$\quad M \longmapsto \pm \deg(\Lambda^{rk(M)} M)$

Remark for any s.es $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$

$$m(\mu(\mathcal{E}_1), \mu(\mathcal{E}_2)) \le \mu(\mathcal{E}) \le \max(\mu(\mathcal{E}_1), \mu(\mathcal{E}_2))$$

$$\min\left(\frac{d_1}{r_1}, \frac{d_2}{r_2}\right) \le \frac{d_1 + d_2}{r_1 + r_2} \le \max\left(\frac{d_1}{r_1}, \frac{d_2}{r_2}\right)$$

- if 2 of 3 have the same slope so is the third

if $\mu(\mathcal{E}_1) < \mu(\mathcal{E}_2) \Rightarrow \mu(\mathcal{E}_1) < \mu(\mathcal{E}) < \mu(\mathcal{E}_2)$
if $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2) \Rightarrow \mu(\mathcal{E}_1) > \mu(\mathcal{E}) > \mu(\mathcal{E}_2)$

If $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_r = \mathcal{E}$ where $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i$ $\quad \mathcal{E}_i \ne \mathcal{E}_{i+1}$

- $r \le rk(\mathcal{E})$ (because $rk(C) = 0 \iff C = 0) \Rightarrow \mathcal{C}$ artinian noetherian
- $\min(\mu(\mathcal{E}_i/\mathcal{E}_{i-1})) \le \mu(\mathcal{E}) \le \max \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$

An object $\mathcal{E}$ in $\mathcal{C}$ is called semistable if $\forall \mathcal{F} \rightarrowtail \mathcal{E}$ we have

$\mu(\mathcal{F}) \leq \mu(\mathcal{E})$

Remark: $\mathcal{E}$ is ss $\iff$ $\forall\ \mathcal{E} \twoheadrightarrow \mathcal{G}$ $\mu(\mathcal{G}) \geq \mu(\mathcal{E})$

Remark: if $\mathcal{E}$ is semistable $\forall\ \underset{\neq 0}{\mathcal{F}} \overset{\alpha}{\hookrightarrow} \mathcal{E}$ $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ since

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[dr, "\text{monic-epic}"'] & \mathcal{E} \\ & \mathrm{Im}\,\alpha \arrow[u] \end{tikzcd} \Rightarrow \mu(\mathcal{F}) \leq \mu(\mathrm{Im}\,\alpha) \leq \mu(\mathcal{E}).$$

Remark: if $\mathcal{E}, \mathcal{G}$ are semistable of slope λ $\mathcal{E} \xrightarrow{f} \mathcal{G}$ $\mathrm{Ker} f, \mathrm{Im} f, \mathrm{Coker} f, \mathrm{CoIm} f$ are either 0 or semistable of slope λ.

Lemma [Ans 11.2]: $\forall\ \mathcal{E}, \mathcal{E}' \in \mathcal{C}$ semistable of slopes $\lambda > \lambda'$

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}') = 0$$

(**Proof:** if $u: \mathcal{E} \to \mathcal{E}'$ $u \neq 0$
$$\begin{tikzcd} \overset{\lambda}{\mathcal{E}} \arrow[r] \arrow[d] & \overset{\lambda'}{\mathcal{E}'} \\ \mathrm{CoIm}\,u \arrow[r, hook, "m.e."'] & \mathrm{Im}\,u \arrow[u] \end{tikzcd}$$
$\Rightarrow \mu(\mathrm{CoIm}\,u) \leq \lambda'$ but $\mu(\mathrm{CoIm}\,u) \geq \lambda > \lambda'$ since $\mathcal{E}$ s.s. $\lightning$.)

