TALK IN THE GEOMETRY SEMINAR - UNIPD

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Abstract. This talk[1] aims to provide an overview of my research journey, beginning with my graduate thesis and culminating in a discussion of ongoing projects. Key themes include Serre's modularity conjectures, the study of (some) Shimura varieties in positive characteristic, modular forms and associated mathematical structures, like the stack $G-\mathtt{Zip}^{\mu}$. Particular emphasis will be placed on θ-operators and their applications, showcasing both classical results and novel insights.

Contents

1. So, what's a θ operator?

Modular curves and modular forms. We fix a prime p, once and for all, and take with it k an algebraic closure of $\mathbf{F}_p$. For this section, consider $N \geq 5$ an integer prime to p and let $Y = Y_1(N)$ be the k-fibre of the modular curve of level $\Gamma_1(N)$. This is an affine, smooth curve over k. Writing $\pi\colon E \to Y$ for the universal object, we can consider

$$\underline{\omega} \coloneqq \pi_*(\Omega^1_{E/Y}),$$

which is an invertible sheaf.

Over the k-algebra $A = k((q^{1/N}))$, one can define the *Tate curve*

$$E_{\mathrm{Tate}} = \mathbf{G}_{m,A}/q^{\mathbf{Z}} = \mathrm{Spec}(A[t^{\pm}])/q^{\mathbf{Z}} \cong \mathrm{Proj}(A[X,Y,Z]/(Y^2Z + XYZ - X^3 - a_4(q)XZ^2 - a_6(q)Z^3)).$$

[Here

$$a_4(q) = -5\sum_{n\geq 1}\frac{n^3q^n}{1-q^n},\ a_6(q) = \sum_{n\geq 0}\frac{7n^5+5n^3}{12}\frac{q^n}{1-q^n},$$

which are elements of $qk[[q]]$.] E_{Tate} is the generic fibre of a curve over $k[[q^{1/N}]]$ (whose special fibre is a *generalised* elliptic curve). This curve is endowed with a canonical differential $\omega_{\mathrm{can}} \in H^0(E_{\mathrm{Tate}}, \Omega^1_{E/A})$[, which can be described as $\omega_{\mathrm{can}} = dt/t$]. Extending the base of E_{Tate} from $k((q))$ to $k((q^{1/N}))$ ensures that all the level $\Gamma_1(N)$-structures are A-rational. They are given by points of order exactly N, namely, fixing $\zeta_N \in k$ a primitive N-th root of 1, the points

$$P = \zeta_N^i q^{j/N},\ i,j \in \mathbf{Z}/N\mathbf{Z},$$

with $\gcd(i,j,N) = 1$. Choose α such a point, hence a level $\Gamma_1(N)$-structure on E_{Tate}. Since $H^0(E_{\mathrm{Tate}}, \Omega^1_{E/A}) = k((q^{1/N}))\cdot\omega_{\mathrm{can}}$, for any section f of $\underline{\omega}^{\lambda} \coloneqq \underline{\omega}^{\otimes\lambda}, \lambda \in \mathbf{Z}$, we can pull f back via the classifying morphism $E_{\mathrm{Tate}} \to Y$ of $(E_{\mathrm{Tate}}, \alpha)$ to obtain some $f_\alpha(q^{1/N})\cdot\omega^{\lambda}_{\mathrm{can}}$. This is called the *$q$-expansion of f at the cusp α*.

Definition 1.1 (Modular forms)**.** *We shall call a* modular form of level $\Gamma_1(N)$, weight $\lambda \in \mathbf{Z}$, with coefficients in k *an element of $H^0(Y_1(N), \underline{\omega}^{\lambda})$ such that the q-expansion $f_\alpha(q^{1/N})$ at each cusp α satisfies $f_\alpha \in k[[q^{1/N}]]$. If we also have that $f_\alpha \in q^{1/N}k[[q^{1/N}]]$, for each α, then we call f a* cusp form.

One usually denotes the k-vector space of modular forms of level $\Gamma_1(N)$ and weight λ by $M_\lambda(\Gamma_1(N), k)$ and also writes

$$M(N,k) = M(\Gamma_1(N),k) \coloneqq \oplus_{\lambda\in\mathbf{Z}} M_\lambda(\Gamma_1(N),k)$$

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[1]These notes cover much more material than I have any hope of presenting in one hour; take it as a compendium to go with the talk.

for the graded k-algebra they form. We write $S_\lambda(\Gamma_1(N),k)$ for the subspace of cusp form as well as

$$S(N,k)=S(\Gamma_1(N),k):=\oplus_{\lambda\in\mathbf{Z}}S_\lambda(\Gamma_1(N),k),$$

which is an ideal of $M(\Gamma_1(N),k)$.

Is this it? Using q-expansions, one can define a graded k-linear differential operator on $M(N,k)$, namely

$$\theta\colon M_\lambda(\Gamma_1(N),k)\longrightarrow S_{\lambda+p+1}(\Gamma_1(N),k),$$
$$f_\alpha(q^{1/N})\cdot\omega_{\mathrm{can}}^\lambda\longmapsto\frac{qd}{dq}(f_\alpha(q^{1/N}))\cdot\omega_{\mathrm{can}}^{\lambda+p+1}$$

From this definition, not much is clear about this operator. Following [9], we can give a more geometric construction.

