How to construct vector bundles on the Fargues-Fontaine curve.

[Lurie 21,26 - Anschütz 11]

Let $X = \mathrm{Proj}\left(\bigoplus_{n \geq 0} B^{\varphi = p^n}\right)$, C^b fixed

$\lambda = \frac{d}{n} \in \mathbb{Q}$, $n > 0$. $\Rightarrow \exists\, \mathcal{E}$ semistable v.b. on X

s.t. $\deg \mathcal{E} = d$, $\mathrm{rk}\, \mathcal{E} = n$

<u>Construction:</u> Let $E/\mathbb{Q}_p$ of deg. n, let $\mathcal{L}$ be a deg. d line bundle on X_E ($\Rightarrow$ semistable since the rank is 1)

If $\pi : X_E \to X$, $\pi_* \mathcal{L}$ is a semistable v.b on X of deg. d and rk n (π_* - multiplies the rank by n)

This uses an equivalence of categories

$$\{\text{v.b. on } X_E\} \simeq \{\text{v.b. on } X \text{ with an action of } E\}$$

This construction is independent of the choice of $\mathcal{L}$ since $\mathrm{Pic}(X) \overset{\deg}{\simeq} \mathbb{Z}$. It is also independent of E.

$\pi \in \mathcal{O}_E$
uniformiser

$$\begin{tikzcd} E \arrow[d, dash, "e", "\text{tot. ram.}"'] \\ E_0 \arrow[d, dash, "d", "\text{unram.}"'] \\ \mathbb{Q}_p \end{tikzcd} \quad n$$

$$E_0 = W(\mathbb{F}_{p^d})\left[\tfrac{1}{p}\right]$$

We have two "versions" of the construction.

- $U \subset X$ affine open, complement of vanishing locus of some $\overset{\text{homogeneous}}{t}$
  $$\Rightarrow \mathcal{E}(U) = \left(B\left[\tfrac{1}{t}\right] \otimes_{E_0} E\right)^{\varphi^d = \pi^m} \qquad m \text{ depends on } t$$

- (isocrystals) $k = \bar{k}$ perfect of char p, $W(k)$ Witt vectors, $K = W(k)\left[\frac{1}{p}\right]$, φ_K Frobenius automorphism on K

An isocristal$/_k$ is a fin. dim. vect. sp. V over K with a Frobenius semilinear automorphism: $\varphi_V \in \mathrm{Aut}(V)$ s.t. $\varphi_V(\lambda v) = \varphi_K(\lambda)\varphi_V(v)$.

Building blocks:

m, n coprime, $V_{\frac{m}{n}} = K^n$

$$\varphi_{V_{\frac{m}{n}}}(x_1, \dots, x_n) = (\varphi_K(x_2), \dots, \varphi_K(x_n), p^m \varphi_K(x_1))$$

Thm (Dieudonné-Manin Classification) (see §8)

- The category of isocrystals$/k$ is semisimple.
- The simple objects are the $V_{m/n}$.

Let $K = \breve{E} = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$. The inclusion $\overline{\mathbb{F}}_p \longrightarrow C^\flat$ extends to $K \to B$.

Let V be an isocrystal. Let $\mathcal{E}_V$ be the sheaf associated to

$$\bigoplus_{n \geq 0} \mathrm{Hom}_K(V, B)^{\varphi = p^n} \qquad \text{(Lurie)}$$

$$\left[\text{equiv.} \quad \bigoplus_{d \geq 0} (B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d} \qquad \text{(Anschütz)} \right]$$

One way to see this: $U \subseteq X$ open affine, complement of the vanishing locus of t $\Rightarrow \mathcal{E}_V(U) = \{\varphi\text{-equivariant } K\text{-linear}: V \to B[\frac{1}{t}]\}$

For $V = V_{\frac{m}{n}}$ we write $\mathcal{O}(\frac{m}{n})$, and we have

$$\mathcal{O}(\tfrac{m}{n})(U) \cong \left(B[\tfrac{1}{t}]\right)^{\varphi^n = p^m} \Rightarrow \mathcal{O}(\tfrac{m}{n})$$ semistable vector bundle of rank n and deg. m.

$\Rightarrow \mathcal{E}_V$ is a vector bundle on X of rank $= \dim_K V$.

We obtain:

Thm (FF) $\forall \mu \in \mathbb{Q}$, there is an equivalence of categories

$$\left\{ \begin{array}{c} \text{Isoclinic isocrystals} \\ \text{of slope } \mu \end{array} \right\}^{op} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Semistable v.b. on } X \\ \text{of slope } \mu \end{array} \right\}$$

$\{$ Isoclinic isocrystals $\}$ $\simeq$ $\{$ Semistable v.b. on X $\}$
of slope μ — of slope μ

direct sum of copies of V_μ

Anschütz's variant:

Thm (F.F.): $\mathcal{E}(-)$ induces a bijection:

$$\varphi\text{-mod}_{\breve{E}}/\text{isom.} \longleftrightarrow \text{Bun}_X/\text{isom.}$$

Bonuses:
$\mathcal{E}(-)$ is compatible with HN filtration + splitting

But this is not an equivalence of categories.

