A NOTE ON EFFECTIVE DESCENT FOR OVERCONVERGENT ISOCRYSTALS

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ABSTRACT. In this short note we prove that proper surjective and faithfully flat maps are morphisms of effective descent for overconvergent isocrystals. We deduce that for an arbitrary variety over a perfect field of characteristic p, the Frobenius pull-back functor is an equivalence on the overconvergent category.

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INTRODUCTION

Let k be a perfect field of characteristic $p > 0$. One of the major problems in arithmetic geometry over the last 50 years or so has been that of describing a 'good' category of coefficients for p-adic cohomology of varieties over k, with behaviour mirroring that of the category of ℓ-adic étale sheaves for $\ell \neq p$. The first attempt at doing so was the category of crystals on a variety introduced by Berthelot in his thesis, following an idea of Grothendieck. However, this category fails one of the basic requirements that one expects of such a 'good' category of coefficients, namely topological invariance. This manifests itself in the fact that the Frobenius pull-back functor

$$F^* : \mathrm{Crys}(X/W)_{\mathbb{Q}} \to \mathrm{Crys}(X/W)_{\mathbb{Q}}$$

on isocrystals is not necessarily an equivalence of categories, even if X is smooth and proper. This problem was rectified by the introduction of the category of convergent isocrystals in [Ogu84], which turns out to be the largest full sub-category of $\mathrm{Crys}(X/W)_{\mathbb{Q}}$ on which F^* is an equivalence. This characterisation is deduced from the fact that the category of convergent isocrystals satsfies descent under proper and surjective morphisms of varieties, which in turn implies the required topological invariance.

When X is not proper, Berthelot introduced in [Ber96] a refinement of the category of convergent isocrystals on X, by considering 'overconvergence conditions' (on both objects and morphisms) along the boundary of some compactification $X \hookrightarrow \overline{X}$. A natural question then arises of whether or not this category of overconvergent isocrystals satisfies proper descent, and is thus a topological invariant. This is what we prove in this note.

Theorem (4.1). *Let $f : X \to Z$ be a proper surjective (or faithfully flat) morphism of varieties over k. Then f is a morphism of effective descent for overconvergent isocrystals.*

Note that the 'full faithfulness' part of descent follows from the more general results on cohomological descent proved in [Tsu03, ZB14], the question is really one of effectivity of descent data. In the expected manner, one can then use this theorem to obtain invariance of $\mathrm{Isoc}^\dagger(X/K)$ under universal homeomorphisms, and in particular the following.

Corollary (5.2). *Let X be a variety over k. Then the Frobenius pull-back functor*

$$F^* : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(X/K)$$

is an equivalence of categories.

For smooth varieties with good compactifications, this follows from Berthelot's theorem on Frobenius descent for arithmetic $\mathscr{D}$-modules [Ber00]. A more general version of this result (and therefore the deduction of Corollary 5.2) forms part of current work in progress of Crew (see the introduction to [Cre17] for details). Our proof via Theorem 4.1, however, is reasonably direct (i.e. does not depend on any results on arithmetic $\mathscr{D}$-modules), and Theorem 4.1 itself is potentially of independent interest.

The strategy of the proof of Theorem 4.1 is to reduce to the following version of flat descent in analytic geometry.

Theorem (2.9). *Let $f : X \to Y$ be a faithfully flat morphism of adic spaces locally of finite type over a complete, discretely valued field. Then f is a morphism of effective descent for coherent sheaves.*

This is essentially just a rephrasing of the descent results of [Con06, §4]; our modest contribution is the rather satisfying observation that Conrad's condition that a flat map of rigid analytic spaces (in the sense of Tate) 'admits local fpqc sections' translates *exactly* into the surjectivity of the associated map on adic spaces. We would therefore like to view this result as yet more evidence (if it were needed) that Huber's theory of adic spaces really is the correct setting in which to do non-archimedean analytic geometry.

Given this analytic descent result the proof of Theorem 4.1 proceeds more or less as expected, the point being that a projective surjective map of varieties can, locally on the base, be extended to a proper flat morphism of frames. One can then show that the induced morphism on suitably small neighbourhoods of the respective tubes is faithfully flat (in the sense of adic geometry) and therefore is a morphism of effective descent for coherent sheaves. Applying this universally in frames mapping to the base Z and using Le Stum's 'site-theoretic' interpretation of $\mathrm{Isoc}^\dagger(X/K)$ given in [LS07, §8] completes the proof.