The geometric construction. We need some ingredients.

(i) We consider the *(degree* 1*) de Rham cohomology* of E over Y, defined as $H^1_{\mathrm{dR}}(E/Y):=R^1\pi_*(\Omega^\bullet_{E/Y})$, where

$$\Omega^\bullet_{E/Y}:\quad 0\to\mathcal{O}_E\xrightarrow{d}\Omega^1_{E/Y}\to 0$$

is the *de Rham complex* of E over Y. The sheaf $H^1_{\mathrm{dR}}(E/Y)$ is locally free of rank 2 over Y.

(ii) The triple $E/Y/k$ gives rise to the *Gauss–Manin connection*

$$\nabla\colon H^1_{\mathrm{dR}}(E/Y)\longrightarrow H^1_{\mathrm{dR}}(E/Y)\otimes_{\mathcal{O}_Y}\Omega^1_{Y/k}$$

which is integrable (trivially, in this case, since Y is a curve, but this is a general propriety of the GM connection) and k-linear.

(iii) Moreover, we have the so-called *Hodge filtration*

$$0\to\underline{\omega}\longrightarrow H^1_{\mathrm{dR}}(E/Y)\longrightarrow\underline{\omega}^{-1}\to 0$$

which is deduced from the degeneration (on the first page) of the *Hodge–de Rham spectral sequence*, which is $E_1^{pq}=R^q\pi_*(\Omega^p_{E/Y})\Rightarrow R^{p+q}\pi_*(\Omega^\bullet_{E/Y})$.

(iv) One can give a natural "stratification" of Y.
- In Y, we can distinguish between the *ordinary locus* Y^{ord}, which is a dense open whose points correspond to ordinary elliptic curves, and its complement Y^{ss}, the *supersingular locus*, whose name has an obvious meaning in terms of the moduli problem.
- Recall that an elliptic curve E_x/k, for some $x\in Y(k)$ is *ordinary* if and only if the *Verschiebung* isogeny $V\colon E_x^{(p)}\to E_x$ is étale; here $E_x^{(p)}$ denotes the *p-twist* of E_x, namely the pullback

$$\begin{array}{ccc} E_x^{(p)} & \longrightarrow & E_x \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(k), \end{array}$$

where $\sigma\colon k\to k,\alpha\mapsto\alpha^p$.
- One can check whether such a V is étale by looking at the cotangent space at the identity, that is, checking whether

$$V^*\colon\underline{\omega}_{E_x/k}=H^0(E_x,\Omega^1)\cong k\longrightarrow\omega^p_{E_x/k}=H^0(E_x,(\Omega^1)^{\otimes p})\cong k$$

is 0 or not.
- This can be globalised to Y by considering the k-linear morphism $V=V^*\colon\underline{\omega}\to\underline{\omega}^p\cong\underline{\omega}^{(p)}$, where $\mathcal{F}^{(p)}:=F_S^*(\mathcal{F})$, for any $\mathcal{F}\in\mathrm{Q}\underline{\mathrm{Coh}}_S$, and $F_S\colon S\to S$ the absolute Frobenius on S.
- This vanishes at x if and only if the corresponding section $h\in H^0(Y,\underline{\omega}^{p-1})$, called the *Hasse invariant*, is zero in x. In fact, we have that $h\in M_{p-1}(N,k)$ and we have proved part of the following.

Lemma 1.2 (Igusa)**.** *We have $Y^{\mathrm{ord}}=D_Y(h),Y^{\mathrm{ss}}=V(h)$. Moreover, the zeros of h are simple.*

(v) There is a natural way to "split" the Hodge filtration in characteristic p.
- The relative Frobenius $F\colon E\to E^{(p)}$ induces by pullback a map $F=F^*\colon H^1_{\mathrm{dR}}(E/Y)^{(p)}\to H^1_{\mathrm{dR}}(E/Y)$. This is identically zero on $\underline{\omega}^p\subseteq H^1_{\mathrm{dR}}(E/Y)^{(p)}$, so that, by the Hodge filtration, we obtain a map $\underline{\omega}^{-p}\to H^1_{\mathrm{dR}}(E/Y)$.
- The image of this map, denote it by $\mathcal{U}$, is an invertible sheaf.
- Moreover, we have that $\mathcal{U}=\ker(V\colon H^1_{\mathrm{dR}}(E/Y)\to H^1_{\mathrm{dR}}(E/Y)^{(p)})$.

