CYCLE CLASSES IN OVERCONVERGENT RIGID COHOMOLOGY AND A SEMISTABLE LEFSCHETZ $(1,1)$ THEOREM

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ABSTRACT. In this article we prove a semistable version of the variational Tate conjecture for divisors in crystalline cohomology, stating that a rational (logarithmic) line bundle on the special fibre of a semistable scheme over $k[\![t]\!]$ lifts to the total space if and only if its first Chern class does. The proof is elementary, using standard properties of the logarithmic de Rham–Witt complex. As a corollary, we deduce similar algebraicity lifting results for cohomology classes on varieties over global function fields. Finally, we give a counter example to show that the variational Tate conjecture for divisors cannot hold with $\mathbb{Q}_p$-coefficients.

CONTENTS

INTRODUCTION

Many of the deepest conjectures in arithmetic and algebraic geometry concern the existence of algebraic cycles on varieties with certain properties. For example, the Hodge and Tate conjectures state, roughly speaking, that on smooth and projective varieties over $\mathbb{C}$ (Hodge) or finitely generated fields (Tate) every cohomology class which 'looks like' the class of a cycle is indeed so. One can also pose variational forms of these conjectures, giving conditions for extending algebraic classes from one fibre of a smooth, projective morphism $f : X \to S$ to the whole space. For divisors, the Hodge forms of both these conjectures (otherwise known as the Lefschetz $(1,1)$ theorem) are relatively straightforward to prove, using the exponential map, but even for divisors the Tate conjecture remains wide open in general.

Applying the principle that deformation problems in characteristic p should be studied using p-adic cohomology, Morrow in [Mor14] formulated a crystalline variational Tate conjecture for smooth and proper families $f : X \to S$ of varieties in characteristic p, and proved the conjecture for divisors, at least when f is projective. The key step of the proof is a version of this result over $S = \mathrm{Spec}\,(k[\![t_1,\ldots,t_n]\!])$, which when $n = 1$ is a direct equicharacteristic analogue of Berthelot and Ogus' theorem [BO83, Theorem 3.8] on lifting line bundles from characteristic 0 to characteristic p.

Morrow's proof of the local statement uses some fairly heavy machinery from motivic homotopy theory, in particular a 'continuity' result for topological cyclic homology. In this article we provide a new proof of the local crystalline variational Tate conjecture for divisors, at least over the base $S = \mathrm{Spec}\,(k[\![t]\!])$, which only uses some fairly basic properties of the de Rham–Witt complex. The point of giving this proof is that it adapts essentially verbatim to the case of semistable reduction, once the corresponding basic properties of the *logarithmic* de Rham–Witt complex are in place.

So let $\mathscr{X}$ be a semistable, projective scheme over $k[\![t]\!]$, with special fibre X_0 and generic fibre X. Then there is an isomorphism

$$H^2_{\mathrm{rig}}(X/\mathscr{R})^{\nabla=0} \cong H^2_{\text{log-cris}}(X_0^\times/K^\times)^{N=0}$$

between the horizontal sections of the Robba ring-valued rigid cohomology of X and the part of the log-crystalline cohomology of X_0 killed by the monodromy operator. The former is defined to be $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)\otimes_{\mathscr{E}^\dagger}\mathscr{R}$, where $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$ is the bounded Robba ring-valued rigid cohomology of X constructed in [LP16]. These groups are (φ,∇)-modules over $\mathscr{R}$ and $\mathscr{E}^\dagger$ respectively. In particular, if $\mathscr{L}$ is a line bundle on X_0, we can view its first Chern class $c_1(\mathscr{L})$ as an element of $H^2_{\mathrm{rig}}(X/\mathscr{R})$. Our main result is then the following semistable version of the local crystalline variational Tate conjecture for divisors.

Theorem (4.5). *$\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$ if and only if $c_1(\mathscr{L})$ lies in $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)\subset H^2_{\mathrm{rig}}(X/\mathscr{R})$.*

There is also a version for logarithmic line bundles on X_0. The general philosophy of p-adic cohomology over $k((t))$ is that the $\mathscr{E}^\dagger$-structure $H^i_{\mathrm{rig}}(X/\mathscr{E}^\dagger)\subset H^i_{\mathrm{rig}}(X/\mathscr{R})$ is the equicharactersitic analogue of the Hogde filtration on the p-adic cohomology of varieties over mixed characteristic local fields. With this in mind, this is the direct analogue of Yamashita's semistable Lefschetz (1,1) theorem [Yam11]. As a corollary, we can deduce a global result on algebraicity of cohomology class as follow. Let F be a function field of transcendence degree 1 over k, and X/F a smooth projective variety. Let v be a place of semistable reduction for X, with reduction X_v. Then there is a map

$$\mathrm{sp}_v : \mathscr{H}^2_{\mathrm{rig}}(X/K)^{\nabla=0} \to H^2_{\text{log-cris}}(X_v^\times/K_v^\times)$$

from the second cohomology of X (see §5) to the log crystalline cohomology of X_v.

Theorem (5.2). *A class $\alpha\in\mathscr{H}^2_{\mathrm{rig}}(X/K)^{\nabla=0}$ is in the image of $\mathrm{Pic}(X)_{\mathbb{Q}}$ under the Chern class map if and only if $\mathrm{sp}_v(\alpha)$ is in the image of $\mathrm{Pic}(X_v)_{\mathbb{Q}}$.*

One might wonder whether the analogue of the crystalline variational Tate conjecture holds for line bundles with $\mathbb{Q}_p$-coefficients (in either the smooth or semistable case). Unfortunately, the answer is no. Indeed, if it were true, then it follows relatively easily that the analogue of Tate's isogeny theorem would hold over $k((t))$, in other words for any two abelian varieties A,B over $k((t))$, the map

$$\mathrm{Hom}(A,B)\otimes\mathbb{Q}_p \to \mathrm{Hom}(A[p^\infty],B[p^\infty])\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

would be an isomorphism. That this cannot be true is well-known, and examples can be easily provided with both A and B elliptic curves.

Let us now summarise the contents of this article. In §1 we show that the cycle class map in rigid cohomology over $k((t))$ descends to the bounded Robba ring. In §2 we recall the relative logarithmic de Rham–Witt complex, and prove certain basic properties of it that we will need later on. In §3 we reprove a special case of the key step in Morrow's article [Mor14], showing the crystalline variational Tate conjecture for smooth and projective schemes over $k[\![t]\!]$. The argument we give is elementary. In §4 we prove the semistable version of the crystalline variational Tate conjecture over $k[\![t]\!]$, more or less copying word for word the argument in §3. In §5 we translate these results into algebraicity lifting results for varieties over *global* function fields. Finally, in §6 we give a counter-example to the analogue of the of crystalline variational Tate conjecture for line bundles with $\mathbb{Q}_p$-coefficients.

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Notations and convenions. Throughout we will let k be a perfect field of characteristic $p>0$, W its ring of Witt vectors and $K=W[1/p]$. In general we will let $F=k(\!(t)\!)$ be the field of Laurent series over k, and $R=k[\![t]\!]$ its ring of integers (although this will not be the case in §5). We will denote by $\mathscr{E}^\dagger,\mathscr{R},\mathscr{E}$ respectively the bounded Robba ring, the Robba ring, and the Amice ring over K, and we will also write $\mathscr{E}^+=W[\![t]\!]\otimes_W K$. For any of the rings $\mathscr{E}^+$, $\mathscr{E}^\dagger$, $\mathscr{R}$, $\mathscr{E}$ we will denote by $\underline{\mathbf{M}\Phi}^\nabla_{(-)}$ the corresponding category of (φ,∇)-modules, i.e. finite free modules with connection and horizontal Frobenius. A variety over a given Noetherian base scheme will always mean a separated scheme of finite type. For any abelian group A and any ring S we will let A_S denote $A\otimes_{\mathbb{Z}} S$.

1. Cycle class maps in overconvergent rigid cohomology

Recall that for varieties X/F over the field of Laurent series $F=k(\!(t)\!)$ the rigid cohomology groups $H^i_{\mathrm{rig}}(X/\mathscr{E})$ are naturally (φ,∇)-modules over the Amice ring $\mathscr{E}$. In the book [LP16] we showed how to canonically descend these cohomology groups to obtain 'overconvergent' (φ,∇)-modules $H^i_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$ over the bounded Robba ring $\mathscr{E}^\dagger$, these groups satisfy all the expected properties of an 'extended' Weil cohomology theory. In particular, there exist versions $H^i_{c,\mathrm{rig}}(X/\mathscr{E})$, $H^i_{c,\mathrm{rig}}(X/\mathscr{E}^\dagger)$ with compact support.

Definition 1.1. Define the (overconvergent) rigid homology of a variety X/F by

$$H_i^{\mathrm{rig}}(X/\mathscr{E}):=H^i_{\mathrm{rig}}(X/\mathscr{E})^\vee,\ \ H_i^{\mathrm{rig}}(X/\mathscr{E}^\dagger):=H^i_{\mathrm{rig}}(X/\mathscr{E}^\dagger)^\vee$$

and the (overconvergent) Borel–Moore homology by

$$H_i^{\mathrm{BM,rig}}(X/\mathscr{E}):=H^i_{c,\mathrm{rig}}(X/\mathscr{E})^\vee,\ \ H_i^{\mathrm{BM,rig}}(X/\mathscr{E}^\dagger):=H^i_{c,\mathrm{rig}}(X/\mathscr{E}^\dagger)^\vee.$$

In [Pet03] the author constructs cycle class maps in rigid cohomology, which can be viewed as homomorphisms

$$A_d(X)\to H_{2d}^{\mathrm{BM,rig}}(X/\mathscr{E})$$

from the group of d-dimensional cycles modulo rational equivalence. Our goal in this section is the following entirely straightforward result.

Proposition 1.2. *The cycle class map descends to a homomorphism*

$$A_d(X)\to H_{2d}^{\mathrm{BM,rig}}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p^d}.$$

Proof. Note that since $H_{2d}^{\mathrm{BM,rig}}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p^d}\subset H_{2d}^{\mathrm{BM,rig}}(X/\mathscr{E})$ it suffices to show that for every integral closed subscheme $Z\subset X$ of dimension d, the cycle class $\eta(Z)\in H_{2d}^{\mathrm{BM,rig}}(X/\mathscr{E})$ actually lies in the subspace $H_{2d}^{\mathrm{BM,rig}}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p^d}$.