Definition: Let $\mathcal{E} \in \mathcal{C}$. A filtration

$\mathcal{F}^{\cdot}(\mathcal{E})$: $0 := \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$ such that

1. $\mathcal{E}_{i-1} \rightarrowtail \mathcal{E}_i$ $\quad \mathcal{E}_i / \mathcal{E}_{i-1} \neq 0$ semistable
2. $\mu_i := \mu(\mathcal{E}_i / \mathcal{E}_{i-1})$ is strictly decreasing.

$$\mu(\mathcal{E}_1) = \mu_1 > \mu_2 > \ldots > \mu(\mathcal{E}_r) = \mu(\mathcal{E}/\mathcal{E}_{r-1}).$$

is called a Harder–Narasimhan filtration.

Rmk: $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2) > \ldots > \mu(\mathcal{E})$ if $\mu(\mathcal{E}) = \mu(\mathcal{E}_1) \Rightarrow \mathcal{E} = \mathcal{E}_1$

Proposition [Ans 11.3]: any $\mathcal{E} \in \mathcal{C}$ has a unique functorial HN filtration.

Dim: (Ans + André Th. 4.2.3 and Lemma 3.3.2).

Induction on $rk(\mathcal{E})$. $\qquad 0 \to \frac{\mathcal{E}_2}{\mathcal{E}_1} \to \frac{\mathcal{E}}{\mathcal{E}_1} \twoheadrightarrow \frac{\mathcal{E}}{\mathcal{E}_2} \to$

Passo 1: $rk\,\mathcal{E} = 1$.

Passo 2: construct functorially $\mathcal{E}_1$ $\quad 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r$

Passo 3: construct f. $\mathcal{F}(\mathcal{E})$ using $\mathcal{F}(\mathcal{E}/\mathcal{E}_1)$ since $rk(\mathcal{E}/\mathcal{E}_1) < rk(\mathcal{E})$.

(P1) If $\mathcal{E}$ is semistable $\mathcal{F}^{\cdot}(\mathcal{E})$: $0 = \mathcal{E}_0 \subset \mathcal{E}_1 = \mathcal{E}$ HN since if $\mathcal{E}_1 \rightarrowtail \mathcal{E}$

$\Rightarrow \mu(\mathcal{E}) \leq \mu(\mathcal{E}_1)$ $\Rightarrow$ but Lemma 11.2 $\Rightarrow \mu(\mathcal{E}) = \mu(\mathcal{E}_1) \Rightarrow \mathcal{E}/\mathcal{E}_1$ ss $\mu(\mathcal{E}/\mathcal{E}_1) = \mu(\mathcal{E})$. NO.

If $rk(\mathcal{E}) = 1 \Rightarrow \mathcal{E}$ has no strict subobject $\mathcal{E}_1 \neq \mathcal{E} \Rightarrow \mathcal{E}$ is semistable. HN.

(P2) If the result holds true the $\mathcal{E}_1 \subset \mathcal{E}$ satisfies the following condition:

$\mathcal{E}_1 \rightarrowtail \mathcal{E}$ $\quad$ $\mathcal{E}_1$ is semistable of maximal slope and maximal rank between them of

If $0 \neq \mathcal{F} \rightarrowtail \mathcal{E} \Rightarrow \mu(\mathcal{F}) \leq \mu(\mathcal{E}_1)$ if $\mu(\mathcal{F}) = \mu(\mathcal{E}_1)$

$$\begin{tikzcd} \mathcal{F} \arrow[r, tail] \arrow[dr, dashed] & \mathcal{E} \\ & \mathcal{E}_1 \arrow[u, tail] \end{tikzcd}$$

because

$$\begin{tikzcd} \mathcal{F} \arrow[r, tail] & \mathcal{E} \\ \underset{ss}{\mathcal{F}_1} \arrow[u, tail] \arrow[r, tail] & \underset{ss}{\mathcal{E}_1} \arrow[u, tail] \end{tikzcd}$$

Lemma $\Rightarrow$ $\begin{array}{c} \mu(\mathcal{F}) \\ \wedge\!\!| \\ \mu(\mathcal{F}_1) \leq \mu(\mathcal{E}_1) \end{array}$