- In fact, $V\colon H^1_{\mathrm{dR}}(E/Y) \to H^1_{\mathrm{dR}}(E/Y)^{(p)}$ factors through $\underline{\omega}^p \subseteq H^1_{\mathrm{dR}}(E/Y)^{(p)}$ to give an isomorphism
$$H^1_{\mathrm{dR}}(E/Y)/\mathcal{U} \xrightarrow{\sim} \underline{\omega}^p.$$
- Over Y^{ord} the composition
$$\underline{\omega}^{-p} \xrightarrow{F} \mathcal{U} \to H^1_{\mathrm{dR}}(E/Y^{\mathrm{ord}})/\underline{\omega} \cong \underline{\omega}^{-1}$$
is an isomorphism.
- Thus, we obtain, via $F\colon H^1_{\mathrm{dR}}(E/Y^{\mathrm{ord}})^{(p)} \to H^1_{\mathrm{dR}}(E/Y^{\mathrm{ord}})$ a splitting of the Hodge filtration, for which $H^1_{\mathrm{dR}}(E/Y^{\mathrm{ord}}) \cong \underline{\omega} \oplus \mathcal{U}$. The projection $p_{\mathrm{ur}}\colon H^1_{\mathrm{dR}}(E/Y^{\mathrm{ord}}) \to \underline{\omega}$ is called the *unit-root splitting* and, in this case it is given by
$$p_{\mathrm{ur}}\colon H^1_{\mathrm{dR}}(E/Y^{\mathrm{ord}}) \xrightarrow{V} \underline{\omega}^p \xrightarrow{V|_{\underline{\omega}}^{-1}} \underline{\omega}.$$
- This splitting does not extend in a natural way to $Y \supseteq Y^{\mathrm{ord}}$.
- However, one can extend $h \cdot p_{\mathrm{ur}}$ simply by
$$V\colon H^1_{\mathrm{dR}}(E/Y^{\mathrm{ord}}) \longrightarrow \underline{\omega}^p.$$

(vi) More generally, one can consider the inclusion $\underline{\omega}^\lambda \subseteq \mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y))$ and deduce from the GM connection another integrable connection
$$\nabla\colon \mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y)) \longrightarrow \mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y)) \otimes_{\mathcal{O}_Y} \Omega^1_{Y/k}.$$
This satisfies an incarnation of the so-called *Griffiths transversality*, namely
$$\nabla(\underline{\omega}^\lambda) \subseteq F(\mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y))) \otimes_{\mathcal{O}_Y} \Omega^1_{Y/k},$$
where
$$F(\mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y))) = \mathrm{im}(\underline{\omega}^{\lambda-1} \otimes H^1_{\mathrm{dR}}(E/Y) \to \mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y))).$$

(vii) The unit-root splitting generalises to a splitting, over the whole Y, given by
$$h \cdot \mathrm{Sym}^\lambda(p_{\mathrm{ur}})\colon F(\mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y))) \to \underline{\omega}^{\lambda+p+1}.$$
Again, while a priori defined only over Y^{ord}, this does in fact extend to Y, namely because the poles along Y^{ss} of $\mathrm{Sym}^\lambda(p_{\mathrm{ur}})$ restricted to $F(\mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y)))$ are at most simple, hence cleared out by h.

(viii) From the GM connection and the Hodge filtration, we can derive the following composition
$$\underline{\omega} \xrightarrow{\nabla} H^1_{\mathrm{dR}}(E/Y) \otimes_{\mathcal{O}_Y} \Omega^1_{Y/k} \longrightarrow \underline{\omega}^{-1} \otimes \Omega^1_{Y/k}.$$
Dualising $\underline{\omega}^{-1}$, we obtain an $\mathcal{O}_S$-linear map $\underline{\mathrm{ks}}\colon \underline{\omega}^2 \to \Omega^1_{Y/k}$ called the *Kodaira–Spencer morphism.* One can show the following, for instance, by Grothendieck–Messing theory.

Lemma 1.3. *The map* $\underline{\mathrm{ks}}$ *is an isomorphism.*

Finally, we have all that is needed to define
$$\theta\colon \underline{\omega}^\lambda \xrightarrow{\nabla} F(\mathrm{Sym}^\lambda(H^1_{\mathrm{dR}}(E/Y))) \otimes \Omega^1_{Y/k} \xrightarrow{(h\cdot\mathrm{Sym}^\lambda(p_{\mathrm{ur}}))\otimes\underline{\mathrm{ks}}^{-1}} \underline{\omega}^{\lambda+p+1},$$
which is the *(geometric, classical) theta operator*. By considering its action on global sections, we obtain the operator mentioned above. Moreover, in [9], Katz shows that θ has some interesting properties.

Proposition 1.4. *We have the following.*

(i) The action of θ on q-expansions is given by $\frac{qd}{dq}$.
(ii) For $f \in M_\lambda(N,k)$, we have $h \mid \theta(f)$ if and only if $h \mid f$ or $p \mid \lambda$.
(iii) If $f \in M_\lambda(N,k)$ is such that $\theta(f) = 0$, then $f = h^r \cdot g^p$, for $0 \le r \le p-1$, with $r \equiv -\lambda [\mathrm{mod}\, p]$ and $g \in M_{\lambda'}(N,k)$.

These can all be proved by some local computations on $U \subseteq Y$ an open trivialising $\underline{\omega}$, $H^1_{\mathrm{dR}}(E/Y)$, etc. or by working on some finite étale cover compactifying an *Igusa curve*, see [5].

2. Sure, but why would I care?

What we have just defined is a weight shifting operator. In fact, one can show, for instance using the formula for the q-expansions and q-expansion principle, see [8, Sec. 1.6], that, for $\ell \neq p$ a prime, we have

$$\ell\theta T_\ell = T_\ell \theta,$$

where T_ℓ is the Hecke operator at ℓ. In particular, if f is a Hecke-eigenform, then we can associate to f a continuous Galois representation $\rho_f \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}_2(k)$ and we have that

$$\rho_{\theta(f)} = \chi \otimes \rho_f,$$

where $\chi \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to k^\times$ is the *cyclotomic character* $\bmod p$. [What do I mean by this, you ask? I mean that if you consider the action of $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on a primitive p-th root of unity $\zeta_p \in \overline{\mathbf{Q}}$, you find $\sigma(\zeta_p) = \zeta_p^{\chi(\sigma)}$.]

