Generalised Theta Operators on Unitary Shimura Varieties

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Motivation

- The theory of the classical *theta operator* $(\bmod\, p)$ was used in the proof of the weight part of *Serre's modularity conjecture*.
- Edixhoven's proof relied, in particular, on the study of the θ*-cycles* of Tate and Jochnowitz.
- The construction of the classical θ has been extended to more general settings (Hilbert and Siegel modular varieties, certain PEL Shimura varieties).
- Many questions remain open, in particular, a clear generalisation of the theory of θ-cycles.

Modular Curves & Modular Forms

- Fix p a prime and $N \geq 5$ an integer prime to p. Write $\mathbb{F}$ for an algebraic closure of $\mathbb{F}_p$.
- Consider the modular curve $Y = Y_1(N)$ of level $\Gamma_1(N)$ over $\mathbb{F}$ and its compactification $X = X_1(N)$.
- Over X we have $\underline{\omega}$, the *Hodge sheaf*. It is an invertible sheaf and on Y we can define it as $\pi_*\Omega^1_{E/Y}$, where $\pi \colon E \to Y$ is the universal elliptic curve.
- The elements of $M_k(N) = H^0(X, \underline{\omega}^k)$ are modular forms of level $\Gamma_1(N)$ weight k with coefficients in $\mathbb{F}$. They form a $\mathbb{Z}$-graded algebra $M(N) = \oplus_k M_k(N)$.

The Hasse Invariant

- In characteristic $p > 0$, we have a special modular form $h \in H^0(X, \underline{\omega}^{p-1})$, called the *(classical) Hasse invariant*.
- The Hasse invariant vanishes with simple zeroes on the *supersingular locus* $Y^{\mathrm{ss}} \subset Y \subset X$. The complement $X^{\mathrm{ord}} = X \setminus Y^{\mathrm{ss}}$ is called the *ordinary locus*.

de Rham Cohomology

- We can consider the (relative) *de Rham cohomology* $H^i_{\mathrm{dR}}(E/Y) := R^i\pi_*(\Omega^\bullet_{E/Y})$ of E over Y. This is a finite locally free $\mathcal{O}_Y$-sheaf.
- On $H = H^1_{\mathrm{dR}}(E/Y)$ we have the *Gauss-Manin connection*
$$\nabla \colon H \longrightarrow H \otimes_{\mathcal{O}_Y} \Omega^1_{Y/\mathbb{F}}.$$
- We also have the *Hodge filtration*
$$0 \longrightarrow \underline{\omega} \longrightarrow H \longrightarrow \underline{\omega}^\vee \longrightarrow 0. \tag{H}$$
- Combining (H) with the GM connection we get a morphism
$$\underline{\omega} \longrightarrow H \xrightarrow{\nabla} H \otimes \Omega^1_{Y/\mathbb{F}} \longrightarrow \underline{\omega}^\vee \otimes \Omega^1_{Y/\mathbb{F}}.$$

  The corresponding map $\underline{\mathrm{ks}} \colon \underline{\omega}^2 \to \Omega^1_{Y/\mathbb{F}}$ is the *Kodaira-Spencer isomorphism.*

Unit-Root Splitting I

- Over the dense open Y^{ord} we can split (H) naturally, using a cosection
$$p_{\mathrm{ur}}\colon H \longrightarrow \underline{\omega}.$$
called the *unit-root splitting*.
- This splitting cannot be extended to Y "naturally" (there are poles along Y^{ss}).

Unit-Root Splitting II

- We can also consider

$$h \cdot p_{\mathrm{ur}} \colon H \longrightarrow \underline{\omega}^{p},$$

which is well defined over all of Y.
- On $\mathrm{Sym}^k(H)$ there is a natural filtration $F^\bullet$ coming from (H), defined by

$$F^i(\mathrm{Sym}^k(H)) = \mathrm{im}(\underline{\omega}^i \otimes \mathrm{Sym}^{k-i}(H) \to \mathrm{Sym}^k(H)).$$

- One can also extend uniquely

$$h \cdot \mathrm{Sym}^k(p_{\mathrm{ur}})|_{F^{k-1}(\mathrm{Sym}^k(H))} \colon F^{k-1}(\mathrm{Sym}^k(H)) \longrightarrow \underline{\omega}^{k+p-1}$$

from Y^{ord} to Y.

The Theta Operator I

- The GM connection induces a connection on $\mathrm{Sym}^k(H)$, still denoted ∇. By construction, we have the transversality property

$$\nabla(F^i(\mathrm{Sym}^k(H))) \subseteq F^{i-1}(\mathrm{Sym}^k(H)) \otimes \Omega^1_{Y/\mathbb{F}}.$$

- We can finally consider the composition

$$\theta\colon \underline{\omega}^k \xrightarrow{\nabla} F^{k-1}(\mathrm{Sym}^k(H)) \otimes \Omega^1_{Y/\mathbb{F}} \longrightarrow \underline{\omega}^{k+p+1}$$

where the second map is $(h \cdot \mathrm{Sym}^k(p_{\mathrm{ur}})) \otimes \underline{\mathrm{ks}}^{-1}$. This is the *(classical) theta operator*.