By construction, $\eta(Z)$ is the image of the fundamental class of Z (i.e. the trace map $\mathrm{Tr}_Z : H^{2d}_{c,\mathrm{rig}}(Z/\mathscr{E}) \to \mathscr{E}(-d)$) under the map

$$H^{\mathrm{BM,rig}}_{2d}(Z/\mathscr{E}) \to H^{\mathrm{BM,rig}}_{2d}(X/\mathscr{E})$$

induced by the natural map $H^{2d}_{c,\mathrm{rig}}(X/\mathscr{E}) \to H^{2d}_{c,\mathrm{rig}}(Z/\mathscr{E})$ in compactly supported cohomology. Hence it suffices to simply observe that both this map and the trace map descend to horizontal, Frobenius equivariant maps on the level of $\mathscr{E}^\dagger$-valued cohomology. Alternatively, we could observe that both $H^{2d}_{c,\mathrm{rig}}(X/\mathscr{E}) \to H^{2d}_{c,\mathrm{rig}}(Z/\mathscr{E})$ and Tr_Z are horizontal and Frobenius equivariant at the level of $\mathscr{E}$-valued cohomology, which gives

$$A_d(X) \to H^{\mathrm{BM,rig}}_{2d}(X/\mathscr{E})^{\nabla=0,\varphi=p^d},$$

then applying Kedlaya's full faithfulness theorem [Ked04, Theorem 5.1] gives an isomorphism

$$H^{\mathrm{BM,rig}}_{2d}(X/\mathscr{E})^{\nabla=0,\varphi=p^d} \cong H^{\mathrm{BM,rig}}_{2d}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p^d}.$$

□

2. Preliminaries on the de Rham–Witt complex

The purpose of this section is to gather together some results we will need on the various de Rham–Witt complexes that will be used throughout the article. These are all generalisations to the logarithmic case of well-known results from [Ill79], and should therefore present no surprises. The reader will not lose too much by skimming this section on first reading and referring back to the results as needed.

We will, as throughout, fix a perfect ground field k of characteristic $p>0$, all (log)-schemes will be considered over k. Given a morphism $(Y,N)\to(S,L)$ of fine log schemes over k, Matsuue in [Mat16] constructed a relative logarithmic de Rham–Witt complex $W_\bullet\omega^*_{(Y,N)/(S,L)}$, denoted $W_\bullet\Lambda^*_{(Y,N)/(S,L)}$ in [Mat16]. This is an étale sheaf on Y equipped with operators F,V satisfying all the usual relations (see for example [Mat16, Definition 3.4(v)]) and which specialises to various previous constructions in particular cases.

(1) When $S=\mathrm{Spec}\,(k)$ and the log structures L and N are trivial, then this gives the (canonical extension of the) classical de Rham–Witt complex $W_\bullet\Omega^*_Y$ (to an étale sheaf on Y).
(2) More generally, when the morphism $(Y,N)\to(S,L)$ is strict, we recover the relative de Rham–Witt complex $W_\bullet\Omega^*_{Y/S}$ of Langer and Zink [LZ04].
(3) When the base (S,L) is the scheme $\mathrm{Spec}\,(k)$ with the log structure of the punctured point, and (Y,N) is of semistable type (i.e. étale locally étale over $k[x_1,\ldots,x_{d+1}]/(x_1\cdots x_c)$ with the canonical log structure) then we obtain the logarithmic de Rham–Witt complex $W\omega^*_Y$ studied in [HK94].
(4) If we take (Y,N) semistable but instead equip $\mathrm{Spec}\,(k)$ with the trivial log structure, the resulting complex is isomorphic to the one denoted $W\tilde{\omega}^*_Y$ in [HK94].

If we are given a morphism of log schemes $(Y,N)\to(S,L)$ over k, then as in [Mat16, §2.2] we can lift the log structure $N\to\mathscr{O}_Y$ to a log structure $W_rN\to W_r\mathscr{O}_Y$, where by definition $W_rN = N\oplus\ker((W_r\mathscr{O}_Y)^*\to\mathscr{O}_Y^*)$ and the map $N\to W_r\mathscr{O}_Y$ is the Techmüller lift of $N\to\mathscr{O}_Y$. Since $W_r\omega^1_{(Y,N)/(S,L)}$ is a quotient of the pd-log de Rham complex $\breve{\omega}^*_{(W_rY,W_rN)/(W_rS,W_rL)}$ (see [Mat16, §3.4]) there is a natural map $d\log : W_rN \to W_r\omega^1_{(Y,N)/(S,L)}$ and hence we obtain maps

$$d\log : N^{\mathrm{gp}} \to W_r\omega^1_{(Y,N)/(S,L)}$$

which are compatible as r varies. We let $W_r\omega^1_{(Y,N)/(S,L),\log}$ denote the image.

When both log structures are trivial, and $Y \to \mathrm{Spec}\,(k)$ is smooth, then [Ill79, Proposition I.3.23.2] says that $d\log$ induces an exact sequence

$$0 \to p^r \mathscr{O}_Y^* \to \mathscr{O}_Y^* \to W_r\Omega^1_{Y,\log} \to 0$$

and our first task in this section to obtain an analogue of this result for semistable log schemes over k. In fact, since we will really only be interested in the case when Y arises as the special fibre of a semistable scheme over $k[\![t]\!]$, we will only treat this special case.

We will therefore let $\mathscr{X}$ denote a semistable scheme over $R = k[\![t]\!]$ (not necessarily proper). We will let L denote the log structure given by the closed point of $\mathrm{Spec}\,(R)$, and write $R^\times = (R, L)$. We will denote by L_n the inverse image log structure on $R_n = k[\![t]\!]/(t^{n+1})$, and write $R_n^\times = (R_n, L_n)$. We will also write $k^\times = (k, L_0)$. We will denote by M the log structure on $\mathscr{X}$ given by the special fibre, and write $\mathscr{X}^\times = (\mathscr{X}, M)$. Similarly we have log structures M_n on $X_n = \mathscr{X} \otimes_R R_n$, and we will write $X_n^\times = (X_n, M_n)$. Finally, when considering the logarithmic de Rham–Witt complex relative to k (with the trivial log structure) we will drop k from the notation, e.g. we will write $W_r\omega^*_{X_0^\times}$ instead of $W_r\omega^*_{X_0^\times/k}$.

Proposition 2.1. *The sequence*

$$0 \to p^r M_0^{\mathrm{gp}} \to M_0^{\mathrm{gp}} \to W_r\omega^1_{X_0^\times,\log} \to 0$$

is exact.

Proof. The surjectivity of the right hand map and the injectivity of the left hand map are by definition, and since $p^r W_r\omega^1_{X_0^\times,\log} = 0$, the sequence is clearly a complex. The key point is then to show exactness in the middle. So suppose that we are given $m \in M_0^{\mathrm{gp}}$ is such that $d\log m = 0$. We will show that $m \in p^r M_0^{\mathrm{gp}}$ by induction on r.

When $r = 1$ we note that the claim is étale local, we may therefore assume $X_0^\times$ to be affine, étale and strict over $\mathrm{Spec}\left(\frac{k[x_1,\ldots,x_d]}{(x_1\cdots x_c)}\right)$, say $X_0 = \mathrm{Spec}\,(A)$. We have

$$\omega^1_{(A,\mathbb{N}^c)} \cong \bigoplus_{i=1}^{c} A \cdot d\log x_i \oplus \bigoplus_{i=c+1}^{d} A \cdot dx_i.$$

Now suppose that we are given a local section $n = u\prod_{i=1}^c x_i^{n_i}$ of N^{gp} for $u \in A^*$ and $n_i \in \mathbb{Z}$. Write

$$d\log u = \sum_{i=1}^{c} a_i d\log x_i + \sum_{i=c+1}^{d} a_i dx_i$$

with $a_i \in A$, note that since $d\log u$ actually comes from an element of Ω^1_A it follows that $a_i \in x_iA$ for $1 \le i \le c$. In particular, we have $n_i = -x_ia_i$ for $1 \le i \le c$, and passing to A/x_iA it therefore follows that $n_i = 0$ in k. Hence each n_i is divisible by p. It follows that $\prod_{i=1}^c x_i^{n_i}$ is in pN^{gp}, and its $d\log$ vanishes. By dividing by this element we may therefore assume that $n = u \in A^*$. Since semistable schemes are of Cartier type, we may apply [Kat89, Theorem 4.12], which tells us that (étale locally) $u \in A^{(p)*}$ (since $d\log u = 0 \Rightarrow du = 0$). Since k is perfect, $A^{(p)*} = (A^*)^p$ and we may conclude.

When $r > 1$ and $d\log n = 0 \in W_r\omega^1_{X_0^\times,\log}$, then in particular $d\log n = 0 \in W_{r-1}\omega^1_{X_0^\times}$; hence by applying the induction hypothesis we obtain $n = p^{r-1}n_1$. But now this implies that $p^{r-1}d\log n_1 = 0 \in W_r\omega^1_{X_0^\times}$, we claim that in fact it follows that $d\log n_1 = 0 \in \omega^1_{X_0^\times}$. Indeed, since $\omega^1_{X_0^\times}$ is a locally free $\mathscr{O}_{X_0}$-module, to prove that a section vanishes it suffices to show that it does so on a dense open subscheme. In particular, by restricting to the smooth locus of X_0 we can assume that X_0 is smooth and the log structure is given by

$\mathscr{O}_{X_0}^* \oplus \mathbb{N}$, $(u,m) \mapsto u.0^m$. We now apply [Ill79, Proposition I.3.4] and [Mat16, Lemma 7.4] to conclude that $d\log n_1 = 0$ as required. Thus applying the case $r = 1$ finishes the proof. $\square$

The following is analogous to [Ill79, Corollaire I.3.27].

Proposition 2.2. *The sequences of pro-sheaves*

$$0 \to \left\{W_r\omega^1_{\mathscr{X}^\times,\log}\right\}_r \to \left\{W_r\omega^1_{\mathscr{X}^\times}\right\}_r \xrightarrow{1-F} \left\{W_r\omega^1_{\mathscr{X}^\times}\right\}_r \to 0,$$
$$0 \to \left\{W_r\omega^1_{X_0^\times/k^\times,\log}\right\}_r \to \left\{W_r\omega^1_{X_0^\times/k^\times}\right\}_r \xrightarrow{1-F} \left\{W_r\omega^1_{X_0^\times/k^\times}\right\}_r \to 0$$

are exact.