This is not just some cute observation. Edixhoven, in [2], used this as one of the steps in his proof of the *weight part of the Serre's conjectures*. Namely, let $\rho \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}_2(k)$ be a continuous, irreducible representation whose determinant is *odd*: let $c \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be a complex conjugation, induced by some embedding $\overline{\mathbf{Q}} \to \mathbf{C}$, then $\det(\rho)(c) = -1$. [As an aside, this parity condition can be tricky to generalise and, according to people who know more than me, it is meant to capture the fact that f is algebraic: namely, representations with $\det(\rho)(c) = 1$, should come from *Maass forms*, which are essentially analytic in nature (with current mathematical technology; possibly not for long).]

Definition 2.1 (Modularity)**.** *We say that ρ is* modular *if there is some cuspidal eigenform f such that $\rho \sim \rho_f$.*

Then, we have the following.

Theorem 2.2 (Edixhoven)**.** *Suppose that ρ is modular. Then, we can take f to have weight $k(\rho)$ and level $N(\rho)$, which are minimal and described by an explicit recipe due to Serre. This recipe only depends on $\rho|_{D_p}$, for some $D_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \leq \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.*

Remark 2.3. The fact that it is "enough to look at $\rho|_{D_p}$" is related to the fact that this is supposed to be a $\bmod p$ incarnation of a *local* Langlands correspondence.

The work of Edixhoven dates back to the early '90s and in the 2000s, Khare and Wintenberger gave a full proof of Serre's conjecture in the classical setting, namely the following.

Theorem 2.4 (Khare–Wintenberger)**.** *Any $\rho \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}_2(k)$ continuous, irreducible representation whose determinant is odd is modular.*

This completes the picture for degree 2 representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Given that, and more broadly the impulse that research in the Langlands programme has experienced since then, many authors considered natural generalisations of the weight part of the Serre conjectures.

Gee–Herzig–Savitt. Generally speaking, the picture is the following[2]. We consider a connected, reductive group G over F a number field. We assume that G_{F_v} is unramified (quasi-split and split over an unramified extension of F_v) at each place $v \mid p$. By Bruhat–Tits theory, this is the same as saying that there is a reductive group $\mathbf{G}_{\mathcal{O}_v}$ whose generic fibre is G_{F_v}. We can consider a compact open subgroup

$$K = K^p K_p, K^p \leq G(\mathbb{A}_F^{\infty,p})(= \mathrm{Res}_{F|\mathbf{Q}}(G)(\mathbb{A}^{\infty,p})), K_p = \mathbf{G}(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p),$$

with K^p sufficiently small. We also take $K_\infty = \prod_{v|\infty} K_v \leq \prod_{v|\infty} G(F_v)$ maximal connected and compact modulo centre (for instance, if $F = \mathbf{Q}$ and $G = \mathbf{GL}_n$, then $K_v = \mathbf{R}_{>0}^\times \cdot \mathbf{SO}_n(\mathbf{R})$ for $F_v = \mathbf{R}$, $K_v = \mathbf{R}_{>0}^\times \cdot \mathbf{U}_n(\mathbf{R})$ for $F_v = \mathbf{C}$). We can then define a locally symmetric space

$$Y_K \coloneqq G(F)\backslash G(\mathbb{A}_F)/KK_\infty.$$

For any irreducible smooth representation W of K_p with k-coefficients, we have that the action of the p-subgroup $U_0 \coloneqq \ker(K_p \to \prod_{v|p} \mathbf{G}(k_v))$ is trivial, so that W is really an irreducible representation of

$$\prod_{v|p} \mathbf{G}(k_v)(= \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}(\mathbf{G})(\mathbf{F}_p)).$$

[This is because, by definition of smoothness, for any $w \in W$, the group $\mathrm{Stab}_{K_p}(w)$ contains a maximal compact open U which is contained and normal in U_0. Therefore, we have an action of $U\backslash U_0$ on $\mathrm{Span}_k \langle g \cdot w, g \in K_p \rangle = W$, by irredicibility of W. If $U\backslash U_0 \neq 1$, then $U\backslash U_0$ is a p-group acting on a vector space of characteristic p, namely W,

[2]What follow is my attempt at generalising what the authors write in [4]. There might be naive mistakes in what I'm doing here; take it with a grain of salt.

so that it must have a fixed vector w'. But U_0 is normal in K_p, so that K_p should preserve the line $\langle w' \rangle_k \leq W$. That would entail that $\langle w' \rangle_k = W$ and the action of U_0 is trivial, contradicting the assumption that $U \neq U_0$. So we must have $U = U_0$.]

Definition 2.5 (Serre weight)**.** *An irreducible k-representation of $\prod_{v|p} \mathbf{G}(k_v)$ is called a* Serre weight *of G.*

Given a Serre weight W, we can consider the k-local system

$$\mathcal{W} \coloneqq ((G(F)\backslash G(\mathbb{A}_F)/K^p K_\infty) \times W)/K_p$$

over Y_K.