If $\mu(\mathcal{F}) = \mu(\mathcal{E}_1) \Rightarrow \mu(\mathcal{E}_1) = \mu(\mathcal{F}) \leq \mu(\mathcal{F}_1) \leq \mu(\mathcal{E}_1) \rightarrow \mu(\mathcal{F}) = \mu(\mathcal{F}_1)$

$\Rightarrow \mathcal{F}_1 = \mathcal{F}$ semistable by functoriality
$$\begin{tikzcd} \mathcal{F} \arrow[r, tail] \arrow[d, equal] & \mathcal{E}. \\ \mathcal{F}_1 \arrow[r, tail] & \mathcal{E}_1 \arrow[u, tail] \end{tikzcd}$$

Such an object $\mathcal{E}_1$ is called a universal destabilizing subobject of $\mathcal{E}$.

If $rk(\mathcal{E}) = 1$ $\quad \mathcal{E}_1 = \mathcal{E}$. By induction on $rk(\mathcal{E})$. If $rk(\mathcal{E}) \geq 2$ let

$Q = \{ P \mid \mathcal{E} \twoheadrightarrow P \quad \mu(P) < \mu(\mathcal{E}) \}$ non empty otherwise $\mathcal{E}$ ss.

Let $P \in Q$ of minimal rank $\Rightarrow$ P is semistable.

$$\boxed{\begin{tikzcd} 0 \arrow[r] & N \arrow[r, tail] & \mathcal{E} \arrow[r, two heads] & P \arrow[r] & 0 \\ & \mathcal{E}_1 := N_1 \arrow[u, tail] \end{tikzcd}}$$

works if N_1 exists since $rk(N) < rk(\mathcal{E})$ $\quad \mathcal{E}_1$ is semistable $\mu(\mathcal{E}_1) > \mu(\mathcal{E})$

If
$$\begin{tikzcd} \mathcal{F} \arrow[r, tail] & \mathcal{E} \\ \mathcal{F}_1 \arrow[u] \arrow[r, tail] & \mathcal{E}_1 \arrow[u] \end{tikzcd}$$
and $\mu(\mathcal{F}) > \mu(\mathcal{E})$ $\quad Hom(\mathcal{F}_1, P) = 0$

$\mu(\mathcal{F}) \leq \mu(\mathcal{F}_1) \leq \mu(\mathcal{E}_1)$. If $\mu(\mathcal{F}) = \mu(\mathcal{E}_1) \Rightarrow \mathcal{F} = \mathcal{F}_1$

(P3)
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}/\mathcal{E}_1 \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[u, equal] & \mathcal{E}' \arrow[r] \arrow[u] & \mathcal{G} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

$\forall\, \mathcal{G}$ strict $\mathcal{G} \subset \mathcal{E}/\mathcal{E}_1 \Rightarrow \mu(\mathcal{G}) < \mu(\mathcal{E}_1)$ otherwise

$\mu(\mathcal{E}') \geq \mu(\mathcal{E}_1)$ and $rk(\mathcal{E}') > rk(\mathcal{E}_1)$ $\lightning$.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}/\mathcal{E}_1 \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[u, equal] & \mathcal{E}_\ell \arrow[r] \arrow[u] & \mathcal{F}_{\ell-1} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[u, equal] & \mathcal{E}_{\ell-2} \arrow[r] \arrow[u] & \mathcal{F}_{\ell-2} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[u, equal] & \mathcal{E}_2 \arrow[r] \arrow[u] & \mathcal{F}_1 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

$\mathcal{E}/\mathcal{E}_1$ has a HN filtration $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell = \mathcal{E}/\mathcal{E}_1$

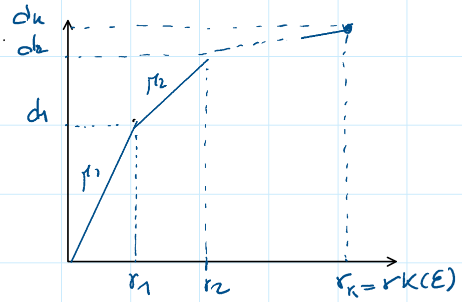
$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_{\ell+1} = \mathcal{E}$