Proof. Let us consider the first sequence. Using Néron–Popescu desingularisation [Pop86, Theorem 1.8] and the fact that the logarithmic de Rham–Witt complex commutes with filtered colimits, we may reduce to considering the analogous question for Y smooth over k with log structure N coming from a normal crossings divisor $D \subset Y$. The claim is étale local, we may therefore assume that Y is étale over $k[x_1,\dots,x_n]$ with D the inverse image of $\{x_1 \cdots x_c = 0\}$. Locally, N is generated by $\mathscr{O}_Y^*$ and x_i for $1 \le i \le c$, so in order to see that the sequence is a complex, or in other words that $(1-F)(d\log n) = 0$, it suffices to check that $(1-F)(d\log x_i) = 0$. This is a straightforward calculation. For the surjectivity of $1-F$ we claim in fact that

$$1-F : W_{r+1}\omega^1_{(Y,N)} \to W_r\omega^1_{(Y,N)}$$

is surjective. For this we note that by [Mat16, §9] there exists an exact sequence

$$0 \to W_r\Omega^1_Y \to W_r\omega^1_{(Y,N)} \to \bigoplus_{i=1}^{c} W_r\mathscr{O}_{D_i} \cdot d\log x_i \to 0$$

for all r, where D_i are the irreducible components of D. Denote the induced map $W_r\omega^1_{(Y,N)} \to W_r\mathscr{O}_{D_i}$ by Res_i. Since $(1-F)(d\log x_i) = 0$ it follows that we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W_{r+1}\Omega^1_Y \arrow[r] \arrow[d, "1-F"'] & W_{r+1}\omega^1_{(Y,N)} \arrow[r] \arrow[d, "1-F"'] & \bigoplus_{i=1}^c W_{r+1}\mathscr{O}_{D_i} \arrow[r] \arrow[d, "1-F"'] & 0 \\
0 \arrow[r] & W_r\Omega^1_Y \arrow[r] & W_r\omega^1_{(Y,N)} \arrow[r] & \bigoplus_{i=1}^c W_r\mathscr{O}_{D_i} \arrow[r] & 0
\end{tikzcd}$$

where $W_r\Omega^1_Y$ is the usual (non-logarithmic) de Rham–Witt complex of Y. It therefore suffices to apply [Ill79, Propositions I.3.26, I.3.28], stating that the left and right vertical maps are surjective. Finally, to show exactness in the middle, suppose that we are given $\omega \in W_{r+1}\omega^1_{(Y,N)}$ such that $(1-F)(\omega) = 0$. Then applying [Ill79, Proposition I.3.28] we can see that

$$\mathrm{Res}_i(\omega) \in \mathbb{Z}/p^{r+1}\mathbb{Z} + \ker(W_{r+1}\mathscr{O}_{D_i} \to W_r\mathscr{O}_{D_i})$$

for all i. Hence after subtracting off an element of $d\log(N^{\mathrm{gp}})$ we may assume that in fact

$$\omega \in W_{r+1}\Omega^1_Y + \ker\left(W_{r+1}\omega^1_{(Y,N)} \to W_r\omega^1_{(Y,N)}\right).$$

Now applying [Ill79, Corollaire I.3.27] tells us that

$$\omega \in d\log(N^{\mathrm{gp}}) + \ker\left(W_{r+1}\omega^1_{(Y,N)} \to W_r\omega^1_{(Y,N)}\right)$$

and hence the given sequence of pro-sheaves is exact in the middle.

For the second sequence, the surjectivity of $1-F$ follows from the corresponding claim for the first sequence, since sections of $W_r\omega^1_{X_0^\times}$ can be lifted locally to $W_r\omega^1_{\mathscr{X}^\times}$. We may also argue étale locally; assuming that $X_0^\times$ is étale and strict over $\mathrm{Spec}\left(\mathbb{N}^c \to \frac{k[x_1,\ldots,x_d]}{(x_1\cdots x_c)}\right)$. The fact that the claimed sequence is a complex follows again from observing that $(1-F)(d\log x_i)=0$ for $1\leq i\leq c$. To see exactness in the middle we use the fact that (again working étale locally) we have an exact sequence

$$0\to\bigoplus_i W_r\Omega^1_{D_i}\to W_r\omega^1_{X_0^\times/k^\times}\to\bigoplus_{ij}W_r\mathscr{O}_{D_{ij}}\to 0$$

by [Mat16, Lemma 8.4], where D_i are the irreducible components of $X_0^\times$ and D_{ij} their intersections. Moreover, this fits into a diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_i \mathscr{O}^*_{D_i} \arrow[r] \arrow[d, "d\log"'] & M_0^{\mathrm{gp}} \arrow[r] \arrow[d, "d\log"'] & \bigoplus_{ij}\underline{\mathbb{Z}}_{D_{ij}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_i W_r\Omega^1_{D_i} \arrow[r] \arrow[d, "1-F"'] & W_r\omega^1_{X_0^\times/k^\times} \arrow[r] \arrow[d, "1-F"'] & \bigoplus_{ij}W_r\mathscr{O}_{D_{ij}} \arrow[r] \arrow[d, "1-F"'] & 0 \\
0 \arrow[r] & \bigoplus_i W_{r-1}\Omega^1_{D_i} \arrow[r] & W_{r-1}\omega^1_{X_0^\times/k^\times} \arrow[r] & \bigoplus_{ij}W_{r-1}\mathscr{O}_{D_{ij}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Exactness of the middle vertical sequence at $W_r\omega^1_{X_0^\times/k^\times}$ now follows from the classical result [Ill79, Corollaire I.3.27, Proposition I.3.28] and a simple diagram chase. $\square$

Next, we will need to understand the kernel of $W_r\omega^1_{X_0^\times,\log}\to W_r\omega^1_{X_0^\times/k^\times,\log}$.

Lemma 2.3. *For all $r\geq 1$ the sequence*

$$0\to\mathbb{Z}/p^r\mathbb{Z}\cdot d\log t\to W_r\omega^1_{X_0^\times,\log}\to W_r\omega^1_{X_0^\times/k^\times,\log}\to 0$$

is exact.

Proof. Note that by [Mat16, Lemma 7.4] it suffices to show that

$$d\log(M_0)\cap W_r\mathscr{O}_{X_0}\cdot d\log t=\mathbb{Z}/p^r\mathbb{Z}\cdot d\log t$$

inside $W_r\omega^1_{X_0^\times}$, the containment $\supset$ is clear. For the other, suppose that we are given an element of the form $g\cdot d\log t\in W_r\omega^1_{X_0^\times}$ which is in the image of $d\log$. Then we know that $\tilde{g}\cdot d\log t=d\log n+c$ in $W_{r+1}\omega^1_{X_0^\times}$, for some $c\in\ker\left(W_{r+1}\omega^1_{X_0^\times}\to W_r\omega^1_{X_0^\times}\right)$ and $\tilde{g}\in W_{r+1}\mathscr{O}_{X_0}$ lifting g. Arguing as in Proposition 2.2 above we can see that $(1-F)(d\log n)=0$, and again applying [Mat16, Lemma 7.4] we can deduce that in fact $g=F(g)$ in $W_r\mathscr{O}_{X_0}$. Hence $g\in\mathbb{Z}/p^r\mathbb{Z}$ as claimed. $\square$

Finally, we will need to know that the logarithmic de Rham–Witt complex computes the log crystalline cohomology of the semistable scheme $\mathscr{X}$.

Proposition 2.4. *There is an isomorphism*

$$H^i_{\mathrm{cont}}(\mathscr{X}_{\text{ét}},W\omega^*_{\mathscr{X}^\times})_{\mathbb{Q}}\xrightarrow{\sim}H^i_{\log\text{-}\mathrm{cris}}(\mathscr{X}^\times/K)$$

for all $i\geq 0$.

Proof. It suffices to show that $H^i(\mathscr{X}_{\text{ét}}, W_r\omega^*_{\mathscr{X}^\times}) \cong H^i_{\text{log-cris}}(\mathscr{X}^\times/W_r)$ where $W_r = W_r(k)$. Arguing locally on $\mathscr{X}$ we may assume in fact that $\mathscr{X}$ is affine, and in particular admits a closed embedding $\mathscr{X} \hookrightarrow \mathscr{P}$ into some affine space over $W_r[\![t]\!]$.

Now applying Néron–Popescu desingularisation [Pop86, Theorem 1.8] to $W_r \to W_r[\![t]\!]$, we may in fact write $\mathscr{X} = \lim_\alpha X_\alpha$ as a limit of smooth k-schemes, such that:

- that there exist compatible normal crossings divisors $D_\alpha \subset X_\alpha$ whose inverse image in $\mathscr{X}$ is precisely the special fibre X_0;
- there exist compatible closed embeddings $X_\alpha \hookrightarrow P_\alpha$ into smooth W_r-schemes such that $\mathscr{P} = \lim_\alpha P_\alpha$.

Since both the de Rham–Witt complex and étale cohomology commute with cofiltered limits of schemes, it suffices to show that the same is true of log-crystalline cohomology, in other words that we have

$$H^i_{\text{log-cris}}(\mathscr{X}^\times/W_r) = \operatorname{colim}_\alpha H^i_{\text{log-cris}}(X_\alpha^\times/W_r),$$

where $X_\alpha^\times$ denotes the scheme X_α endowed with the log structure given by D_α. By [Kat89, Theorem 6.4], $H^i_{\text{log-cris}}(X_\alpha^\times/W_r)$ is computed as the de Rham cohomology of the log-PD envelope of $X_\alpha^\times$ inside P_α. Since log-PD envelopes commute with cofiltered limits of schemes (i.e. filtered colimits of rings), it suffices to show that $H^i_{\text{log-cris}}(\mathscr{X}^\times/W_r)$ can be computed as the de Rham cohomology of the log-PD envelope of $\mathscr{X}^\times$ inside $\mathscr{P}$.