There is a finite set of places Σ_0, containing those above p and ∞, such that for $w \notin \Sigma_0$ the reductive group G_{F_w} is unramified and $K_w = \mathbf{G}(\mathcal{O}_w)$. For any finite place $w \notin \Sigma_0$, we can consider the spherical Hecke algebra

$$\mathcal{H}_w \coloneqq \mathcal{H}(\mathbf{G}(\mathcal{O}_w)\backslash G(F_w)/\mathbf{G}(\mathcal{O}_w), \overline{\mathbf{Z}}_p),$$

which, by the *Satake isomorphism*, see [6], is commutative. We can identify $\mathcal{H}_w$ with $\mathcal{H}(K\backslash KG(F_w)K/K, \overline{\mathbf{Z}}_p)$, which is naturally a sub-algebra of

$$\mathcal{H}(K\backslash G(\mathbb{A}_F^\infty)/K, \overline{\mathbf{Z}}_p),$$

which has a natural action on the cohomology $H^\bullet_{\mathrm{Betti}}(Y_K, \mathcal{W})$.

Let now $\rho\colon \mathrm{Gal}(\overline{F}/F) \to {}^L\mathbf{G}(k)$ be a continuous, irreducible representation. Here ${}^L\mathbf{G}$ denotes a $\mathrm{mod}\, p$ incarnation of the *Langlands dual group*. If $G = \mathbf{GL}_n$, then we can just take ${}^L\mathbf{G} = \mathbf{GL}_n$. This ρ is ramified [meaning that the action of $I_{\tilde{v}} \leq D_{\tilde{v}} \leq \mathrm{Gal}(\overline{F}/F)$, the inertia $I_{\tilde{v}}$ at some place $\tilde{v}\colon \overline{F} \to \overline{F}_v$ above v a place of F, acts non-trivially via ρ] only at a finite set of places of F. Let Σ be a finite set of places of F containing Σ_0 and the ramification of ρ. We can consider the unramified Hecke algebra

$$\mathbb{T}_\Sigma \coloneqq \otimes'_{w\notin\Sigma} \mathcal{H}_w,$$

which, again, acts on $H^\bullet_{\mathrm{Betti}}(Y_K, \mathcal{W})$. We can define a maximal ideal $\mathfrak{m} = \mathfrak{m}(\rho, K, \Sigma)$ by requiring that, for all places $w \notin \Sigma$, the semisimplification of $\rho(\mathrm{Frob}_w^{-1})$ matches via the *twisted Satake isomorphism* the $\mathcal{H}_w$-eigenvalues determined by $\mathfrak{m}$. [The Satake isomorphism gives

$$\mathcal{H}_w \xrightarrow{\sim} \overline{\mathbf{Z}}_p[X^\bullet(\hat{T}) \cong X_\bullet(T)] \cong R(\hat{G}) \otimes \overline{\mathbf{Z}}_p,$$

so that an Hecke-eigenform gives a character of the representation ring $R(\hat{G}) \otimes \overline{\mathbf{Z}}_p$, whose reduction $\mathrm{mod}\, p$ corresponds to some semisimple conjugacy class $s_{f,w}$ in $\hat{\mathbf{G}}(k) \leq {}^L\mathbf{G}(K)$. We are requiring $\rho(\mathrm{Frob}_w^{-1})^{\mathrm{ss}} = s_{f,w}$.]

Definition 2.6 (Automorphy)**.** *We say that ρ is* automorphic *if there are some W, U, Σ such that $H^\bullet(Y_K, \mathcal{W})_\mathfrak{m} \neq 0$.*

Definition 2.7 (Set of Serre weights)**.** *We call*

$$W(\rho) \coloneqq \left\{ W \quad \begin{matrix} \textit{smooth, irreducible} \\ k\textit{-representation of} \end{matrix} \quad \prod_{v|p} \mathbf{G}(k_v) \,\middle|\, H^\bullet_{\mathrm{Betti}}(Y_K, \mathcal{W})_\mathfrak{m} \neq 0 \right\}$$

the set of Serre weights of *ρ.*

Question 2.8. Can we describe $W(\rho)$ in terms of certain invariants of ρ and, in particular, of $\rho_{D_{\tilde{v}}}$, for all $v \mid p$?

One of the approaches to this question is that of looking for *entailments*, namely, given $W \in W(\rho)$, we want to describe general procedures to *shift the weight*, that is, ways to produce from W some other $W' \in W(\rho)$. In settings where Y_K is algebraic, or somehow related to another algebraic locally symmetric space $Y'_{K'}$, see [11], with a model above some E, we can look at

$$H^\bullet_{\text{ét}}(Y_K, \mathcal{W})$$

and use techniques from algebraic geometry. This is, for instance, the case of $G/\mathbf{Q}$ coming from a Shimura datum (G, X). If, by some vanishing results, we can relate *in a precise way*, reminiscent of the Eichler–Shimura isomorphism, $H^\bullet_{\text{ét}}(Y_K, \mathcal{W})$ to the coherent cohomology $H^\bullet(Y_K, \mathcal{V}(\lambda))$, we might have some hope of addressing the question by looking at $\mathcal{V}(\lambda)$, commonly called *automorphic sheaves.* This is the context in which one looks for generalisations of the classical θ operator.