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Notations and conventions. We will let $\mathcal{V}$ be a complete, discrete valuation ring with fraction field K of characteristic 0 and perfect residue field k of characteristic $p > 0$. We will let ϖ be a uniformiser of $\mathcal{V}$. A variety over k will be a separated scheme of finite type, and the category of these objects will be denoted $\mathbf{Var}_k$. All formal schemes over $\mathcal{V}$ will be assumed to be of finite type.

An analytic space over K will be an adic space locally of finite type over $\mathrm{Spa}(K, \mathcal{V})$. It will be called an analytic variety over K if in addition the structure morphism to $\mathrm{Spa}(K, \mathcal{V})$ is separated. Similarly, we will refer to a rigid analytic space in the sense of Tate [BGR84, §9.3.1] as a 'rigid space', and when it is separated over $\mathrm{Sp}(K)$ we will call it a rigid variety. Thus by [Hub96, §1.1.11] there is a fully faithful functor from rigid spaces over K to analytic spaces over K, which induces an equivalence on the full-subcategories of quasi-separated objects. We will denote the category of analytic spaces over K by $\mathbf{An}_K$.

1. The formalism of descent

In this section we will very briefly recall the formalism of descent, and introduce the three examples that particularly interest us, namely coherent sheaves on analytic spaces over K, $j^\dagger$-modules on frames over $\mathscr{V}$, and overconvergent isocrystals on algebraic varieties over k. So suppose that we have a fibred category

$$\mathscr{F} \to \mathscr{C}$$

over some base category $\mathscr{C}$. That is, for every object $X \in \mathscr{C}$ we have a category $\mathscr{F}_X$, and for every morphism $f : X \to Y$ a pull-back functor $f^* : \mathscr{F}_Y \to \mathscr{F}_X$, which are compatible under composition. Then for any morphism $f : X \to Y$ in $\mathscr{C}$ we have two pull-back functors

$$\pi_0^*, \pi_1^* : \mathscr{F}_X \to \mathscr{F}_{X\times_Y X}$$

associated to the two projections $\pi_i : X \times_Y X \to X$. Similarly, we have three projections

$$\pi_{01}, \pi_{12}, \pi_{02} : X \times_Y X \times_Y X \to X \times_Y X$$

giving rise to corresponding pull-back functors.

Definition 1.1. If $E \in \mathscr{C}_X$ then *descent data* on E relative to f is an isomorphism $\alpha : \pi_0^* E \xrightarrow{\sim} \pi_1^* E$ such that

$$\pi_{02}^*(\alpha) = \pi_{12}^*(\alpha) \circ \pi_{01}^*(\alpha).$$

The category of objects in $\mathscr{F}_X$ equipped with descent data is denoted $\mathscr{F}_{X\times_Y X \rightrightarrows X}$, pull-back by f induces

$$f^* : \mathscr{F}_Y \to \mathscr{F}_{X\times_Y X \rightrightarrows X}.$$

Definition 1.2. We say that f is a morphism of descent for $\mathscr{F}$ if the functor

$$f^* : \mathscr{F}_Y \to \mathscr{F}_{X\times_Y X \rightrightarrows X}$$

is fully faithful. We say that f is a morphism of effective descent for $\mathscr{F}$ if f^* is an equivalence of categories.

The three key examples of fibred categories we will consider in this note are the following.

Example 1.3.
(1) As in [Cre92, §1] (but fixing the ground field K) we will view the category of overconvergent isocrystals $\mathbf{Isoc}^\dagger$ as a fibred category

$$\mathbf{Isoc}^\dagger \to \mathbf{Var}_k$$

over the category of k-varieties. That is, for every $X \in \mathbf{Var}_k$ we have the category $\mathrm{Isoc}^\dagger(X/K)$ of overconvergent isocrystals on X/K, and for every morphism $f : X \to Y$ there is a pull-back functor $f^* : \mathrm{Isoc}^\dagger(Y/K) \to \mathrm{Isoc}^\dagger(X/K)$, compatibly with composition.

(2) Similarly, we may view the category $\mathbf{Coh}$ of coherent sheaves as a fibred category

$$\mathbf{Coh} \to \mathbf{An}_K$$

over the category of analytic spaces over K.