In other words, what we require a logarithmic analogue of [Kat91, Theorem 1.7], or equivalently a log-p-basis analogue of [Kat89, Theorem 6.4]. But this follows from Proposition 1.2.18 of [CV15]. □

3. Morrow's variational Tate conjecture for divisors

The goal of this section is to offer a simpler proof of a special case of [Mor14, Theorem 3.5] for smooth and proper schemes $\mathscr{X}$ over the power series ring $R = k[\![t]\!]$. This result essentially states that a line bundle on the special fibre of $\mathscr{X}$ lifts iff its its first Chern class in H^2_{cris} does, and should be viewed as an equicharacteristic analogue of Berthelot and Ogus's theorem [BO83, Theorem 3.8] stating that a line bundle on the special fibre of a smooth proper scheme over a DVR in mixed characteristic lifts iff its Chern class lies in the first piece of the Hodge filtration. We will also give a slightly different interpretation of this result that emphasises the philosophy that in equicharacteristic the 'correct' analogue of a Hodge filtration is an $\mathscr{E}^\dagger$-structure. Our proof is simpler in that it does not depend on any results from topological cyclic homology, but only on fairly standard properties of the de Rham–Witt complex. As such, it is far more readily adaptable to the semistable case, which we shall do in §4 below.

Throughout this section, $\mathscr{X}$ will be a smooth and proper $R = k[\![t]\!]$-scheme. Let R_n denote $k[\![t]\!]/(t^{n+1})$ and set $X_n = \mathscr{X} \otimes_R R_n$. Write X for the generic fibre of $\mathscr{X}$ and $\mathfrak{X}$ for its formal (t-adic) completion. Since all schemes in this section will have trivial log structure, we will use the notation $W_\bullet\Omega^*$ for the de Rham–Witt complex instead of $W_\bullet\omega^*$. The key technical calculation we will make is the following.

Lemma 3.1. *Fix $n \geq 0$, write $n = p^m n_0$ with $(n_0, p) = 1$, and let $r = m+1$. Then the map*

$$d\log : 1 + t^n \mathscr{O}_{X_n} \to W_r\Omega^1_{X_n,\log}$$

is injective.

Proof. We may assume that $X_n = \operatorname{Spec}(A_n)$ is affine, moreover étale over $R_n[x_1, \ldots, x_d]$. In this case since deformations of smooth affine schemes are trivial, we have $A_n \cong A_0 \otimes_k R_n$. Hence $1 + t^n A_n = 1 + t^n A_0$, and our problem therefore reduces to showing that if $a \in A_0$ is such that $d\log[1+at^n]=0$, then in fact $a=0$. But vanishing of a may be checked over all closed points of A_0, so by functoriality of the $d\log$ map we may in fact assume that A_0 is a finite extension of k, enlarging k we may moreover assume that $A_0=k$; in other words we need to show that the map

$$d\log : 1+t^nk \to W_r\Omega^1_{R_n}$$

is injective. Since k is perfect, any $1+at^n \in 1+t^nk$ can be written uniquely as $(1+t^{n_0}b)^{p^m}$ for some $b\in k$, hence $d\log[1+at^n]=p^m d\log(1+t^{n_0}b)$. It follows that if $d\log[1+at^n]=0$, then $p^m n_0 bt^{n_0-1}dt=0$ in $W_r\Omega^1_{R_n}$. Since any non-zero such b is invertible, the lemma will follow if we can show that $p^m t^{n_0-1}dt$ is non-zero in $W_r\Omega^1_{R_n}$. This can be checked easily using the exact sequence

$$\frac{W_r((t^{n+1}))}{W_r((t^{n+1})^2)} \xrightarrow{d} W_r\Omega^1_{k[t]}\otimes_{W_r(k[t])} W_rR_n \to W_r\Omega^1_{R_n}\to 0$$

from [LZ05]. □

From this we deduce the following.

Proposition 3.2. *For $r\gg 0$ (depending on n) there is a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & 1+t\mathscr{O}_{X_n} \arrow[r] \arrow[d, equal] & \mathscr{O}^*_{X_n} \arrow[r] \arrow[d, "d\log"] & \mathscr{O}^*_{X_0} \arrow[r] \arrow[d, "d\log"] & 1 \\
1 \arrow[r] & 1+t\mathscr{O}_{X_n} \arrow[r, "d\log"] & W_r\Omega^1_{X_n,\log} \arrow[r] & W_r\Omega^1_{X_0,\log} \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Proof. It is well-known that the top row is exact, and the diagram is clearly commutative, it therefore suffices to show that for all n the sequence

$$1\to 1+t\mathscr{O}_{X_n}\to W_r\Omega^1_{X_n,\log}\to W_r\Omega^1_{X_0,\log}\to 0$$

is exact for $r\gg 0$. From the definition of $W_r\Omega^1_{X_n,\log}$ and the exactness of the sequence

$$1\to 1+t\mathscr{O}_{X_n}\to \mathscr{O}^*_{X_n}\to \mathscr{O}^*_{X_0}\to 1$$

it is immediate that $W_r\Omega^1_{X_n,\log}\to W_r\Omega^1_{X_0,\log}$ is surjective and the composite $1+t\mathscr{O}_{X_n}\to W_r\Omega^1_{X_0,\log}$ is zero. Given $\alpha\in\mathscr{O}^*_{X_n}$ mapping to 0 in $W_r\Omega^1_{X_0,\log}$, it follows from [Ill79, Proposition I.3.23.2] that there exists $\beta\in\mathscr{O}^*_{X_n}$ and $\gamma\in 1+t\mathscr{O}_{X_n}$ such that $\alpha=\beta^{p^r}+\gamma$, and hence $d\log\alpha=d\log\gamma$ in $W_r\Omega^1_{X_n,\log}$. The sequence

$$1+t\mathscr{O}_{X_n}\to W_r\Omega^1_{X_n,\log}\to W_r\Omega^1_{X_0,\log}\to 0$$

is therefore exact, and it remains to show that

$$1+t\mathscr{O}_{X_n}\overset{d\log}{\to} W_r\Omega^1_{X_n,\log}$$

is injective for $r\gg 0$. By induction on n this follows from Lemma 3.1 above. □

We now set

$$W_r\Omega^i_{\mathfrak{X},\log} := \lim_n W_r\Omega^i_{X_n,\log}$$

as a sheaf on $\mathfrak{X}_{\text{ét}}$ and define

$$H^j_{\text{cont}}(\mathfrak{X}_{\text{ét}}, W\Omega^i_{\mathfrak{X},\log}) := H^j(\mathbf{R}\lim_r \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, W_r\Omega^i_{\mathfrak{X},\log})).$$

As an essentially immediate corollary of Proposition 3.2, we deduce the key step of Morrow's proof of the variational Tate conjecture in this case.

Corollary 3.3. *Let $\mathscr{L} \in \mathrm{Pic}(X_0)$, with first Chern class $c_1(\mathscr{L}) \in H^1_{\mathrm{cont}}(X_{0,\text{ét}}, W\Omega^1_{X_0,\log})$. Then $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X})$ if and only if $c_1(\mathscr{L})$ lifts to $H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\Omega^1_{\mathscr{X},\log})$*

Proof. One direction is obvious. For the other direction, assume that the first Chern class $c_1(\mathscr{L})$ lifts to $H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\Omega^1_{\mathscr{X},\log})$, in particular it therefore lifts to $H^1_{\mathrm{cont}}(\mathfrak{X}_{\text{ét}}, W\Omega^1_{\mathfrak{X},\log})$. Hence by Proposition 3.2 it follows that $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathfrak{X})$, and we may conclude using Grothendieck's algebrisation theorem that it lifts to $\mathrm{Pic}(\mathscr{X})$. □

From this the form **(crys-ϕ)** form of the variational Tate conjecture follows as in [Mor14].

Corollary 3.4. *Let $\mathscr{L} \in \mathrm{Pic}(X_0)_{\mathbb{Q}}$, with first Chern class $c_1(\mathscr{L}) \in H^2_{\mathrm{cris}}(X/K)^{\varphi=p}$. Then $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$ if and only if $c_1(\mathscr{L})$ lifts to $H^2_{\mathrm{cris}}(\mathscr{X}/K)^{\varphi=p}$.*

Proof. Let us first assume that k is algebraically closed. By [Mor14, Proposition 3.2] the inclusions $W\Omega^1_{\mathscr{X},\log}[-1] \to W\Omega^*_{\mathscr{X},\log}$ and $W\Omega^1_{X_0,\log}[-1] \to W\Omega^*_{X_0,\log}$ induce an isomorphism

$$H^1_{\mathrm{cont}}(X_{0,\text{ét}}, W\Omega^1_{X_0,\log})_{\mathbb{Q}} \xrightarrow{\sim} H^2_{\mathrm{cris}}(X_0/K)^{\varphi=p}$$

and a surjection

$$H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\Omega^1_{\mathscr{X},\log})_{\mathbb{Q}} \twoheadrightarrow H^2_{\mathrm{cris}}(\mathscr{X}/K)^{\varphi=p}.$$

The claim follows. In general, we argue as in [Mor14, Theorem 1.4]: the claim for k algebraically closed shows that $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$ after making the base change $k[\![t]\!] \to \bar{k}[\![t]\!]$. Let $k[\![t]\!]^{\mathrm{sh}}$ denote the strict Henselisation of $k[\![t]\!]$ inside $\bar{k}[\![t]\!]$, by Néron–Popescu desingularisation there exists some smooth local $k[\![t]\!]^{\mathrm{sh}}$-algebra A such that $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$ after making the base change $k[\![t]\!] \to A$. But the map $k[\![t]\!]^{\mathrm{sh}} \to A$ has a section, from which it follows that in fact $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$ after making some finite field extension $k \to k'$. But now simply taking the pushforward via $\mathscr{X} \otimes_k k' \to \mathscr{X}$ and dividing by $[k' : k]$ gives the result. □

To finish off this section, we wish to give a slightly different formulation of Corollary 3.4. After [LP16] we can consider the 'overconvergent' rigid cohomology $H^i_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$ of the generic fibre X, which is a (φ,∇)-module over the bounded Robba ring $\mathscr{E}^\dagger$. Set $H^i_{\mathrm{rig}}(X/\mathscr{R}) := H^i_{\mathrm{rig}}(X/\mathscr{E}^\dagger) \otimes_{\mathscr{E}^\dagger} \mathscr{R}$. By combining Dwork's trick with smooth and proper base change in crystalline cohomology we have an isomorphism

$$H^i_{\mathrm{rig}}(X/\mathscr{R})^{\nabla=0} \cong H^i_{\mathrm{rig}}(X_0/K)$$

for all i. In particular, for any $\mathscr{L} \in \mathrm{Pic}(X_0)_{\mathbb{Q}}$ we can consider $c_1(\mathscr{L})$ as an element of $H^i_{\mathrm{rig}}(X/\mathscr{R})^{\nabla=0} \subset H^i_{\mathrm{rig}}(X/\mathscr{R})$. One of the general philosophies of p-adic cohomology in equicharacteristic is that while the cohomology groups $H^i_{\mathrm{rig}}(X/\mathscr{R})$ in some sense only depend on the special fibre X_0, the 'lift' X of X_0 is seen in the $\mathscr{E}^\dagger$-lattice $H^i_{\mathrm{rig}}(X/\mathscr{E}^\dagger) \subset H^i_{\mathrm{rig}}(X/\mathscr{R})$. The correct equicharacteristic analogue of a Hodge filtration, therefore, is an $\mathscr{E}^\dagger$-structure. With this in mind, then, a statement of the variational Tate conjecture for divisors which is perhaps slightly more transparently analogous to that in mixed characteristic is the following.