3. OK, BUT WHAT HAVE *YOU* DONE?

In my thesis, in particular, I considered the case of a unitary group $\mathbf{G}$ relative to some quadratic imaginary extension $E/\mathbf{Q}$ with signature $(n-1,1)$ at the only place at ∞ of E, that is, $\mathbf{G}(\mathbf{R}) \cong \mathrm{GU}(n-1,1)$, for some $n \geq 3$. This is similar, in some essential ways, to the setting of the Kottwitz–Harris–Taylor "simple" Shimura varieties, considered in [7]. Most notably, $\mathbf{G}_{\mathbf{Q}_p} \cong \mathbf{GL}_n \times \mathbf{G}_m$. In this setting, for p an odd prime split in E, the geometric special fibre S_K of the corresponding Shimura variety, at a neat p-hyperspecial level $K \leq \mathbf{G}(\mathbb{A}_{\mathbf{Q}})$, has a particularly nice geometry. Namely, this is a *PEL Shimura variety*, which means that it is a (component of a) moduli space of abelian varieties with additional structure (a prime-to-p polarisation, an action of $\mathcal{O}_E$, as well as a level K-structure), much like the modular curve $Y_1(N)$ that parametrised elliptic curves with level $\Gamma_1(N)$-structure. Whereas for the modular curve we had a distinction between ordinary and supersingular elliptic curves E_x, which can be read off from the p-torsion $E_x[p]$ of E_x, here looking at the isomorphism classes of $A_x[p]$ (with the additional structure carried around), we obtain a *stratification* of the points of $x \in S_K$ called the *Ekedahl–Oort stratification.* The EO stratification, which is in general very complicated for most PEL Shimura varieties, becomes fairly simple in this case.

Theorem 3.1. *For $0 \leq r \leq n-1$, we have locally closed subschemes (with the reduced-induced structure) $S_{K,r} \subseteq S_K$ such that:*

(i) for each geometric point x of $S_{K,r}$ the isomorphism class of $A_x[p]$ is constant,
(ii) $S_{K,r}$ has dimension r and is smooth,
(iii) we have the closure relations $\overline{S}_{K,r} = \cup_{r' \leq r} S_{K,r'}$ and these closures are also smooth.

Remark 3.2. It is not by accident that this stratification is reminiscent of

$$\mathbf{P}_k \backslash \mathbf{G}_k \cong \mathbf{P}_k^{n-1} \cong \mathbf{A}_k^{n-1} \sqcup \mathbf{A}_k^{n-2} \sqcup \ldots \sqcup \mathbf{A}_k^1 \sqcup \mathbf{A}_k^0.$$

While it was known, from the works of Eischen–Mantovan, see for instance [3], and Goren–de Shalit, see [1], that one can define a theta operator on $S_K = \overline{S}_{K,n-1}$, from the point of view of the Serre's conjectures, it looked as though that something was missing. Namely, the operators defined by these authors produced weight shifts of the form

$$\theta_{n-1} \colon \mathcal{V}(\lambda) \longrightarrow \mathcal{V}(\lambda + \underline{\Delta}_{n-1}),$$

where now the *automorphic weight* $\lambda \in \mathbf{Z}^n$ is a n-tuple of integers, subject to certain conditions $[\lambda_1 \geq \ldots \geq \lambda_{n-1}]$, and $\underline{\Delta}_{n-1} = (p+1, p, \ldots, p, 1)$. The shape of $\underline{\Delta}_{n-1}$ resembled that of certain Serre weights for $\mathbf{G}_k = \mathbf{GL}_{n,k} \times \mathbf{G}_{m,k}$, but, at the same time, looking at these weights one[3] would expect to find some other weight shifting operators, producing a shift of the form

$$\underline{\Delta}_r = (p+1, p, \ldots, p, \overbrace{1, \ldots, 1}^{n-r-1 \text{ times}}, 1).$$

The main result of my thesis is that these weight shifts can be realised geometrically, by working on some deeper EO strata. Namely, we have the following.

Theorem 3.3 (LLP)**.** *Let $1 \leq r \leq n-1$ be an integer and λ an automorphic weight. There exists a differential operator*

$$\theta_r \colon \mathrm{gr}^{\bullet,r}(\mathcal{V}(\lambda)) \longrightarrow \mathrm{gr}^{\bullet,r}(\mathcal{V}(\lambda + \underline{\Delta}_r)),$$

defined on the (closure of the) Ekedahl-Oort stratum $\overline{S}_{K,r}$, with

$$\underline{\Delta}_r = (p+1, p, \ldots, p, 1, \ldots, 1, 1)$$

where exactly the last $n-r$ entries are 1. *We have the following properties.*

(i) The operator θ_r is A_r-linear, that is, $\theta_r(A_r) = 0$, where A_r is the partial Hasse invariant *defined in* [10].
(ii) The operator θ_r is Hecke-equivariant up to a cyclotomic twist.
(iii) Let $f \in H^0(\overline{S}_{K,r}, \mathrm{gr}^{\bullet,r}(\mathcal{V}(\lambda)))$ and write it as $f = \sum_{\underline{a}} f_{\underline{a}}$, for the decomposition described in [10]. *If $r = 1, 2$, then $\theta_r(f)$ is divisible by the Hasse invariant A_r if and only if for each component $f_{\underline{a}}$ either $A_r \mid f_{\underline{a}}$ or $p \mid a_1 + \lambda_n$.*

4. ALRIGHT, WHAT'S NEXT?

4.1. **Smoothness.** A fundamental point needed in the construction from my thesis was the smoothness of the closures $\overline{S}_{K,r}$. This can be proved using Grothendieck–Messing theory and, in fact, in the process, one shows that a certain generalisation of the Kodaira–Spencer map is an isomorphism. The smoothness of a *union* of distinct EO strata is, in general, a rare occurence[4]. This leads naturally to the following question.