(3) Let $\mathbf{Frame}_{\mathscr{V}}$ denote the category of frames over $\mathscr{V}$, that is triples $(X, \overline{X}, \mathfrak{X})$ consisting of an open immersion of k-varieties $X \to \overline{X}$ and a closed immersion $\overline{X} \to \mathfrak{X}$ of separated formal $\mathscr{V}$-schemes. Then taking $(X, \overline{X}, \mathfrak{X})$ to the category of coherent $j_X^\dagger \mathscr{O}_{]\overline{X}[_{\mathfrak{X}}}$-modules (in the sense of adic geometry) gives rise to a fibred category

$$\mathbf{Coh}_{j^\dagger} \to \mathbf{Frame}_{\mathscr{V}}.$$

2. Flat descent for analytic spaces

The purpose of this section is to give a careful discussion of flat descent for analytic spaces, and in particular rephrasing the results of [Con06] in terms of adic spaces. One particularly pleasing aspect of this reformulation is that it gives a very natural interpretation of the condition appearing in [Con06, §4] that a flat map of rigid spaces 'admits local fpqc sections' - it simply means that the induced map on adic spaces is surjective. This will then let us deduce a simple-to-state version of flat descent for analytic spaces over K.

Definition 2.1. Let $f : X \to Y$ be a morphism of analytic spaces over K.

(1) We say that f is *flat* if for all $x \in X$ the ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is flat.
(2) We say that f is *faithfully flat* if in addition f is surjective.
(3) We say that f is *fpqc* if it is faithfully flat and quasi-compact.

Note that the second condition is stronger than simply requiring surjectivity on rigid points, as the following example shows.

Example 2.2. Let X be the disjoint union of the open unit disc and the closed annulus of radius 1. Let Y be the closed unit disc, and $f : X \to Y$ the obvious map. Then f is flat and surjective on rigid points, but not faithfully flat in our sense.

Since we will be comparing with the situation of rigid spaces, let us recall the following definitions.

Definition 2.3. Let $f_0 : X_0 \to Y_0$ be a morphism of rigid spaces over K.

(1) We say that f_0 is *flat* if for all $x \in X_0$ the ring homomorphism $\mathscr{O}_{Y_0,f_0(x)} \to \mathscr{O}_{X_0,x}$ is flat.
(2) We say that f_0 is *fpqc* if it is flat, quasi-compact and surjective.

Note that we have deliberately avoided giving the definition of a faithfully flat map of rigid spaces *without additional quasi-compactness hypotheses*. We will first need to check various compatibilities of these notions. Note that it follows immediately from the definitions that a morphism $f_0 : X_0 \to Y_0$ of rigid spaces over K is flat if the associated morphism $f : X \to Y$ of analytic spaces over K is so, and the converse follows from the following result.

Proposition 2.4. *Let $f : X \to Y$ be a morphism of analytic spaces over K. If the ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is flat for all rigid points of X, then f is flat.*

Proof. The question is local on X and Y which we may assume to be affinoid, say $X = \mathrm{Spa}(B, B^+)$ and $Y = \mathrm{Spa}(A, A^+)$. By [Sch99, Theorem 4.8] the assumption on 'flatness at rigid points' implies that (after possibly further localising) $A \to B$ is flat as a morphism of affinoid K-algebras. Since analytic localisations are flat the claim then follows. □

Let us from now on abbreviate 'quasi-compact, quasi-separated' as qcqs.

Corollary 2.5. *Let $f_0 : X_0 \to Y_0$ be a morphism of qcqs rigid spaces over K, with induced morphism $f : X \to Y$ of analytic spaces over K. Then the following are equivalent.*

(1) *f_0 is flat (resp. fpqc);*

(2) *f is flat (resp. fpqc);*

(3) *there exists a flat (resp. fpqc) morphism $\mathfrak{X} \to \mathfrak{Y}$ of admissible formal schemes over $\mathscr{V}$ whose induced morphism on rigid generic fibres is f_0, and on adic generic fibres is f.*

Proof. The flat case follows from Proposition 2.4 above together with [Bos09, Theorem 7.1]. In the fpqc case the implication (1)⇒(2) is clear, and the equivalence (1)⇔(3) follows from the proof of [Bos09, Corollary 7.2]. It remains to prove that if $\mathfrak{X} \to \mathfrak{Y}$ is an fpqc map of admissible formal schemes over $\mathscr{V}$, then the induced map on adic generic fibres is surjective.

