Singularities in the Ekedahl–Oort stratification

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- We work with **flag spaces** over **stacks of G-zips**.
- Results pull back to Shimura varietes via **zip period maps**.
- On the stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$, many results hold also for Shimura-like zip data of **exceptional type**. The zip period map is missing.

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- They are linked to the existence of certain **Hasse invariants**.
- They are linked to extensions of the **canonical filtration**.
- The **rigidity** of $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ makes these links “tighter”.

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  - a couple (G, μ) over k to a Shimura datum (G, μ) (over $\mathbb{Q}$ and the reflex field).

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- $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ is also a moduli space of G**-zips of type** $\mathcal{Z}$: $\underline{\mathcal{I}} = (\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \iota)$, $\mathcal{I}$ is a G-torsor, $\mathcal{I}_P, \mathcal{I}_Q \subseteq \mathcal{I}$ are P, Q-torsors, $\iota \colon (\mathcal{I}_P/R_u)^{(p)} \xrightarrow{\sim} \mathcal{I}_Q/R_u$.

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- For (G,μ) "Shimura-like", often we can define a **smooth, surjective** period map $\zeta \colon S \to G\text{-}\mathtt{Zip}^{\mu}$.

The modular curve

Example (Modular curve)

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    - the **supersingular locus**.
- $(\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \iota)$ are given by **Hodge** and **conjugate filtrations**, with the **Cartier isomorphism** on $H^1_{\mathrm{dR}}(E/X)$.

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- For a given $w \in {}^I W$, there is a special choice $P_0 = P_w$, the **canonical parabolic**.

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- Via $G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}_{P_0} \to G\text{-}\mathtt{Zip}^{\mathcal{Z}_0}$ and $G\text{-}\mathtt{Zip}^{\mathcal{Z}_0} \to [P_0 \backslash G_k / Q_0]$ we obtain stratifications on $G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}_{P_0}$.

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- We want to compare these stratifications to ${}^I W$ on $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ via $G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}_{P_0} \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$.

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- We want to understand the morphism

$$\pi_{P_w} : \overline{G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}_{P_w,w}} \to \overline{G\text{-}\mathtt{Zip}^{\mathcal{Z}}_w}.$$

Some technical definitions

- We say $\mathcal{U} \subseteq G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ is w-**open** if it is open in $\overline{G\text{-}\mathtt{Zip}}^{\mathcal{Z}}_{w}$. Notice that $\mathcal{U}$ corresponds to $\Gamma_{\mathcal{U}} \subseteq {}^{I}W$.

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- A canonical cover is **separating (SCC)** if $\mathcal{U}^{(P_w)} = \overline{G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}_{P_w,w}} \cap \pi_{P_w}^{-1}(\mathcal{U})$.
- A w-open is **elementary** if $\Gamma_{\mathcal{U}} = \{w, w'\}$ and $G\text{-}\mathtt{Zip}^{\mathcal{Z}}_{w'}$ has codimension one in $\overline{G\text{-}\mathtt{Zip}_w^{\mathcal{Z}}}$.

Theorem A

Theorem (2025, J.-S. Koskivirta, LLP, S. Reppen)

Let $\mathcal{U} \subseteq G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ be a w-open, for $w \in {}^I W$. TFAE:

1. *The map $s_w : G\text{-}\mathtt{Zip}^{\mathcal{Z}}_w \to G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}_{P_w,w}$ extends to a section $\tilde{s}_w : \mathcal{U} \to G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}_{P_w}$ of π_{P_w}.*
2. *$\mathcal{U}$ admits a **SCC** and is **normal**.*
3. *$\mathcal{U}$ admits a **SCC** and is w-**bounded**.*
4. *$\mathcal{U}$ admits a **SCC** and $\pi_{P_w} : \mathcal{U}^{(P_w)} \to \mathcal{U}$ is an **isomorphism**.*

In that case, $\mathcal{U}$ is Cohen–Macaulay and the extension $\tilde{s}_w$ is unique.
*If $\mathcal{U}$ is elementary, we **can remove SCC** from 2.*

Theorem B

Theorem (2025, J.-S. Koskivirta, LLP, S. Reppen)

Let S be the special fiber of an abelian type Shimura variety with good reduction. For any one-dimensional Ekedahl–Oort stratum S_w, there is a description of the smooth locus of $\overline{S}_w$.

1. *In type* A, *this depends uniquely on the signature.*
2. *In type* B, C *or* D, *the smooth locus of $\overline{S}_w$ is always S_w.*

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Example

For simple type A of signature (r, s), $\overline{S}_w$ is smooth iff $\gcd(r, s) = 1$.

Theorem C

Theorem (2025, J.-S. Koskivirta, LLP, S. Reppen)

Let S be the special fiber of a Shimura variety of simple B_n-type with good reduction. Let $S = \overline{S}_0 \supseteq \overline{S}_1 \supseteq \ldots \supseteq \overline{S}_{2n-1}$ denote the EO-stratification of S.

1. *For $0 \leq j \leq n-1$:*
   - i. *The* ***smooth locus*** *of $\overline{S}_j$ is $\bigcup_{i=j}^{2n-1-j} S_i$.*
   - ii. *The closure $\overline{S}_j$ is* ***normal*** *and a* ***local complete intersection****.*
   - iii. *The closure $\overline{S}_j$ admits a* ***reduced strata Hasse invariant*** *of weight $(p^{j+1}-1)\eta_\omega$, where η_ω is the Hodge character.*
2. *For $j \geq n$, the smooth and normal locus of $\overline{S}_j$ is S_j.*

Thank you for your attention!

Questions?

Some applications and developments

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- Some relations with characteristic zero.

The modular curve II

Example (Modular curve)

To an elliptic scheme $A/S/k$ we can associate the torsors

$$\begin{aligned} \mathcal{I} &= \mathrm{isom}(H^1_{\mathrm{dR}}(A/S), \mathcal{O}_S^2), \\ \mathcal{I}_P &= \mathrm{isom}(H^1_{\mathrm{dR}}(A/S) \supset \underline{\omega}, \mathcal{O}_S^2 \supset \mathcal{O}_S), \\ \mathcal{I}_Q &= \mathrm{isom}(\ker(V) \subset H^1_{\mathrm{dR}}(A/S), \mathcal{O}_S \subset \mathcal{O}_S^2), \end{aligned}$$

with the isomorphisms given by **Frobenius** and **Verschiebung**:

$$\begin{aligned} F &\colon (H^1_{\mathrm{dR}}(A/S)/\underline{\omega})^{(p)} \to \mathrm{im}(F) = \ker(V), \\ V &\colon H^1_{\mathrm{dR}}(A/S)/\ker(V) \to \underline{\omega}^{(p)}. \end{aligned}$$