Question 4.1. Suppose that S_K is now the geometric special fibre of some other Shimura variety at p-hyperspecial level K, relative to some datum (G, X). When is some union of EO strata smooth (or just normal)?

[3]Fred Diamond.

[4]An intuitive reason why is that the same is true for Schubert strata of $\mathbf{P} \backslash \mathbf{G}$ for general $\mathbf{G}$ and $\mathbf{P}$.

This question makes sense, thanks to the existence and smoothness of integral models, for Hodge type Shimura varieties, at least. In that context, one can define the EO stratification by looking at the fibres of the smooth surjective map

$$\zeta\colon S_K \longrightarrow \mathbf{G}_{\mathbf{F}_p}\text{-}\mathtt{Zip}^{\mu},$$

where $\mathbf{G}_{\mathbf{F}_p}\text{-}\mathtt{Zip}^{\mu}$ is the *stack of* $\mathbf{G}_{\mathbf{F}_p}$*-zips.* It is a 0-dimensional stack that, roughly speaking, parametrises certain families of $\mathbf{G}_{\mathbf{F}_p}$-torsors with additional data that one can naturally associate with points of the special fibre of the Shimura variety. In particular, ζ should be thought of as a mod p *period map.* Many problems regarding the geometry of the special fibre (vanishing of cohomology, existence of sections, smoothness of strata, etc.) can be restated in terms of the stack of $\mathbf{G}_{\mathbf{F}_p}$-zips, where they usually become group-theoretic problems, relating to the algebraic representation theory of $\mathbf{G}_{\mathbf{F}_p}$. In the case of Question 4.1, Jean-Stefan Koskivirta, Stefan Reppen and I have been working on some special cases of a conjectural answer.

Let $w \in {}^IW \cong W_I\backslash W$ be the index corresponding to some EO stratum $\mathcal{X}_w \subseteq \mathbf{G}_{\mathbf{F}_p}\text{-}\mathtt{Zip}^{\mu}$ and let $w' \in {}^IW$ be the index of a *lower neighbour* $\mathcal{X}_{w'}$ of $\mathcal{X}_w$ (that is, $\mathcal{X}_{w'} \subseteq \overline{\mathcal{X}}_w$ with codimension 1). We can associate to w, w' two parabolic subgroups $P_w, P_{w'}$ of $\mathbf{G}_k$, sometimes called *canonical parabolics.*

Conjecture 4.2 (Koskivirta)**.** *The union $\mathcal{X}_w \cup \mathcal{X}_{w'}$ is smooth if and only if $P_{w'} \subseteq P_w$.*

We can prove some special cases, for instance, we have the following.

Theorem 4.3 (Koskivirta, Reppen, LP)**.** *Suppose that G is a unitary group with $G(\mathbf{R}) = \mathrm{GU}(r,s)$, with* $\gcd(r,s) = 1$*. Then, the closure of the unique one-dimensional EO stratum in $S_{K,k}(G,X)$ is smooth. In this case the canonical parabolics are Borel subgroups.*

We can compute explicitly many more examples, mainly in the case of unitary and orthogonal groups, and we are currently working on a general result.

4.2. **Vanishing.** In another on-going project with Koskivirta we are looking at the possible generalisations of Goldring's and Koskivirta's *cone conjectures.* Namely, the automorphic bundles $\mathcal{V}(\lambda)$ can be defined naturally on $\mathbf{G}_{\mathbf{F}_p}\text{-}\mathtt{Zip}^{\mu}$, in the sense that they are pullbacks of some bundles via ζ. In fact, since ζ is smooth (hence flat) and surjective, we have an inclusion

$$H^0(\mathbf{G}_{\mathbf{F}_p}\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\lambda)) \hookrightarrow H^0(S_K, \mathcal{V}(\lambda)).$$

The cone conjectures say that, up to replacing λ by a positive multiple, the bigger space is non-zero if and only if the smaller one is. This gives some control on the *automorphic cone* of S_K, that is,

$$\mathcal{C}_{\mathrm{auto}} \coloneqq \{\lambda \in \mathbf{Z}^n \mid \exists N > 0, H^0(S_K, \mathcal{V}(N\lambda)) \neq 0\}.$$

Working on the side of the stack $\mathbf{G}_{\mathbf{F}_p}\text{-}\mathtt{Zip}^{\mu}$ we can describe generators of this cone, hence obtaining vanishing results for the coherent cohomology of automorphic sheaves, which are important in the study of general Serre conjectures. The work of my thesis leads naturally to the following question.

Question 4.4. Does the natural analogue of the cone conjectures hold for closures of strata $\overline{\mathcal{X}}_w$? Can the relative cones be described explicitly?