Applying [Sch12, Theorem 2.22], we will divide the map $X \to Y$ into two parts. First of all, let $\mathscr{C}$ denote the category of admissible blow-ups of $\mathfrak{Y}$, and $\mathscr{D}$ that of $\mathfrak{X}$. By [Aut17, Tag 080F] there is therefore a canonical functor $\mathscr{C} \to \mathscr{D}$ taking $\mathfrak{Y}' \to \mathfrak{Y}$ to its *base change* $\mathfrak{X}' \to \mathfrak{X}$, the induced map $\mathfrak{X}' \to \mathfrak{Y}'$ is therefore faithfully flat. We now consider the maps

$$X = \varprojlim_{\mathfrak{X}' \to \mathfrak{X} \in \mathscr{D}} \mathfrak{X}' \to \varprojlim_{\mathfrak{Y}' \to \mathfrak{Y} \in \mathscr{C}} \mathfrak{X}' \to \varprojlim_{\mathfrak{Y}' \to \mathfrak{Y} \in \mathscr{C}} \mathfrak{Y}' = Y,$$

and can conclude by applying [FK13, Theorem 0.2.2.13] twice. □

Corollary 2.6. *Let $f : X \to Y$ be a flat morphism of analytic spaces over K. Then f is open.*

Proof. The question is local on both Y and X, we may therefore assume them both to be qcqs. It moreover suffices to show that the image $f(X)$ is open. We know that there exists a flat formal model $\mathfrak{X} \to \mathfrak{Y}$ of f, and arguing as in [Bos09, Corollary 7.2] we can see that this map has to factor as a fpqc map $\mathfrak{X} \to \mathfrak{U}$ followed by an open immersion $\mathfrak{U} \to \mathfrak{Y}$. We can now apply Corollary 2.5 above. □

This then begs the question of what the 'rigid' analogue of faithful flatness is, and rather pleasingly this turns out to be exactly the descent condition appearing in [Con06, Theorem 4.2.8].

Definition 2.7. We say that a flat map $f_0 : X_0 \to Y_0$ of rigid spaces 'admits local fpqc sections' if there exists an admissible cover $Y_0 = \bigcup_i Y_{0,i}$ of Y_0, fpqc maps $Z_{0,i} \to Y_0$, and for each i factorisations $Z_{0,i} \to X_0 \to Y_0$ of $Z_{0,i} \to Y_0$.

Theorem 2.8. *Let $f_0 : X_0 \to Y_0$ be a flat morphism of rigid spaces over K, with $f : X \to Y$ the induced morphism of analytic spaces over K. Then f_0 admits local fpqc sections if and only if f is faithfully flat.*

Proof. First suppose that f_0 admits fpqc local sections, we must show that f is surjective. This is clearly local for an admissible covering of Y_0, hence we may assume that Y_0 is affinoid, and that there exists an fpqc map $Z_0 \to Y_0$ and a commutative diagram

$$\begin{array}{ccc} & & X_0 \\ & \nearrow & \downarrow \\ Z_0 & \longrightarrow & Y_0. \end{array}$$

Hence by simple functoriality, it suffices to show that if $Z_0 \to Y_0$ is fpqc, then the induced map $Z \to Y$ is surjective. This follows from Corollary 2.5 above.

Conversely, let us suppose that f is faithfully flat, we wish to show that f_0 admits local fpqc sections. This question is clearly local for an admissible cover of Y_0, which we may therefore assume to be affinoid, and in particular qcqs. Thus for any qcqs open $V \subset X$ we know that $f(V)$ is a qcqs open in Y. Hence the fpqc map $V \to f(V)$ between qcqs analytic spaces over K has to come from an fpqc map $V_0 \to f_0(V_0)$ of Tate spaces over K. As V ranges over an open cover of X by qcqs opens, the images $f(V)$ form an open cover of Y. Hence the images $f_0(V_0)$ form an admissible open cover of Y_0, and f_0 admits fpqc local sections. □

By following the proof of this theorem, it is now straightforward to deduce the descent result we require from [Con06, Theorem 4.2.8].