$\dfrac{\mathcal{E}_i}{\mathcal{E}_{i-1}} \simeq \dfrac{\mathcal{F}_{i-1}}{\mathcal{F}_{i-2}}$ $\quad i \geq 2$ decr. slopes

$\mu(\mathcal{E}_1) > \mu\left(\frac{\mathcal{E}_2}{\mathcal{E}_1}\right) = \mu(\mathcal{F}_1)$ $\quad$ if $\mu(\mathcal{E}_1) = \mu(\mathcal{F}_1)$

Newton Polygon

The HN polygon is the unique concave polygon in $\mathbb{R}^2$ $(0,0)$ slopes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ with mult. $rk(\mathcal{E}_i/\mathcal{E}_{i-1})$ (mult. of the break)



$\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2) > \dots \quad \mu(\mathcal{E})$

$(rk\mathcal{E}_i, \deg \mathcal{E}_i)$ extremal points

Convex hull (x,y) $\quad x = rk\,\mathcal{E}_i \quad y = \deg \mathcal{E}_i$

$(r_{i-1}, d_{i-1}) \to (r_i, d_i)$ slope μ_i

$0 \to \mathcal{E}_{i-1} \to \mathcal{E}_i \to \dfrac{\mathcal{E}_i}{\mathcal{E}_{i-1}} \to 0$

$\mu_i = \dfrac{d_i - d_{i-1}}{r_i - r_{i-1}}$

Proposition [Ans 11.4, André 3.2.4 and 3.2.5]: Let $\lambda \in \mathbb{Q} \cup \infty$ $\mathcal{C}_\lambda^{sst} = \mathcal{C}(\lambda)$ be the full subcategory of semistable objects of $\mathcal{C}$ of slope λ (or ∞). $\mathcal{C}_\lambda^{sst}$ is abelian of finite length i.e any object has a finite filtration by simple objects.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "u"] \arrow[d, two heads] & \mathcal{F} \\ \mathrm{Co\,Im}\,u = \overline{\mathcal{E}} \arrow[r, "\overline{u}"] & \mathrm{Im}\,u = \overline{\mathcal{F}} \arrow[u, hook] \end{tikzcd}$$

$0 \to \mathrm{Ker}\,u \to \mathcal{E} \to \mathrm{CoIm}\,u \to 0$

$F(\overline{u})$ iso equality $\Rightarrow \overline{u}$ is mono-epic

$\mu(\lambda)$

Example: $X = \mathbb{P}^1_K$ or X Riemann sphere (complete curve $\overset{\text{degree } 1}{\infty} \in X$: $X \setminus \infty$ is affine and $\mathrm{Pic}(X \setminus \{\infty\}) = 0$ $\quad H^1(X, \mathcal{O}_X(-\infty)) = 0$

$\deg : \mathrm{Pic}(X) \xrightarrow{\simeq} \mathbb{Z} \qquad \mathcal{O}_X(k) := \mathcal{O}_X(k\infty)$

If $k \in \mathbb{Z}$ $\quad \mathcal{E}$ loc. free $\quad \mu(\mathcal{E}(k)) = \mu(\mathcal{E}) + k \qquad \mu(\mathcal{O}_X(k)) = k$

$H^0(X, \mathcal{O}_X(k)) = 0$ for $k < 0$

$H^1(X, \mathcal{O}_X(k)) = 0$ for $k \geq -1 \qquad H^1(X, \mathcal{O}_X) = 0$

Let X be $\mathbb{P}^1_K$, X Riemann sphere (smooth connected projective curve $/\mathbb{C}$)

$\mathcal{C} = \mathrm{Bun}_X = \{$vector bundles$\} = \{\mathcal{O}_X$ locally free modules of finite rank$\}$

$\mathcal{E} \in \mathrm{Bun}_X$

$\deg \mathcal{E} = \deg(\Lambda^r \mathcal{E}) \in \mathbb{Z} \qquad rk(\mathcal{E}) = rk(\mathcal{E}_\eta)$.