Theorem 4.5 (Koskivirta, LP)**.** *In the case of a unitary group of signature* $(2,1)$ *(i.e. for the* Picard modular surface*) the cone conjecture holds for all strata and one can give an explicit formula describing the non-vanishing cones.*

We are currently working on generalising this result to $\mathrm{GU}(n-1,1)$ and all the strata therein. Even just describing the cone over the whole special fibre is challenging (the combinatorics becomes rather involved quite quickly) and that is the goal of a separate project of Goldring, Koskivirta and Yang (Deding).

4.3. **Divisibility criteria and simplicity of zeros.** Point (ii) in Proposition 1.4 and point (iii) of Theorem 3.3 are essential when looking at applications to the Serre conjectures, because they lead to entailments where the weight is shifted "by a small amount" (in the classical case, if $h \nmid f$, but $p \mid \lambda$, then $\theta(f)/h$ is a modular form of weight $\lambda + 2$ and 2 is smaller than $p+1$). Such a divisibility is called a *drop in the weight filtration.* Giving criteria describing when such a drop happens is essential and intimately linked to the following question.

Question 4.6. When do sensible generalisations of the classical Hasse invariant have simple zeros?

I think I can give a partial answer, which would entail the following.

Conjecture 4.7. *In the context of* [3]*, that is, for PEL Shimura varieties of type A and C and p totally split in the reflex field E, the relevant Hasse invariants h_τ have simple zeros.*

Moreover, denoting by θ the operator defined by the authors, for $f \in H^0(S_K, \mathcal{V}(\lambda))$, we have that $h_\tau \mid \theta(f)_\tau$ if and only if $h_\tau \mid f_\tau$ or $p \mid \lambda_{\tau,1} + \lambda_{\tau^*,1} = \langle \lambda, \delta^\vee_{\tau,1} + \delta^\vee_{\tau^*,1} \rangle$, in case (A), $p \mid \lambda_{\tau,1}$, in case (C).

Some technical details are still missing, but similar results have been proved by other authors in unpublished work. Moreover, the same techniques involving flag spaces should be applicable to the setting of Theorem 3.3, where it would allow me to remove the hypothesis $r = 1, 2$ in point (iii).

4.4. **Theta operators on the stack of G-zips.** Another question that arose naturally in conversation with Wushi Goldring and Jean-Stefan Koskivirta is whether one could define theta operators already on the period stack $\mathbf{G}_{\mathbf{F}_p}\text{−}\mathtt{Zip}^\mu$. While this seems intuitively natural on the one hand, for instance, because one can define Hasse invariants and their generalisations on $\mathbf{G}_{\mathbf{F}_p}\text{−}\mathtt{Zip}^\mu$, on the other the stack $\mathbf{G}_{\mathbf{F}_p}\text{−}\mathtt{Zip}^\mu$, being of dimension 0, is not "big enough" to have the kind of differential structure (e.g. the sheaf of differentials Ω^1) that one needs to construct θ operators, at least not in a naive sense.

Recently, building on ideas of Wushi and Jean-Stefan, I have started exploring a direction of research that might bypass this kind of issues. In short, when stating their aforementioned cone conjectures, Goldring and Koskivirta do not limit themselves to period maps ζ coming from a Shimura variety. They consider more generally morphisms of k-stacks

$$\xi \colon S \longrightarrow G\text{−}\mathtt{Zip}^\mu$$

such that:

(i) S is a nice enough scheme over k (quasi-projective variety),
(ii) ξ is smooth (hence S is),
(iii) ξ is surjective from each connected component of S.

One can then pullback automorphic bundles from $G\text{−}\mathtt{Zip}^\mu$ to S, consider sections on S and the saturated cones $\mathcal{C}_{\mathrm{zip}} \subseteq \mathcal{C}_S$.

Conjecture 4.8. *One has* $\mathcal{C}_{\mathrm{zip}} = \mathcal{C}_S$.

My idea is to impose further conditions on ξ as above, namely:

(iv) ξ factors through some enrichment $G\text{−}\mathtt{Zip}^{\mu,\nabla}$ of $G\text{−}\mathtt{Zip}^\mu$ that gives an integrable connection on something like $H^1_{\mathrm{dR}}(A/S)$,
(v) the connection from the previous point gives rise to a Kodaira–Spencer-like isomorphism.

These conditions are very restrictive. For instance, point (iv) forces the dimension of S to be that of $\mathbf{P}\backslash\mathbf{G}$, where $\mathbf{P}$ is the *Hodge parabolic*. One interesting twist that this approach allows, is that one could consider this set-up for stacks of G-zips that *do not come* from Shimura data. In a basic example, one can take $S = \mathbf{P}^1_k, \mathcal{Z} = (G = \mathbf{GL}_{2,\mathbf{F}_p}, P = B^+, Q = B^+, L = M = T)$ and define naturally a map

$$\xi \colon \mathbf{P}^1_k \longrightarrow \mathbf{GL}_2\text{−}\mathtt{Zip}^{\mathcal{Z}}$$

satisfying the conditions above (several, in fact). From this, one can recover, through this abstract construction of a zip-theoretic theta operator, the *Dickson invariant* $X_0X_1^p - X_0^pX_1$. This is an invariant that is known to be related to the theta operator in the representation theoretic approach the classical Serre's conjectures. This construction seems to make the analogy between theta and $X_0X_1^p - X_0^pX_1$ more explicit in geometric terms.

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