Theorem 3.5. *Assume that $\mathscr{X}$ is projective over R. Then a line bundle $\mathscr{L} \in \mathrm{Pic}(X_0)_{\mathbb{Q}}$ lifts to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$ if and only if $c_1(\mathscr{L}) \in H^2_{\mathrm{rig}}(X/\mathscr{R})$ lies in $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$.*

Proof. This is simply another way of stating the condition **(flat)** in [Mor14, Theorem 3.5]. □

Remark 3.6. It seems entirely plausible that the methods of this section can be easily adapted to give a proof of [Mor14, Theorem 3.5] in general, i.e. over $k[\![t_1,\ldots,t_n]\!]$ rather than just $k[\![t]\!]$.

4. A SEMISTABLE VARIATIONAL TATE CONJECTURE FOR DIVISORS

In this section we will prove a semistable version of Theorem 3.5, or equivalently an equicharacteristic analogue of [Yam11, Theorem 0.1]. The basic set-up will be to take a proper, semistable scheme $\mathscr{X}/R$, as before we will consider the semistable schemes X_n/R_n as well as the smooth generic fibre X/F. The special fibre of $\mathscr{X}$ defines a log structure M, and pulling back via the immersion $X_n \to \mathscr{X}$ defines a log structure M_n on each X_n. For each n we will put a log structure L_n on R_n via $\mathbb{N} \to R_n$, $1 \mapsto t$, note that for $n = 0$ this is the log structure of the punctured point on k. We will let L denote the log structure on R defined by the same formula. As before we will write $R^\times = (R, L)$, $R_n^\times = (R_n, L_n)$, $\mathscr{X}^\times = (\mathscr{X}, M)$, $X_n^\times = (X_n, M_n)$ and $k^\times = (k, L_0)$. The logarithmic version of Proposition 3.2 is then the following.

Proposition 4.1. *For $r \gg 0$ (depending on n) there is a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & 1+t\mathscr{O}_{X_n} \arrow[r] \arrow[d, equal] & \mathscr{O}_{X_n}^* \arrow[r] \arrow[d] & \mathscr{O}_{X_0}^* \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1+t\mathscr{O}_{X_n} \arrow[r] \arrow[d] & M_n^{\mathrm{gp}} \arrow[r] \arrow[d, "d\log"] & M_0^{\mathrm{gp}} \arrow[r] \arrow[d, "d\log"] & 0 \\
1 \arrow[r] & \mathscr{K}_{n,r} \arrow[r] & W_r\omega^1_{X_n^\times,\log} \arrow[r] & W_r\omega^1_{X_0^\times/k^\times,\log} \arrow[r] & 0.
\end{tikzcd}$$

with exact rows. Moreover each $\mathscr{K}_{n,r}$ fits into an exact sequence of pro-sheaves on $X_{n,\text{ét}}$

$$1 \to 1+t\mathscr{O}_{X_n} \to \{\mathscr{K}_{n,r}\}_r \to \{\mathbb{Z}/p^r\mathbb{Z}\}_r \to 0$$

which is split compatibly with varying n.

Proof. We first claim that if we replace $W_r\omega^1_{X_0^\times/k^\times,\log}$ by $W_r\omega^1_{X_0^\times,\log}$ then we obtain an exact sequence

$$1 \to 1+t\mathscr{O}_{X_n} \to W_r\omega^1_{X_n^\times,\log} \to W_r\omega^1_{X_0^\times,\log} \to 0$$

for $r \gg 0$. Using Proposition 2.1 the proof of the exactness of

$$1+t\mathscr{O}_{X_n} \to W_r\omega^1_{X_n^\times,\log} \to W_r\omega^1_{X_0^\times,\log} \to 0$$

is exactly as in Proposition 3.2. In fact, to check exactness on the left we can even apply Proposition 3.2: to check a section of $1+t\mathscr{O}_{X_n}$ vanishes it suffices to do on a dense open subscheme of X_n, we may therefore étale locally replace X_n by the canonical thickening of the smooth locus of the special fibre. But now we are in the smooth case, so we apply Proposition 3.2 (which holds locally).

Applying Lemma 2.3 we know that the kernel of

$$W_r\omega^1_{X_0^\times,\log} \to W_r\omega^1_{X_0^\times/k^\times,\log}$$

is isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$, generated by $d\log t$. The snake lemma then shows that, defining $\mathscr{K}_{n,r}$ to be the kernel of $W_r\omega^1_{X_n^\times,\log} \to W_r\omega^1_{X_0^\times/k^\times,\log}$, we have the exact sequence

$$1 \to 1+t\mathscr{O}_{X_n} \to \mathscr{K}_{n,r} \to \mathbb{Z}/p^r\mathbb{Z} \to 0$$

for $r \gg 0$. To see that it splits compatibly with r and n it therefore suffices to show that there exist compatible classes $\omega_r \in W_r\omega^1_{X_n^\times}$ whose image in $W_r\omega^1_{X_0^\times,\log}$ generate the kernel of $W_r\omega^1_{X_0^\times,\log} \to W_r\omega^1_{X_0^\times/k^\times,\log}$; as we have already observed the classes of $d\log t$ will suffice. $\square$

As before, we therefore obtain the following. Let $\mathrm{Pic}(X_0^\times) = H^1(X_{0,\text{ét}}, M_0^{\mathrm{gp}})$ and $\mathrm{Pic}(\mathscr{X}^\times) = H^1(\mathscr{X}_{\text{ét}}, M^{\mathrm{gp}})$.

Corollary 4.2. *Let $\mathscr{L} \in \mathrm{Pic}(X_0^\times)$ (resp. $\mathrm{Pic}(X_0)$). Then $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X}^\times)$ (resp. $\mathrm{Pic}(\mathscr{X})$) iff $c_1(\mathscr{L}) \in H^1_{\mathrm{cont}}(X_{0,\text{ét}}, W\omega^1_{X_0^\times/k^\times,\log})$ lifts to $H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^1_{\mathscr{X}^\times,\log})$.*

Proof. This is similar to the proof of Corollary 3.3, although a little more care is needed in taking the limits in n and r. Again, one direction is clear, so we assume that we are given a (logarithmic) line bundle whose Chern class lifts. First we note that we have isomorphisms

$$\mathrm{Pic}(\mathfrak{X}) \cong H^1_{\mathrm{cont}}(X_{0,\text{ét}}, \{\mathscr{O}^*_{X_n}\}_n),\ \ \mathrm{Pic}(\mathfrak{X}^\times) \cong H^1_{\mathrm{cont}}(X_{0,\text{ét}}, \{M_n^{\mathrm{gp}}\}_n)$$

and hence the obstruction to lifting (in either case) can be viewed as an element of $H^2_{\mathrm{cont}}(X_{0,\text{ét}}, \{1 + t\mathscr{O}_{X_n}\}_n)$. The fact that the Chern class lifts implies that this obstruction vanishes in

$$H^2_{\mathrm{cont}}(X_{0,\text{ét}}, \{\mathscr{K}_{n,r}\}_{n,r}) := H^2(\mathbf{R}\lim_n \mathbf{R}\lim_r \mathbf{R}\Gamma(X_{0,\text{ét}}, \mathscr{K}_{n,r}))$$

and hence the fact that the exact sequence of pro-sheaves

$$1 \to 1 + t\mathscr{O}_{X_n} \to \{\mathscr{K}_{n,r}\}_r \to \{\mathbb{Z}/p^r\mathbb{Z}\}_r \to 0$$

splits, compatibly with n, shows that the obstruction must itself vanish in $H^2_{\mathrm{cont}}(X_{0,\text{ét}}, \{1 + t\mathscr{O}_{X_n}\}_n)$. Finally, we need to see that we have isomorphisms $\mathrm{Pic}(\mathfrak{X}) \cong \mathrm{Pic}(\mathscr{X})$ and $\mathrm{Pic}(\mathfrak{X}^\times) \cong \mathrm{Pic}(\mathscr{X}^\times)$. The first is Grothendieck's algebrization theorem, to see the second we note that $\mathrm{Pic}(\mathscr{X}^\times) \cong \mathrm{Pic}(X)$, the Picard group of the generic fibre of $\mathscr{X}$, similarly $\mathrm{Pic}(\mathfrak{X}^\times) \cong \mathrm{Pic}(X^{\mathrm{an}})$, the Picard group of its analytification. The two are isomorphic by rigid analytic GAGA. $\square$

To relate this to log crystalline cohomology, we use the following.

Lemma 4.3. *The inclusions $W_r\omega^1_{\mathscr{X}^\times,\log}[-1] \to W_r\omega^*_{\mathscr{X}^\times}$ and $W_r\omega^1_{X_0^\times/k^\times,\log}[-1] \to W_r\omega^*_{X_0^\times/k^\times}$ induce surjections*

$$H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^1_{\mathscr{X}^\times,\log})_{\mathbb{Q}} \twoheadrightarrow H^2_{\text{log-cris}}(\mathscr{X}^\times/K)^{\varphi=p}$$
$$H^1_{\mathrm{cont}}(X_{0\text{ét}}, W\omega^1_{X_0^\times/k^\times,\log})_{\mathbb{Q}} \twoheadrightarrow H^2_{\text{log-cris}}(X_0^\times/K^\times)^{\varphi=p}$$

Where φ is the semilinear Frobenius operator. If k is algebraically closed, then the latter is in fact an isomorphism.