S.e.s. are s.e.s. in $\mathrm{Coh}(\mathcal{O}_X)$ $\quad 0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$

$0 \to \mathcal{O}_X(-1) \xrightarrow{t} \mathcal{O}_X \to \underbrace{\mathcal{O}_y}_{\text{Torsion sheaf}} \to 0 \quad \Rightarrow \mathcal{O}_X(-1) \xrightarrow{t} \mathcal{O}_X$ monic-epic

Narasimhan Seshadri stable bundles monodromy repr.
" " of degree 0 $\Longleftrightarrow$ irreducible unitary rep. $\pi_1(X,\bar{\alpha})$

$\mathcal{E} \in Bun_X$ $\quad \mathcal{E} = \bigoplus_{i=1}^{K} \mathcal{O}_X(d_i)^{n_i}$ $\quad \mathcal{E}$ is semistable $\Longleftrightarrow$ $\mathcal{E} = \mathcal{O}_X(d)^n$

let suppose $d_1 > d_2 > \dots > d_k$

$$\mathcal{O}_X(d_1)^{n_1} \subset \dots \subset \bigoplus_{i=1}^{K-2} \mathcal{O}_X(d_i)^{n_i} \subset \bigoplus_{i=1}^{K-1} \mathcal{O}_X(d_i)^{n_i} \subset \mathcal{E}$$

Theorem Grothendieck:

① $\mathcal{E}$ semistable $\Longleftrightarrow$ $\mathcal{E} \simeq \mathcal{O}_X(d)^n$ $\quad d \in \mathbb{Z}$ $\quad n \geq 0$

② $\mathcal{F}^i(\mathcal{E})$ splits

③ $\{(d_1, \dots, d_n) \in \mathbb{Z}^n \mid d_1 \geq d_2 \geq \dots \geq d_n\} \xrightarrow{1:1} \{\text{vector bundles of rank } n\}/\simeq$

$$(d_1, \dots, d_n) \longmapsto \bigoplus_{i=1}^{n} \mathcal{O}_X(d_i)$$

Proof: ① $\Rightarrow$ ② since $Ext^1(\mathcal{O}_X(d_2), \mathcal{O}_X(d_1)) = 0$ if $d_1 > d_2$

① + ② $\Rightarrow$ ③ $\quad \forall\ \mathcal{E}$ $\quad rk(\mathcal{E}) = n$ $\quad$ consider its HN filtration

$$\mathcal{E}_1 = \mathcal{O}_X(d_1)^{n_1} \hookrightarrow \mathcal{O}_X(d_1)^{n_1} \oplus \mathcal{O}_X(d_2)^{n_2} \hookrightarrow \dots \hookrightarrow \prod_{i=1}^{k} \mathcal{O}_X(d_i)^{n_i} = \mathcal{E}$$

$$\sum_{i=1}^{k} n_i = rk(\mathcal{E})$$

$\mathcal{O}_X(d)$ have rk 1 $\Rightarrow$ ss. $\quad \Rightarrow \mathcal{O}_X(d)^n$ ss.

Example: $\mathcal{C} = \mathcal{A} = \varphi\text{-}Mod_{\mathcal{A}}$

$\mathcal{C} := \mathbb{A} := \varphi\text{-}Mod_{\breve{E}}$ $\qquad rk(M) := \dim_{\breve{E}} M$

$$\deg(M) := -\deg(\Lambda^{rk(M)} M)$$

whose ss objects coincide.

M is called F-isocrystal, semistable isocrystals are called isoclinic.