Galois representations. [Anschütz 14]

Let K be a discretely valued non-arch. ext. of $\mathbb{Q}_p$ with k perfect residue field

$K_0 = W(k)[1/p] \subseteq K$

$\overline{K}$ alg. closure of K

$$\begin{array}{c} K \\ | \\ K_0 \\ | \\ \mathbb{Q}_p \end{array}$$

$G_K = \text{Gal}(\overline{K}/K)$ acting on:

$\overline{K}$, $C = \widehat{\overline{K}}$ & $F = C^\flat$

X = F.F. curve associated with F

∞ "=" C ($\Leftarrow$ vanishing locus of $\mathbb{Q}_p t \subseteq B^{\varphi=p}$ ...)

We have a G_K-action on everything!

- on $\mathbb{Q}_p t$ by $\chi_{cycl}: G_K \to \mathbb{Z}_p^\times$
- on $B_{dR}^+ := \hat{\mathcal{O}}_{X,\infty}$ & B_{dR}
- on B_{crys}^+, B_{crys}, $B_e := B_{crys}^{\varphi=\mathrm{Id}} = H^0(X \setminus \{\infty\}, \mathcal{O}_X)$

... what is this? (more details in §10)

A working definition:

$$\mathbb{A}_{crys} = \mathbb{A}_{inf}\left(\frac{\xi^n}{n!} \,\middle|\, n \geq 0\right)^\wedge_p = \left\{ \sum_{n\geq 0} a_n \frac{\xi^n}{n!} \right\}$$

$a_n \in \mathbb{A}_{inf}$
$a_n \to 0$ for the p-adic top.

$\hookrightarrow a_n \in A_{inf}$
$a_n \to 0$ for the p-adic top.

$$B_{crys}^+ = A_{crys}\left[\tfrac{1}{p}\right]$$
$$B_{crys} = B_{crys}^+\left[\tfrac{1}{t}\right]$$

Goal: There are fully faithful embeddings

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{Bun}_X^{G_K} \qquad (G_K\text{-equivariant v.b.})$$

(filtered) $\quad \varphi\text{-FilMod}_{K/K_0} \longrightarrow \mathrm{Bun}_X^{G_K}$

Let $\mathcal{C}$ be the intersection of the essentiae images

- in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ it gives crystalline reps.
- in φ-FilMod it gives weakly admissible filtered φ-mod.

Def.: $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is called crystalline if

$$(V \otimes_{\mathbb{Q}_p} B_{crys})^{G_K} \otimes_{K_0} B_{crys} \xrightarrow{\simeq} V \otimes_{\mathbb{Q}_p} B_{crys}.$$

E.g. the p-adic Tate module of an abelian variety (with good red.)

V crys. $\iff \dim_{K_0} (V \otimes_{\mathbb{Q}_p} B_{crys})^{G_K} = \dim_{\mathbb{Q}_p} V.$

Def.: Let $\mathcal{E} \in \mathrm{Bun}_X$. A G_K-action on $\mathcal{E}$ is the data of isomorphisms $\quad c_\sigma : \sigma^* \mathcal{E} \xrightarrow{\simeq} \mathcal{E} \quad \forall \sigma \in G_K$

s.t. $\quad c_{\sigma\tau} = c_\tau \circ \tau^*(c_\sigma) \quad \forall \sigma, \tau \in G_K$

One can use B_{dR}^+ to characterise topology and continuity.

A G_K-equivariant vector bundle is a pair $(\mathcal{E}, (c_\sigma)_{\sigma \in G_K})$ s.t. the G_K-action on $\mathcal{E} \otimes_{\mathcal{O}_X} B_{dR}^+$ is continuous.

Thm (Corollary of classification):

$$\mathrm{Rep}\ G_K \longrightarrow \mathrm{Bun}^{G_K}$$

Thm (Corollary of classification):

$$\mathrm{Rep}_{\mathbb{Q}_p} G_K \to \mathrm{Bun}_X^{G_K}$$
$$V \mapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$$

is a fully faithful functor.

The essential image consists of G_K-equiv. v.b. that are semistable of slope 0.

Rmk: Slope 0 means sum of copies of $\mathcal{O}_X$

Def.: A filtered φ-module is an element of $\varphi\text{-Mod}_{K_0}$ with a filtration on the base-change to K.

There is a HN formalism on this category (with fiber functor to Vect_K)

A filtered φ-module is weakly admissible if it is semistable of slope 0 via the HN formalism.

Lemma: The functor $\mathcal{E}(-): \varphi\text{-FilMod}_{K/K_0} \to \mathrm{Bun}_X^{G_K}$ preserves degree and HN filtrations.

We have a diagram:

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \cong \mathrm{Bun}_X^{G_K, sst, 0} \arrow[r] & \mathrm{Bun}_X^{G_K} \\
\varphi\text{-FilMod}_{K/K_0}^{wa} \arrow[r] \arrow[u] \arrow[ur, phantom, "\lrcorner"] & \varphi\text{-FilMod}_{K/K_0} \arrow[u]
\end{tikzcd}$$

Magic:

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \times_{\mathrm{Bun}_X^{G_K}} \varphi\text{-FilMod}_{K/K_0} \cong$$

$$\cong \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \times_{\mathrm{Bun}_X^{G_K}} \left(\mathrm{Bun}_X^{G_K} \times_{\mathrm{Rep}_{B_e}(G_K)} \varphi\text{-Mod}_{K/K_0} \right) \cong$$

$$\cong \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \times_{\mathrm{Rep}_{B_e}(G_K)} \varphi\text{-Mod}_{K/K_0} \cong \mathrm{Rep}_{\mathbb{Q}_p}^{crys}(G_K)$$

This shows:

Thm (Colmez-Fontaine): The category of crystalline Galois representations of G_K is equivalent to the category of weakly admissible filtered φ-modules for K.