The Theta Operator II

- We defined θ on Y, but taking global sections one actually obtains
$$\theta\colon M_k(N) \longrightarrow M_{k+p+1}(N),$$
the theta operator *on modular forms*.
- This θ is an $\mathbb{F}$-linear differential graded operator of degree $p+1$ of the graded algebra $M(N)$ into itself. Moreover, $\theta(h) = 0$.
- We have the relations
$$\theta T_l = l T_l \theta$$
for T_l the Hecke operator at the prime $l \neq p$.

Picard Modular Surfaces I

- Let $E/\mathbb{Q}$ be a quadratic imaginary field. We write $\mathrm{Hom}(E, \mathbb{C}) = \{\sigma, \overline{\sigma}\}$. Assume $p > 2$ and that it splits in E.
- We denote by $S/\mathbb{F}$ the geometric special fibre of the *Picard modular surface*. S is a moduli space parametrising $\underline{A} = (A/T, \lambda, \iota, \eta)$, where:
  1. A is an abelian scheme of relative dimension 3 over $T \in \underline{\mathrm{Sch}}_{\mathbb{F}}$.
  2. λ is a prime-to-p polarisation of A.
  3. ι is an $\mathcal{O}_E$-action on A, of signature $(2, 1)$.
  4. η is a (p-hyperspecial and neat) level structure on A.
- S is smooth, quasi-projective of dimension 2 with universal object $\pi\colon A \to S$.

Picard Modular Surfaces II

- The Hodge sheaf $\underline{\omega} = \pi_* \Omega^1_{A/S}$ is now locally free of rank 3 with an action of $\mathcal{O}_E$ which splits it as
$$\underline{\omega} = \underline{\omega}_\sigma \oplus \underline{\omega}_{\overline{\sigma}}.$$
- The summands $\underline{\omega}_\sigma$ and $\underline{\omega}_{\overline{\sigma}}$ are both locally free, with ranks 2 and 1, respectively.
- The Hodge filtration also splits according to the $\mathcal{O}_E$-action as
$$0 \longrightarrow \underline{\omega}_\tau \longrightarrow H_\tau \longrightarrow \underline{\omega}_{\overline{\tau}}^\vee \longrightarrow 0, \tag{H'}$$
where $H = H^1_{\mathrm{dR}}(A/S)$, $\tau \in \{\sigma, \overline{\sigma}\}$.

Automorphic Weights & Sheaves

- An automorphic weight will be a couple $(\underline{k}, w)$, where $\underline{k} = (k_1, k_2) \in \mathbb{Z}^2, w \in \mathbb{Z}$, such that $k_1 \geq k_2$.
- The automorphic sheaf of weight $(\underline{k}, w)$ will be

$$\underline{\omega}^{\underline{k},w} := \mathrm{Sym}^{k_1-k_2}(\underline{\omega}_\sigma) \otimes (\wedge^2 \underline{\omega}_\sigma)^{k_2} \otimes \delta^w,$$

  where $\delta = \delta_\sigma := \det H_\sigma$.
- We will also consider

$$H^{\underline{k},w} := \mathrm{Sym}^{k_1-k_2}(H_\sigma) \otimes \mathrm{Sym}^{k_2}(\wedge^2 H_\sigma) \otimes \delta^w,$$

  where $k_2 \geq 0$.

The Ekedahl-Oort Stratification

- We have 3 EO strata in this case:
  1. The *ordinary locus* S^{μ}, open and dense in S.
  2. The *almost ordinary locus* S^{ao}, locally closed of dimension 1.
  3. The *core locus* S^{core}, closed of dimension 0.
- We will work on the *non-ordinary locus*

$$S^{\mathrm{no}} = S \setminus S^{\mu} = S^{\mathrm{ao}} \sqcup S^{\mathrm{core}} = \overline{S^{\mathrm{ao}}}.$$

It is smooth of dimension 1 (this depends on p split!).

The GM Connection & KS Morphism

- We have as before a Kodaira-Spencer morphism

$$\underline{\mathrm{KS}}\colon \underline{\omega} \otimes \underline{\omega} \longrightarrow \Omega^1_{S/\mathbb{F}}.$$

- $\underline{\mathrm{KS}}$ is not an isomorphism, but its $\mathcal{O}_E$-components are:

$$\underline{\mathrm{ks}} = \underline{\mathrm{KS}}_\sigma \colon \underline{\omega}_\sigma \otimes \det \underline{\omega}_\sigma \otimes \delta^{-1} \longrightarrow \Omega^1_{S/\mathbb{F}}.$$

Ordinary Theta

- The GM connection induces natural connections

$$\nabla\colon H^{\underline{k},w} \longrightarrow H^{\underline{k},w} \otimes \Omega^1_{S/\mathbb{F}}.$$

- These satisfy a natural transversality property

$$\nabla(\underline{\omega}^{\underline{k},w}) \subseteq F(H^{\underline{k},w}) \otimes \Omega^1_{S/\mathbb{F}},$$

  the penultimate step of a natural filtration.