Theorem 2.9. *Let $f : X \to Y$ be a faithfully flat morphism of analytic spaces over K. Then f is a morphism of effective descent for coherent sheaves.*

Proof. Let $\{V_i\}$ be an open cover of X by qcqs opens, and let $U_i = f(V_i)$. Then we have a commutative diagram

$$\begin{array}{ccc} \coprod_i V_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_i U_i & \longrightarrow & Y \end{array}$$

and since we know effective descent for open covers, it suffices to show that each $f_i : V_i \to U_i$ is of effective descent for coherent sheaves. But now f_i is an fpqc morphism between qcqs analytic spaces, in particular it comes from an fpqc morphism $V_{0,i} \to U_{0,i}$ of rigid spaces. Hence we may apply [Con06, Theorem 4.2.8] to conclude. □

3. Descent for coherent $j^\dagger$-modules

The strategy of proof of Theorem 4.1 below will be to follow that of Ogus [Ogu84, Theorem 4.6] in the convergent case, and just as the key component of the proof there is a version of flat descent for coherent sheaves on rigid spaces, so we will need a version of flat descent for coherent $j^\dagger$-modules on frames.

Theorem 3.1. *Let $f : (X, \overline{X}, \mathfrak{X}) \to (T, \overline{T}, \mathfrak{T})$ be a morphism of frames such that:*

1. *$X \to T$ is proper surjective;*
2. *$\overline{X} \to \overline{T}$ is proper;*
3. *$\mathfrak{X} \to \mathfrak{T}$ is flat.*

Then f is a morphism of effective descent for coherent $j^\dagger$-modules (taken in the sense of adic geometry).

Unsurprisingly, the idea will be to reduce to flat descent for rigid analytic varieties, and the key lemma that will enable us to do so is the following.

Lemma 3.2. *Let $f : (X, \overline{X}, \mathfrak{X}) \to (T, \overline{T}, \mathfrak{T})$ be a morphism of frames. Then for every neighbourhood*

$$]X \times_T X[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}} \subset W \subset]\overline{X} \times_{\overline{T}} \overline{X}[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}}$$

of $]X \times_T X[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}}$ in $]\overline{X} \times_{\overline{T}} \overline{X}[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}}$ there exists a neighbourhood

$$]X[_{\mathfrak{X}} \subset V \subset]\overline{X}[_{\mathfrak{X}}$$

of $]X[_{\mathfrak{X}}$ in $]\overline{X}[_{\mathfrak{X}}$ such that $V \times_{\mathfrak{T}_K} V \subset W$.

Proof. The question is local on both $\mathfrak{T}$ and $\mathfrak{X}$, hence we may assume that they are both affine. In particular, we may choose functions $f_1,\ldots,f_r,g_1,\ldots,g_s\in\Gamma(\mathfrak{X},\mathscr{O}_{\mathfrak{X}})$ such that $\overline{X}=\bigcap_i V(f_i)\subset\mathfrak{X}_k$ and $X=\bigcup_j D(g_j)\subset\overline{X}$. Hence we have

$$\overline{X}\times_{\overline{T}}\overline{X}=\bigcap_{i,i'}V(f_i\otimes f_{i'})\subset\mathfrak{X}_k\times_{\mathfrak{T}_k}\mathfrak{X}_k$$

and

$$X\times_T X=\bigcup_{j,j'}D(g_j\otimes g_{j'})\subset\overline{X}\times_{\overline{T}}\overline{X}.$$

By [Ber96, §1.2.4] we may assume that there exists an increasing sequence m_n of integers such that

$$W=\bigcup_n\left\{x\in\mathfrak{X}_K\times_{\mathfrak{T}_K}\mathfrak{X}_K\,\middle|\,v_x(\varpi^{-1}f_i^n\otimes f_{i'}^n)\le 1\ \forall i,i',\ \exists j,j'\text{ s.t. }v_x(\varpi^{-1}g_j^{m_n}\otimes g_{j'}^{m_n})\ge 1\right\}.$$

Again applying [Ber96, §1.2.4] we may construct the neighbourhood

$$V=\bigcup_n\left\{x\in\mathfrak{X}_K\,\middle|\,v_x(\varpi^{-1}f_i^n)\le 1\ \forall i,i',\ \exists j\text{ s.t. }v_x(\varpi^{-1}g_j^{2m_n})\ge 1\right\}$$

of $]X[_{\mathfrak{X}}$ inside $]\overline{X}[_{\mathfrak{X}}$ which clearly satisfies $V\times_{\mathfrak{T}_K}V\subset W$. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1. We may assume that X and T are dense in $\overline{X}$ and $\overline{T}$ respectively, from which we deduce that $\overline{X}\to\overline{T}$ is also proper and surjective. In particular, we can see that the square