Proof. Let us first consider $\mathscr{X}^\times$. Define the map $\mathscr{F} : \{W_r\omega^{\geq 1}_{\mathscr{X}^\times}\}_r \to \{W_r\omega^{\geq 1}_{\mathscr{X}^\times}\}_r$ to be $p^{i-1}F$ in degree i, note that in degrees > 1 it is a contracting operator, and hence $1 - \mathscr{F}$ is invertible on $W_r\omega^{>1}_{\mathscr{X}^\times}$. Similarly, the map $1 - V : \{W_r\mathscr{O}_{\mathscr{X}}\} \to \{W_r\mathscr{O}_{\mathscr{X}}\}$ is an isomorphism. From this and Proposition 2.2 it follows that the triangle

$$0 \to \{W_r\omega^1_{\mathscr{X}^\times,\log}\}_r \to \{W_r\omega^{\geq 1}_{\mathscr{X}^\times}\}_r \stackrel{1-\mathscr{F}}{\to} \{W_r\omega^{\geq 1}_{\mathscr{X}^\times}\}_r \to 0$$

of complexes of pro-sheaves is exact. Since $p\mathscr{F} = \varphi$ on $W_r\omega^{\geq 1}_{\mathscr{X}^\times}$, we deduce an exact sequence

$$0 \to \frac{H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{\geq 1}_{\mathscr{X}^\times})_{\mathbb{Q}}}{\mathrm{im}(\varphi - p)} \to H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^1_{\mathscr{X}^\times,\log})_{\mathbb{Q}} \to H^2_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{\geq 1}_{\mathscr{X}^\times})^{\varphi=p}_{\mathbb{Q}} \to 0.$$

For a complex of K-modules C^* with semilinear Frobenius, let $\mathbf{R}_{\varphi=p}(C^*)$ denote the mapping cone $\mathrm{Cone}(C^* \stackrel{\varphi-p}{\to} C^*)$ and by $H^n_{\varphi=p}(C^*)$ its cohomology groups. Then since $1-V=$"$1-p\varphi^{-1}$" is invertible on $\{W_r\mathscr{O}_{\mathscr{X}}\}_r$ we deduce that

$$\mathbf{R}_{\varphi=p}(\mathbf{R}\Gamma_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{\geq 1}_{\mathscr{X}^\times})_{\mathbb{Q}}) \cong \mathbf{R}_{\varphi=p}(\mathbf{R}\Gamma_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{*}_{\mathscr{X}^\times})_{\mathbb{Q}})$$

from which we extract the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{\geq 1}_{\mathscr{X}^\times})_{\mathbb{Q}}}{\mathrm{im}(\varphi-p)} & \longrightarrow & H^2_{\varphi=p}(\mathbf{R}\Gamma_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{\geq 1}_{\mathscr{X}^\times})_{\mathbb{Q}}) & \longrightarrow & H^2_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{\geq 1}_{\mathscr{X}^\times})_{\mathbb{Q}}^{\varphi=p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{*}_{\mathscr{X}^\times})_{\mathbb{Q}}}{\mathrm{im}(\varphi-p)} & \longrightarrow & H^2_{\varphi=p}(\mathbf{R}\Gamma_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{*}_{\mathscr{X}^\times})_{\mathbb{Q}}) & \longrightarrow & H^2_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^{*}_{\mathscr{X}^\times})_{\mathbb{Q}}^{\varphi=p} & \longrightarrow & 0
\end{array}$$

with exact rows, such that the middle vertical arrow is an isomorphism, the left vertical arrow is an injection and the right vertical arrow is a surjection. Now applying Proposition 2.4 we see that the map

$$H^1_{\mathrm{cont}}(\mathscr{X}_{\text{ét}}, W\omega^1_{\mathscr{X}^\times,\log})_{\mathbb{Q}} \twoheadrightarrow H^2_{\log\text{-cris}}(\mathscr{X}^\times/K)^{\varphi=p}$$

is surjective as claimed. An entirely similar argument works for $X_0^\times$, replacing Proposition 2.4 with [Mat16, Theorem 7.9], and in fact shows that

$$H^1_{\mathrm{cont}}(X_{0\text{ét}}, W\omega^1_{X_0^\times/k^\times,\log})_{\mathbb{Q}} \twoheadrightarrow H^2_{\log\text{-cris}}(X_0^\times/K^\times)^{\varphi=p}$$

is an isomorphism if and only if $(\varphi-p)$ is surjective on $H^1_{\log\text{-cris}}(X_0^\times/K^\times)$. If k is algebraically closed, this follows from semisimplicity of the category of φ-modules over K. $\square$

This enables us to deduce the following.

Corollary 4.4. *Let $\mathscr{L} \in \mathrm{Pic}(X_0^\times)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(X_0)_{\mathbb{Q}}$). Then $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X}^\times)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$) iff $c_1(\mathscr{L}) \in H^2_{\log\text{-cris}}(X_0^\times/K^\times)^{\varphi=p}$ lifts to $H^2_{\log\text{-cris}}(\mathscr{X}^\times/K)^{\varphi=p}$.*

Proof. Exactly as in the proof of Corollary 3.4. $\square$

Let us now rephrase this more closely analogous to Yamashita's criterion in [Yam11]. Note that thanks to [LP16, Corollary 5.8] we have an isomorphism

$$H^i_{\mathrm{rig}}(X/\mathscr{R}) \cong H^i_{\log\text{-cris}}(X_0^\times/K^\times) \otimes \mathscr{R}$$

of (φ,∇)-modules over $\mathscr{R}$, which induces an isomorphism

$$H^i_{\mathrm{rig}}(X/\mathscr{R})^{\nabla=0} \cong H^i_{\log\text{-cris}}(X_0^\times/K^\times)^{N=0}.$$

By [Yam11, Proposition 2.2] (whose proof does not use the existence of a lift to characteristic 0), the first Chern class $c_1(\mathscr{L})$ of any $\mathscr{L}$ in $\mathrm{Pic}(X_0^\times)_{\mathbb{Q}}$ or $\mathrm{Pic}(X_0)_{\mathbb{Q}}$ satisfies $N(c_1(\mathscr{L}))=0$. Hence we may view $c_1(\mathscr{L})$ as an element of $H^2_{\mathrm{rig}}(X/\mathscr{R})$.

Theorem 4.5. *Assume that $\mathscr{X}$ is projective over R. Then $\mathscr{L}$ lifts to $\mathrm{Pic}(\mathscr{X}^\times)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}}$) iff $c_1(\mathscr{L}) \in H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger) \subset H^2_{\mathrm{rig}}(X/\mathscr{R})$.*

Proof. Note that if $c_1(\mathscr{L}) \in H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$, it is automatically in the subspace $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p}$. Now consider the Leray spectral sequence for log crystalline cohomology

$$E_2^{p,q} = H^q_{\log\text{-cris}}\left(\mathrm{Spec}\left(R^\times\right), \mathbf{R}^p f_* \mathscr{O}^{\mathrm{cris}}_{\mathscr{X}^\times/K}\right) \Rightarrow H^{p+q}_{\log\text{-cris}}(\mathscr{X}^\times/K).$$

where $f : \mathscr{X}^\times \to \operatorname{Spec}(R^\times)$ denotes the structure map. Since $\mathscr{X}$ is projective we obtain maps

$$u^i : \mathbf{R}^{d-i} f_* \mathscr{O}^{\mathrm{cris}}_{\mathscr{X}^\times/K} \to \mathbf{R}^{d+i} f_* \mathscr{O}^{\mathrm{cris}}_{\mathscr{X}^\times/K}$$

of log-F-isocrystals over $R^\times$ by cupping with the class of a hyperplane section, we claim that u^i is an isomorphism. To check this, we note that we can identify the category of log-F-isocrystals over $R^\times$ with the category $\underline{\mathbf{M\Phi}}^{\nabla,\log}_{\mathscr{E}^+}$ of log-(φ,∇)-modules over the ring $\mathscr{E}^+ := W[\![t]\!] \otimes_W K$ as considered in [LP16, §5.3]. We now note that the functor of 'passing to the generic fibre', i.e. tensoring with $\mathscr{E} := \mathscr{E}^+\langle t^{-1}\rangle$ is fully faithful, by [Ked04, Theorem 5.1] (together with a simple application of the 5 lemma), and hence by the hard Lefschetz theorem in rigid cohomology (together with standard comparison theorems in crystalline cohomology) the isomorphy of u^i follows. Hence applying the formalism of [Mor14, §2] we obtain surjective maps

$$H^2_{\log\text{-cris}}(\mathscr{X}^\times/K) \to H^0_{\log\text{-cris}}\left(\operatorname{Spec}(R^\times), \mathbf{R}^2 f_* \mathscr{O}^{\mathrm{cris}}_{\mathscr{X}^\times/K}\right)$$
$$H^2_{\log\text{-cris}}(\mathscr{X}^\times/K)^{\varphi=p} \to H^0_{\log\text{-cris}}\left(\operatorname{Spec}(R^\times), \mathbf{R}^2 f_* \mathscr{O}^{\mathrm{cris}}_{\mathscr{X}^\times/K}\right)^{\varphi=p}$$

as the edge maps of degenerate Leray spectral sequences (see in particular [Mor14, Lemma 2.4, Theorem 2.5]). Finally we note that again applying Kedlaya's full faithfulness theorem, together with the proof of [LP16, Proposition 5.45], we can see that

$$H^0_{\log\text{-cris}}\left(\operatorname{Spec}(R^\times), \mathbf{R}^2 f_* \mathscr{O}^{\mathrm{cris}}_{\mathscr{X}^\times/K}\right)^{\varphi=p} \cong H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p}$$

and the claim follows. $\square$

We will now give one final reformulation of this result.