- Over S we can define

$$\theta_1\colon \underline{\omega}^{\underline{k},w} \xrightarrow{\nabla} F(H^{\underline{k},w}) \otimes \Omega^1_{S/\mathbb{F}} \longrightarrow \underline{\omega}^{\underline{k}+\underline{\Delta}_1,w-1}$$

  where the second arrow is $h_\sigma \cdot (p_{\mathrm{ur},\sigma})^{\underline{k},w} \otimes \underline{\mathrm{ks}}^{-1}$ and $\underline{\Delta}_1 = (p+1,p)$. This is the *"ordinary" theta operator.*

Filtrations on S^{no}

- On S^{no} we can consider $\underline{\omega}_0 = \underline{\omega}_\sigma[V]$. It is an invertible sheaf and fits into a short exact sequence

$$0 \longrightarrow \underline{\omega}_0 \longrightarrow \underline{\omega}_\sigma \longrightarrow \underline{\omega}_\mu \longrightarrow 0,$$

  which defines the invertible sheaf $\underline{\omega}_\mu$.
- Moreover, we have that $\underline{\omega}_0 = \ker(V : H_\sigma \to H_\sigma^{(p)})$. This also fits in the ses

$$0 \longrightarrow \underline{\omega}_0 \longrightarrow H_\sigma \longrightarrow H_\mu \longrightarrow 0.$$

- We have $A_2 \in H^0(S^{\mathrm{no}}, \underline{\omega}_\mu^{p-1})$, the *(partial) almost-ordinary Hasse invariant.* It vanishes on S^{core} and nowhere on S^{ao}.

The Generalised Splitting I

- Over S^{ao} we have a natural morphism
$$p_{\mathrm{ur},2}\colon H_\mu \longrightarrow \underline{\omega}_\mu.$$
- The morphism $p_{\mathrm{ur},2}$ splits the ses
$$0 \longrightarrow \underline{\omega}_\mu \longrightarrow H_\mu \longrightarrow \underline{\omega}_\sigma^\vee \longrightarrow 0. \tag{H-no}$$
- We cannot extend $p_{\mathrm{ur},2}$ to S^{core} naturally, but we can extend
$$A_2 \cdot p_{\mathrm{ur},2}\colon H_\mu \longrightarrow \underline{\omega}_\mu^p.$$

The Generalised Splitting II

- Over S^{no}, we can define filtrations on $H^{\underline{k},w}$ and $\underline{\omega}^{\underline{k},w}$, starting from $\underline{\omega}_0 \subseteq \underline{\omega}_\sigma$ and $\underline{\omega}_0 \subseteq H_\sigma$.
- We can moreover define a filtration on $\mathrm{gr}^\bullet(H^{\underline{k},w})$, starting from (H-no), whose penultimate step we call $F(\mathrm{gr}^\bullet(H^{\underline{k},w}))$ and whose last step is $\mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w})$.
- Finally, we can define over the whole S^{no}, starting from $p_{\mathrm{ur},2}$ and $\mathrm{id}_{\underline{\omega}_0}$, a morphism

$$A_2 \cdot \mathrm{gr}^\bullet(p_{\mathrm{ur},2})^{\underline{k},w}|_F \colon F(\mathrm{gr}^\bullet(H^{\underline{k},w})) \longrightarrow \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k}+(p-1,0),w}).$$

The Generalised Operator I

- Since $\underline{\omega}_0 = H_\sigma[V]$, the GM connection on S^{no} (which is smooth) has the property

$$\nabla(\underline{\omega}_0) \subseteq \underline{\omega}_0 \otimes \Omega^1_{S^{\mathrm{no}}/\mathbb{F}}.$$

- In particular, we also have the induced connection

$$\nabla \colon H_\mu \longrightarrow H_\mu \otimes \Omega^1_{S^{\mathrm{no}}/\mathbb{F}}.$$

- The Kodaira-Spencer morphism induces on S^{no} an isomorphism

$$\underline{\mathrm{ks}}_\mu \colon \underline{\omega}_\mu \otimes \det \underline{\omega}_\sigma \otimes \delta^{-1} \longrightarrow \Omega^1_{S^{\mathrm{no}}/\mathbb{F}}.$$

The Generalised Operator II

- Finally, we can consider the composition

  $$\theta_2 \colon \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) \xrightarrow{\nabla} F(\mathrm{gr}^\bullet(H^{\underline{k},w})) \otimes \Omega^1_{S^{\mathrm{no}}/\mathbb{F}} \longrightarrow \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k}+\underline{\Delta}_2,w-1})$$

  where the second map is $A_2 \cdot \mathrm{gr}^\bullet(p_{\mathrm{ur},2})^{\underline{k},w} \otimes \underline{\mathrm{ks}}_\mu^{-1}$ and $\underline{\Delta}_2 = (p+1,1)$. We call this the *almost-ordinary "generalised" theta operator*.
- The action of θ_2 on global sections is Hecke equivariant.
- θ_2 is a differential operator and $\theta_2(A_2) = 0$.

Thank you for your attention!

Questions?