$$\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \big\downarrow & & \big\downarrow \\ T & \longrightarrow & \overline{T} \end{array}$$

is Cartesian. Thus we have $f(]\overline{X}\setminus X[_{\mathfrak{X}})=]\overline{T}\setminus T[_{\mathfrak{T}}$ and hence using Lemma 2.6 we can deduce that if

$$V\subset]\overline{X}[_{\mathfrak{X}}$$

is a neighbourhood of $]X[_{\mathfrak{X}}$, then

$$f(V)\subset]\overline{T}[_{\mathfrak{T}}$$

must be a neighbourhood of $]T[_{\mathfrak{T}}$. Moreover, if $\{V\}$ forms a cofinal system of neighbourhoods of $]X[_{\mathfrak{X}}$ in $]\overline{X}[_{\mathfrak{X}}$, then $\{f(V)\}$ forms a cofinal system of neighbourhoods of $]T[_{\mathfrak{T}}$ in $]\overline{T}[_{\mathfrak{T}}$.

In particular, for any such V we may consider the category

$$\mathrm{Coh}(V\times_{f(V)}V\rightrightarrows V)$$

of coherent $\mathscr{O}_V$-modules together with descent data relative to $V\to f(V)$. Since $V\times_{f(V)}V$ is a neighbourhood of $]X\times_T X[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}}$ in $]\overline{X}\times_{\overline{T}}\overline{X}[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}}$ we therefore obtain a pull-back functor

$$\mathrm{Coh}(V\times_{f(V)}V\rightrightarrows V)\to\mathrm{Coh}(j_X^\dagger\mathscr{O}_{]\overline{X}[_{\mathfrak{X}}}\rightrightarrows j_{X\times_T X}\mathscr{O}_{]\overline{X}\times_{\overline{T}}\overline{X}[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}}})$$

and hence a functor

$$2\text{-colim}_V\mathrm{Coh}(V\times_{f(V)}V\rightrightarrows V)\to\mathrm{Coh}(j_X^\dagger\mathscr{O}_{]\overline{X}[_{\mathfrak{X}}}\rightrightarrows j_{X\times_T X}\mathscr{O}_{]\overline{X}\times_{\overline{T}}\overline{X}[_{\mathfrak{X}\times_{\mathfrak{T}}\mathfrak{X}}}).$$

It follows from Lemma 3.2 together with [LS07, Proposition 6.1.15] that this functor is an equivalence of categories. By Theorem 2.9 above we have an equivalence of categories

$$\mathrm{Coh}(V\times_{f(V)}V\rightrightarrows V)\cong\mathrm{Coh}(f(V)),$$

and hence once more applying [LS07, Proposition 6.1.15] finishes the proof. □

4. Effective descent for isocrystals

In this section we will prove our main result.

Theorem 4.1. *Let $f : X \to Z$ be a proper surjective or faithfully flat map of k-varieties. Then f is a morphism of effective descent for overconvergent isocrystals.*

Note that by the results of [Tsu03, ZB14] f is a morphism of descent, the problem is to show effectivity of descent data. Throughout the proof, we will use Le Stum's 'site-theoretic' characterisation of overconvergent isocrystals [LS07, §8].

Proof. Let us first treat the case of a proper surjective map $f : X \to Z$. As usual, we may by Chow's lemma assume that f is projective, and since the question is also local on Z, we may assume that we have some closed immersion $X \hookrightarrow \mathbb{P}^n_Z$.

Let $E \in \mathrm{Isoc}^\dagger(X/K)$ be equipped with descent data relative to f; we wish to produce an overconvergent isocrystal on Z/K, and we will do so by constructing its realisations on any frame $(T, \overline{T}, \mathfrak{T})$ equipped with a map $T \to Z$. Indeed, in this situation we may base change $X \hookrightarrow \mathbb{P}^n_Z$ by $T \to Z$ to obtain

$$X_T \hookrightarrow \mathbb{P}^n_T$$

and hence we may extend $X_T \to T$ to a morphism of frames

$$\begin{array}{ccccc} X_T & \longrightarrow & \overline{X}_T & \longrightarrow & \widehat{\mathbb{P}}^n_{\mathfrak{T}} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & \overline{T} & \longrightarrow & \mathfrak{T} \end{array}$$