Definition 4.6.
(1) We say that a cohomology class in $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$ is *algebraic* if it is in the image of $\operatorname{Pic}(X)_{\mathbb{Q}}$ under the Chern class map.
(2) We say that a cohomology class in $H^2_{\log\text{-cris}}(X_0^\times/K)$ is *log-algebraic* if it is in the image of $\operatorname{Pic}(X_0^\times)_{\mathbb{Q}}$ under the Chern class map.
(3) We say that a cohomology class in $H^2_{\log\text{-cris}}(X_0^\times/K)$ is *algebraic* if it is in the image of $\operatorname{Pic}(X_0)_{\mathbb{Q}}$ under the Chern class map.

Let

$$\mathrm{sp} : H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)^{\nabla=0} \hookrightarrow H^2_{\mathrm{rig}}(X/\mathscr{R})^{\nabla=0} \xrightarrow{\sim} H^2_{\log\text{-cris}}(X_0^\times/K)^{N=0} \hookrightarrow H^2_{\log\text{-cris}}(X_0^\times/K)$$

denote the composite homomorphism.

Theorem 4.7. *Assume that $\mathscr{X}$ is projective, and let $\alpha \in H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$. The following are equivalent.*

(1) *α is algebraic.*
(2) *$\nabla(\alpha) = 0$ and $\mathrm{sp}(\alpha)$ is log-algebraic.*
(3) *$\nabla(\alpha) = 0$ and $\mathrm{sp}(\alpha)$ is algebraic.*

Proof. Note that since sp is injective, the hypotheses in both (2) and (3) imply that $\varphi(\alpha) = p\alpha$. It therefore suffices to observe that the map $\operatorname{Pic}(\mathscr{X})_{\mathbb{Q}} \to \operatorname{Pic}(X)_{\mathbb{Q}}$ is surjective, the claim then following immediately from Theorem 4.5. $\square$

5. GLOBAL RESULTS

In this section we will deduce some global algebraicity results more closely analogous to the main results of [Mor14]. We will therefore change notation and let F denote a function field of transcendence degree one over our perfect field k of characteristic p. We will let v denote a place of F with completion F_v and residue field k_v. Let $\mathscr{C}$ denote the unique smooth, proper, geometrically connected curve over k with function field F. Let F^{sep} denote a fixed separable closure of F with Galois group G_F.

Definition 5.1. Define $F\text{-Isoc}(F/K) := 2\text{-colim}_U F\text{-Isoc}(U/K)$, the colimit being taken over all open subschemes $U \subset \mathscr{C}$.

Note that by [Ked07, Theorem 5.2.1], for any $E \in F\text{-Isoc}(F/K)$, defined on some $U \subset \mathscr{C}$, the zeroeth cohomology group

$$E^{\nabla=0} = H^0_{\mathrm{rig}}(U/K, E)$$

is a well-defined (i.e. independent of U) F-isocrystal over K. For any smooth and *projective* variety X/F we have cohomology groups $\mathscr{H}^i_{\mathrm{rig}}(X/K) \in F\text{-Isoc}(F/K)$ obtained by choosing a smooth proper model over some $U \subset \mathscr{C}$ and taking the higher direct images and applying [Laz16, Corollary 5.4]. As constructed in [Pál15, §6] (see in particular Propositions 6.17 and 7.2) there is a p-adic Chern class map

$$c_1 : \mathrm{Pic}(X)_{\mathbb{Q}} \to \mathscr{H}^2_{\mathrm{rig}}(X/K)^{\nabla=0}$$

and we will call elements in the image *algebraic*.

Assume now that X has semistable reduction at v, denote the associated log smooth scheme over $k_v^\times$ by $X_v^\times$. Let $\mathscr{E}_v^\dagger$ denote a copy of the bounded Robba ring 'at v', so that by [Tsu98, §6.1] there is a functor

$$\mathbf{i}_v^* : F\text{-Isoc}(F/K) \to \underline{\mathbf{M\Phi}}^{\nabla}_{\mathscr{E}_v^\dagger}.$$

Thanks to the proof of [LP16, Proposition 5.52] this functor sends $\mathscr{H}^2_{\mathrm{rig}}(X/K)$ to $H^2_{\mathrm{rig}}(X_{F_v}/\mathscr{E}_v^\dagger)$. In particular we obtain a map

$$r_v : \mathscr{H}^2_{\mathrm{rig}}(X/K)^{\nabla=0} \to H^2_{\mathrm{rig}}(X_{F_v}/\mathscr{E}_v^\dagger)^{\nabla=0}$$

and composing with the specialisation map considered at the end of §4 we obtain a homomorphism

$$\mathrm{sp}_v : \mathscr{H}^2_{\mathrm{rig}}(X/K)^{\nabla=0} \to H^2_{\log\text{-cris}}(X_v^\times/K_v^\times)$$

where $K_v = W(k_v)[1/p]$.

Theorem 5.2. *Assume that X is projective, and let $\alpha \in \mathscr{H}^2_{\mathrm{rig}}(X/K)^{\nabla=0}$. The following are equivalent.*

(1) *α is algebraic.*
(2) *$\mathrm{sp}_v(\alpha)$ is algebraic.*
(3) *$\mathrm{sp}_v(\alpha)$ is log-algebraic.*

Proof. As before the hypotheses in (2) and (3) imply that $\varphi(\alpha) = p\alpha$. By Theorem 4.7 we clearly have (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3), and if (2) or (3) hold then there exists a line bundle $\mathscr{L} \in \mathrm{Pic}(X_{F_v})_{\mathbb{Q}}$ such that $r_v(\alpha) = c_1(\mathscr{L})$ in $H^2_{\mathrm{rig}}(X_{F_v}/\mathscr{E}_v^\dagger)^{\nabla=0}$. To descend $\mathscr{L}$ to $\mathrm{Pic}(X)_{\mathbb{Q}}$ we follow the proof of Corollary 3.4. Specifically, applying Néron–Popescu desingularisation to the extension $F_v^h \to F_v$ from the Henselisation to the completion at v and arguing exactly as before we can in fact assume that $\mathscr{L}$ descends to $X_{F_v^h}$, and hence to $X_{F'}$ for some finite, separable extension F'/F. Again taking the pushforward and dividing by the degree gives the result. □

6. A COUNTER-EXAMPLE

A natural question to ask is whether or not the analogue of Corollary 3.4 or Corollary 4.4 holds with $\mathrm{Pic}(-)_{\mathbb{Q}}$ replaced by $\mathrm{Pic}(-)_{\mathbb{Q}_p}$. We will show in the section that when k is a finite field this cannot be the case, since it would imply Tate's isogeny theorem for elliptic curves over $k[\![t]\!]$. Let us return to the previous notation of writing $F = k(\!(t)\!)$ and $R = k[\![t]\!]$ for its ring of integers.

We first need to quickly recall some material on Dieudonné modules of abelian varieties over k, R and F. As before, we will let W denote the ring of Witt vectors of k, set $\Omega = W[\![t]\!]$ and let Γ be the p-adic completion of $\Omega[t^{-1}]$, so that we have $\mathscr{E} = \Gamma[1/p]$. Fix compatible lifts σ of absolute Frobenius to $W \subset \Omega \subset \Gamma$. By [dJ95, Main Theorem 1] there are covariant equivalences of categories

$$\mathbf{D} : \mathbf{BT}_k \overset{\sim}{\to} \mathbf{DM}_W, \ \ \mathbf{D} : \mathbf{BT}_R \overset{\sim}{\to} \mathbf{DM}_\Omega, \ \ \mathbf{D} : \mathbf{BT}_F \overset{\sim}{\to} \mathbf{DM}_\Gamma$$

between p-divisible groups over k (resp. R, F) and finite free Dieudonné modules over W (resp. Ω, Γ). In particular, if $\mathscr{A}$ is an abelian variety over any of these rings, we will let $\mathbf{D}(\mathscr{A})$ denote the (covariant) Dieudonné module of its p-divisible group $\mathscr{A}[p^\infty]$. It follows essentially from the construction (see [BBM82]) together with the comparison between crystalline and rigid cohomology that when A/F is an abelian variety we have $\mathbf{D}(A) \otimes_\Gamma \mathscr{E} \cong H^1_{\mathrm{rig}}(A/\mathscr{E})^\vee(-1)$ as (φ, ∇)-modules over $\mathscr{E}$, and from [Ked00, Theorem 7.0.1] that $\mathbf{D}(A) \otimes_\Gamma \mathscr{E}$ canonically descends to a (φ, ∇)-module $\mathbf{D}^\dagger(A) \cong H^1_{\mathrm{rig}}(A/\mathscr{E}^\dagger)^\vee(-1)$ over $\mathscr{E}^\dagger$. The results of [BBM82, §5.1] give a canonical isomorphism $\mathbf{D}^\dagger(A^\vee) \cong \mathbf{D}^\dagger(A)^\vee(-1)$ of (φ, ∇)-modules over $\mathscr{E}^\dagger$. In particular, if E is an elliptic curve then we have a canonical isomorphism $E \cong E^\vee$ and hence an isomorphism $\mathbf{D}^\dagger(E) \cong \mathbf{D}^\dagger(E)^\vee(-1)$.

We can now proceed to the construction of our counter-example. It will be a smooth projective relative surface $\mathscr{X}$ over R, obtained as a product $\mathscr{E}_1 \times_R \mathscr{E}_2^\vee$ $(= \mathscr{E}_1 \times_R \mathscr{E}_2)$ where $\mathscr{E}_i$ are elliptic curves over R (to be specified later on). Let X denote the generic fibre of $\mathscr{X}$ and X_0 the special fibre. As a product of elliptic curves, we know that the Tate conjecture for divisors holds for X_0. Functoriality of Dieudonné modules induces a homomorphism

$$\mathbf{D}^\dagger_{E_1,E_2} : \mathrm{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to \mathrm{Hom}_{\underline{\mathbf{M}\Phi}^\nabla_{\mathscr{E}^\dagger}}(\mathbf{D}^\dagger(E_1), \mathbf{D}^\dagger(E_2))$$

which is injective by standard results.