$\Rightarrow$ HN filtration splits if $\mu(\mathcal{E}_1) \neq \mu(\mathcal{E}_2)$ $\quad Hom(\mathcal{E}_1, \mathcal{E}_2) = 0$

$\lambda \in \mathbb{Q}$ $\quad \lambda = \frac{d}{r}$ $\quad d \in \mathbb{Z}$ $\quad r > 0$ coprime $\rightarrow (D(\lambda), \varphi_{D(\lambda)})$

$D(\lambda) := \breve{E}^r \quad \to \quad a = \begin{pmatrix} 0 & \cdots & 0 & \pi^d \\ 1 & \ddots & & 0 \\ & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \qquad P_a(X) = \lambda_a(X) = X^r - \pi^d \qquad \varphi^r = \pi^d$

matrice compagna base ciclica

Remark: $\begin{tikzcd} D(\lambda) \arrow[r, "\gamma"] \arrow[dr] & M \\ & M \arrow[u, "\varphi^r = \pi^d"'] \end{tikzcd}$ since $\begin{tikzcd} D(\lambda) \arrow[r] \arrow[d, "\pi^d = \varphi^r"'] \arrow[dr, phantom, "\not\circlearrowleft"] & M \arrow[d, "\varphi^r"] \\ D(\lambda) \arrow[r] & M \end{tikzcd}$

Theorem: [Ans 11.7] Dieudonné-Manin classification: $\varphi\text{-Mod}_{\breve{E}}$ is semisimple with simple objects given, up to isom., by $D(\lambda)$ for $\lambda \in \mathbb{Q}$.
The division algebra $\mathrm{End}_{\varphi\text{-Mod}_{\breve{E}}}(D(\lambda))$ over E is central of invariant $-[\lambda] \in Br(E) \simeq \mathbb{Q}/\mathbb{Z}$.

Bun_X

$\breve{E} := W_{\mathcal{O}_E}(\overline{\mathbb{F}_q})[1/\pi] \subseteq B$

$\mathcal{E}(): \mathcal{A} = \varphi\text{-Mod}_{\breve{E}} \longrightarrow Bun_X \qquad X = \mathrm{Proj}(P) \quad$ FF curve

$(D, \varphi_D) \longmapsto \bigoplus \widetilde{(B \otimes_{\breve{E}} D)}^{\varphi \otimes \varphi_D = \pi^d}$

$D(-d) \longmapsto \mathcal{O}_X(d) = \widetilde{B}^{\varphi = \pi^d} = \widetilde{P_d}$

der $\underset{r=1}{D(-d) = \breve{E}} \qquad \varphi_{D(d)} = \pi^{-d}\varphi \qquad a = (\pi^{-d})$

Theorem [Ans 11.14] the previous functor induces a bijection

$\underset{\text{isom}}{\underline{\varphi\text{-Mod}_{\breve{E}}}} \xrightarrow{1:1} \underset{\text{isom}}{\underline{Bun_X}} \qquad$ (i.e. $\mathcal{E}(\_)$ is essentially surjective)

$\left\{ \begin{array}{c} \text{isoclinic isocrystals} \\ \text{of slope } -\lambda \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{Semistable objects in } Bun_X \\ \text{of slope } \lambda \end{array} \right\}$

HN filtration on Bun_X splits.

HN Formalism $\quad X = \mathrm{Proj}\left(\bigoplus_{d \geq 0} P_d\right)$ FF curve.

$|X| = |Y|/\varphi^{\mathbb{Z}}$

$\mathcal{C} = Bun_X$

$$\mathrm{rk}: Sk(Bun_X) \longrightarrow \mathbb{N}$$
$$\mathcal{E} \longmapsto \dim_{\mathcal{O}_{X,\eta}} \mathcal{E}_\eta$$

$$\deg: Sk(Bun_X) \longrightarrow \mathbb{Z}$$
$$\mathcal{E} \longmapsto \deg(\hat{\Lambda}\mathcal{E})$$
$$\mathrm{Pic}(X) \underset{\deg}{\simeq} \mathbb{Z}$$

Clearly $\mathrm{rk}(\mathcal{E}) = 0 \Leftrightarrow \mathcal{E} = 0$.

rk additive on s.e.s.

degree additive

$$F: Bun_X \longrightarrow \mathcal{O}_{X,\eta}\text{-Vect}$$
$$\mathcal{E} \longmapsto \mathcal{E}_\eta$$