where $\overline{X}_T$ is simply the closure of X_T inside $\mathbb{P}^n_{\overline{T}}$. Since proper surjective maps are stable by base change, this morphism satisfies the conditions of Theorem 3.1. We may realise E on $(X_T, \overline{X}_T, \widehat{\mathbb{P}}^n_{\mathfrak{T}})$ to obtain a coherent $j^\dagger_{X_T}\mathscr{O}_{]\overline{X}_T[_{\widehat{\mathbb{P}}^n_{\mathfrak{T}}}}$-module E_{X_T}. Moreover, the descent data for E relative to $X \to Z$ gives rise to descent data for E_{X_T} relative to

$$(X_T, \overline{X}_T, \widehat{\mathbb{P}}^n_{\mathfrak{T}}) \to (T, \overline{T}, \mathfrak{T}).$$

Hence by Theorem 3.1 we obtain a coherent $j^\dagger_T\mathscr{O}_{]\overline{T}[_{\mathfrak{T}}}$-module F_T whose pullback to $(X_T, \overline{X}_T, \widehat{\mathbb{P}}^n_{\mathfrak{T}})$ is E_{X_T}. Note that once our original embedding $X \to \mathbb{P}^n_Z$ was fixed the construction of F_T is completely canonical, and does not depend on any further choices. Thus one easily checks using the corresponding properties of E together with Theorem 3.1 that if $g : (T', \overline{T}', \mathfrak{T}') \to (T, \overline{T}, \mathfrak{T})$ is a morphism of frames over Z, then there is a corresponding isomorphism $g^\dagger F_T \xrightarrow{\sim} F_{T'}$, and these moreover satisfy the cocycle condition. Hence there is a unique overconvergent isocrystal F on Z/K whose realisation on each $(T, \overline{T}, \mathfrak{T})$ is exactly F_T. This completes the proof in the proper surjective case.

We can now deduce the faithfully flat case using [Aut17, Tag 05WN]. This implies that if $f : X \to Z$ is faithfully flat, then there exists a composite $Z' \to Z$ of Zariski covers and finite faithfully flat maps such that $X' := X \times_Z Z'$ admits a section. By the usual arguments, together with the fact that we know effective descent for Zariski covers, we may therefore reduce the faithfully flat case to the *finite* faithfully flat case, and hence to the proper surjective case already handled. □

5. Topological invariance and equivalence of Frobenius pull-back

The main application of Theorem 4.1 we have in mind is the following.

Theorem 5.1. *Let $f : X \to Z$ be a universal homeomorphism (i.e. f is finite, surjective and radicial). Then*

$$f^* : \mathrm{Isoc}^\dagger(Z/K) \to \mathrm{Isoc}^\dagger(X/K)$$

is an equivalence of categories.

Proof. We follow the proof of [Ogu84, Corollary 4.10]. Under the given assumptions on f, the diagonal

$$X \to X \times_Z X$$

is a nilpotent immersion. Since the tube of a k-sub-scheme T inside some formal $\mathscr{V}$-scheme $\mathfrak{T}$ only depends on the underlying set of T, we can thus deduce directly from the definitions that we have an equivalence of categories

$$\mathrm{Isoc}^\dagger(X \times_Z X/K) \cong \mathrm{Isoc}^\dagger(X/K)$$

and hence an equivalence

$$\mathrm{Isoc}^\dagger(X/K) \cong \mathrm{Isoc}^\dagger(X \times_Z X \rightrightarrows X/K).$$

In other words, every $E \in \mathrm{Isoc}^\dagger(X/K)$ is equipped with a canonical descent data relative to $f : X \to Z$. Since f is finite and surjecitve, we may therefore apply Theorem 4.1. □

Now let us suppose that we have chosen a lift σ to K of the q-power Frobenius on k. Thus we obtain a q-power Frobenius pull-back functor

$$F^* : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(X/K).$$

Corollary 5.2. *Let X be any k-variety. Then F^* is an equivalence of categories.*

Proof. Let $X^{(q)}$ be the pull-back of X by the q-power Frobenius of k. Since k is perfect, the corresponding (semi-linear) pull-back functor

$$\mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(X^{(q)}/K)$$

is an equivalence of categories. It therefore suffices to observe that the relative Frobenius $F_{X/k} : X \to X^{(q)}$ is a universal homeomorphism and apply Theorem 5.1. □

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