Theorem 6.1. *Assume that an element $\mathscr{L} \in \mathrm{Pic}(X_0)_{\mathbb{Q}_p}$ lifts to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}_p}$ if its first Chern class $c_1(\mathscr{L}) \in H^2_{\mathrm{rig}}(X/\mathscr{R})$ lies in $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$ (in other words, that the $\mathbb{Q}_p$-analogue of Corollary 3.4 holds). Then the map $\mathbf{D}^\dagger_{E_1,E_2}$ is an isomorphism.*

Proof. This is essentially well-known. To start with, we note that under the conditions of Theorem 6.1 the natural map

$$c_1 : \mathrm{Pic}(X)_{\mathbb{Q}_p} \to H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)^{\nabla=0, \varphi=p}$$

is surjective, and induces an isomorphism between $\mathrm{NS}(X)_{\mathbb{Q}_p}$ and $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p}$. It follows from the Künneth formula [LP16, Corollary 3.78] that

$$H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger) \cong \mathscr{E}^\dagger(-1) \oplus H^1_{\mathrm{rig}}(E_1/\mathscr{E}^\dagger) \otimes H^1_{\mathrm{rig}}(E_2^\vee/\mathscr{E}^\dagger) \oplus \mathscr{E}^\dagger(-1)$$

where the terms on either end are $H^0 \otimes H^2$ and $H^2 \otimes H^0$ respectively. Since $H^1_{\mathrm{rig}}(E_1/\mathscr{E}^\dagger) \cong \mathbf{D}^\dagger(E_1)$ and $H^1_{\mathrm{rig}}(E_2^\vee/\mathscr{E}^\dagger) \cong \mathbf{D}^\dagger(E_2)^\vee(-1)$ we have that

$$\begin{aligned} H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)^{\nabla=0,\varphi=p} &= \mathbb{Q}_p \oplus \left(\mathbf{D}^\dagger(E_1) \otimes_{\mathscr{E}^\dagger} \mathbf{D}^\dagger(E_2)^\vee\right)^{\nabla=0,\varphi=\mathrm{id}} \oplus \mathbb{Q}_p \\ &= \mathbb{Q}_p \oplus \mathrm{Hom}_{\underline{\mathbf{M\Phi}}^\nabla_{\mathscr{E}^\dagger}}(\mathbf{D}^\dagger(E_1), \mathbf{D}^\dagger(E_2)) \oplus \mathbb{Q}_p. \end{aligned}$$

Next, let $\mathrm{DC}_{\mathrm{alg}}(E_1, E_2^\vee)$ denote the group of divisorial correspondences from E_1 to $E_2^\vee$ modulo algebraic equivalence, in other words line bundles on $E_1 \times E_2^\vee$ whose restriction to both $E_1 \times \{0\}$ and $\{0\} \times E_2^\vee$ is trivial. Then we have shown that the map

$$\mathrm{DC}_{\mathrm{alg}}(E_1, E_2^\vee)_{\mathbb{Q}_p} \to \mathrm{Hom}_{\underline{\mathbf{M\Phi}}^\nabla_{\mathscr{E}^\dagger}}(\mathbf{D}^\dagger(E_1), \mathbf{D}^\dagger(E_2))$$

is an isomorphism, and since $\mathrm{DC}_{\mathrm{alg}}(E_1, E_2^\vee)_{\mathbb{Q}} \cong \mathrm{Hom}(E_1, E_2)_{\mathbb{Q}}$, it follows that the map

$$\mathrm{Hom}(E_1, E_2)_{\mathbb{Q}_p} \to \mathrm{Hom}_{\underline{\mathbf{M\Phi}}^\nabla_{\mathscr{E}^\dagger}}(\mathbf{D}^\dagger(E_1), \mathbf{D}^\dagger(E_2))$$

is also an isomorphism. This completes the proof. □

In other words, to produce our required counter-example $\mathscr{X}$ we need to produce elliptic curves $\mathscr{E}_1$ and $\mathscr{E}_2$ as above such that $\mathbf{D}^\dagger_{E_1,E_2}$ is not surjective. So let $k = \mathbb{F}_{p^2}$ and let E_0/k be a supersingular elliptic curve such that $\mathrm{Frob}_{p^2} = [p] \in \mathrm{End}_k(E_0)$ (such elliptic curves exist by Honda–Tate theory). It easily follows that any $\bar{k}$-endomorphism of E_0 has to commute with Frob_{p^2}, and is hence defined over k. By the p-adic version of Tate's isogeny theorem the p-divisible group functor induces an isomorphism:

$$\mathrm{End}(E_0) \otimes \mathbb{Z}_p \longrightarrow \mathrm{End}(E_0[p^\infty]).$$

Lemma 6.2. *There is an isomorphism $\phi : E_0[p^\infty] \to E_0[p^\infty]$ such that the $\mathbb{Q}_p$-linear span of ϕ in $\mathrm{End}(E_0[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ cannot be spanned by an element in*

$$\mathrm{End}(E_0) \otimes \mathbb{Q} \subset \mathrm{End}(E_0) \otimes \mathbb{Q}_p = \mathrm{End}(E_0[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof. Since $\mathrm{End}(E_0[p^\infty])$ is an order in a quaternion algebra over $\mathbb{Q}_p$ by [Sil86, Ch. V, Theorem 3.1], so its group of invertible elements is a p-adic Lie group of dimension 3. Therefore the $\mathbb{Q}_p$-linear spans of elements of $\mathrm{End}(E_0[p^\infty])^*$ is uncountable. As $\mathrm{End}(E_0) \otimes \mathbb{Q}$ is countable, there is a $\phi \in \mathrm{End}(E_0[p^\infty])^*$ whose $\mathbb{Q}_p$-linear span cannot be spanned by the left hand side of the inclusion above. □

Let $\mathscr{E}_1$ be an elliptic curve over R whose special fibre is E_0 and whose generic fibre E_1 over $F = k((t))$ is ordinary. Via the isomorphism ϕ in the lemma above we can consider $\mathscr{E}_1[p^\infty]$ as a deformation of $E_0[p^\infty]$. By the Serre–Tate theorem [Mes72, V. Theorem 2.3] there is a deformation $\mathscr{E}_2$ of E_0 over R corresponding to this deformation p-divisible groups. Let E_2 denote the generic fibre of $\mathscr{E}_2$ over F.

Proposition 6.3. *The map*

$$\mathbf{D}^\dagger_{E_1,E_2} : \mathrm{Hom}(E_1, E_2) \otimes \mathbb{Q}_p \longrightarrow \mathrm{Hom}_{\underline{\mathbf{M\Phi}}^\nabla_{\mathscr{E}^\dagger}}(\mathbf{D}^\dagger(E_1), \mathbf{D}^\dagger(E_2))$$

is not surjective.

Proof. Assume for contradiction that in fact $\mathbf{D}^\dagger_{E_1,E_2}$ is an isomorphism. By construction $\mathscr{E}_1[p^\infty] \cong \mathscr{E}_2[p^\infty]$ so by the functoriality of Dieudonné modules $\mathrm{Hom}(\mathbf{D}(\mathscr{E}_1), \mathbf{D}(\mathscr{E}_2))$ is non-zero, and by pull-back $\mathrm{Hom}(\mathbf{D}(E_1), \mathbf{D}(E_2))$ is also non-zero. As

$$\mathbf{D}^\dagger(E_i) \otimes_{\mathscr{E}^\dagger} \mathscr{E} = \mathbf{D}(E_i) \otimes_\Gamma \mathscr{E},$$

we get that $\mathrm{Hom}(\mathbf{D}^\dagger(E_1),\mathbf{D}^\dagger(E_2))$ is also non-zero, by Kedlaya's full faithfullness theorem [Ked04, Theorem 5.1]. So by our assumptions $\mathrm{Hom}(E_1,E_2)$ is also non-zero, and the elliptic curves E_1 and E_2 are isogeneous.

As $\mathscr{E}_1$ is generically ordinary but has a supersingular special fibre, it is not constant, that is, the j-invariant of its generic fibre $j(E_1)\notin\overline{\mathbb{F}}_p$. Therefore $\mathrm{End}(E_1)=\mathbb{Z}$, so by the above $\mathrm{Hom}(E_1,E_2)\otimes\mathbb{Q}_p$ is one-dimensional. Therefore the same holds for $\mathrm{Hom}(\mathbf{D}^\dagger(E_1),\mathbf{D}^\dagger(E_2))$, too. We have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathscr{E}_1,\mathscr{E}_2)\otimes\mathbb{Q}_p \arrow[r] \arrow[d] & \mathrm{Hom}(E_1,E_2)\otimes\mathbb{Q}_p \arrow[d] \\
\mathrm{Hom}(\mathbf{D}(\mathscr{E}_1),\mathbf{D}(\mathscr{E}_2))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \arrow[r] & \mathrm{Hom}(\mathbf{D}^\dagger(E_1),\mathbf{D}^\dagger(E_2)).
\end{tikzcd}$$

The lower horizontal map is an isomorphism by de Jong's full faithfullness theorem [dJ98], the upper horizontal map is an isomorphism, since any abelian scheme is the Néron model of its generic fibre, and the right vertical map is an isomorphism by assumption. So the left vertical map is an isomorphism, too. Specialisation furnishes us with another commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathscr{E}_1,\mathscr{E}_2)\otimes\mathbb{Q}_p \arrow[r] \arrow[d] & \mathrm{End}(E_0)\otimes\mathbb{Q}_p \arrow[d] \\
\mathrm{Hom}(\mathbf{D}(\mathscr{E}_1),\mathbf{D}(\mathscr{E}_2))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \arrow[r] & \mathrm{End}(\mathbf{D}(E_0))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.
\end{tikzcd}$$

By construction the image of the lower horizontal map in

$$\mathrm{End}(\mathbf{D}(E_0))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=\mathrm{End}(E_0[p^\infty])\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

contains the span of ϕ. Since the domain of this map is one-dimensional, we get that its image is the span of ϕ. Since the left vertical map is an isomorphism by the above, we get that the span of ϕ is spanned by the specialisation of any non-zero isogeny $\mathscr{E}_1\to\mathscr{E}_2$. This is a contradiction. $\square$

We therefore arrive at the following.

Corollary 6.4. *There exists a smooth, projective relative surface $\mathscr{X}/R$ with generic fibre X and special fibre X_0, and a class $\mathscr{L}\in\mathrm{Pic}(X_0)_{\mathbb{Q}_p}$ whose Chern class $c_1(\mathscr{L})\in H^2_{\mathrm{rig}}(X/\mathscr{R})$ lies inside $H^2_{\mathrm{rig}}(X/\mathscr{E}^\dagger)$ but which does not lift to $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}_p}$.*

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