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Rational homotopy theory in arithmetic geometry, applications to rational points

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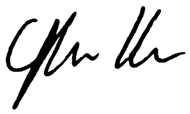
Declaration of Originality

I herewith certify that all material in this dissertation which is not my own work has been properly acknowledged.



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Abstract

In this thesis I study various incarnations of rational homotopy theory in the world of arithmetic geometry. In particular, I study unipotent crystalline fundamental groups in the relative setting, proving that for a smooth and proper family of geometrically connected varieties $f : X \to S$ in positive characteristic, the rigid fundamental groups of the fibres X_s glue together to give an affine group scheme in the category of overconvergent F-isocrystals on S. I then use this to define a global period map similar to the one used by Minhyong Kim to study rational points on curves over number fields.

I also study rigid rational homotopy types, and show how to construct these for arbitrary varieties over a perfect field of positive characteristic. I prove that these agree with previous constructions in the (log-)smooth and proper case, and show that one can recover the usual rigid fundamental groups from these rational homotopy types. When the base field is finite, I show that the natural Frobenius structure on the rigid rational homotopy type is mixed, building on previous results in the log-smooth and proper case using a descent argument.

Finally I turn to ℓ-adic étale rational homotopy types, and show how to lift the Galois action on the geometric ℓ-adic rational homotopy type from the homotopy category $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_\ell})$ to get a Galois action on the dga representing the rational homotopy type. Together with a suitable lifted p-adic Hodge theory comparison theorem, this allows me to define a crystalline obstruction for the existence of integral points. I also study the continuity of the Galois action via a suitably constructed category of cosimplicial $\mathbb{Q}_\ell$-algebras on a scheme.

Contents

Introduction

The topic of this thesis is the study of rational homotopy theory in arithmetic geometry, and my particular interest in the homotopy theory of arithmetic schemes comes from the wish to study rational or integral points. An early indication that homotopy theoretical invariants could give information about rational points was Grothendieck's anabelian section conjecture, which says (in particular) that for hyperbolic curves over number fields, rational points are determined by the étale fundamental group. In his paper [34], Minhyong Kim proposed using a unipotent version of the étale fundamental group, and a 'section map' entirely analogous to the one appearing in Grothendieck's conjecture, to study integral points on hyperbolic curves over number fields.

In the first chapter I look at developing a function field analogue of Kim's methods, and the main focus is on looking at how the unipotent rigid fundamental group varies in families. If $\mathcal{O}_{K,S}$ is some ring of S-integers in a number field, and $f:\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_{K,S})$ is smooth and proper, then the fact that the Galois action on the (p-adic) unipotent étale fundamental group of the generic fibre is unramified away from p and crystalline at places above p can be viewed as saying that this group scheme forms some form of 'p-adic local system' on $\mathrm{Spec}(\mathcal{O}_{K,S})$. Moreover, the fibres of this local system at geometric points of $\mathrm{Spec}(\mathcal{O}_{K,S})$ will exactly be the unipotent fundamental group of the fibre of f over that point (of course, some care needs to be taken as to what exactly is meant here at places above p).

This suggests the following analogue in positive characteristic. Let C/K be a smooth and proper variety over a global function field K, and spread out to some smooth and proper morphism $f:\mathcal{X}\to S$ where S is some smooth curve over a finite field. Then the rôle played in characteristic zero by the (unramified, crystalline) Galois action on the unipotent étale fundamental group of the generic fibre should be a certain 'group scheme of overconvergent F-isocrystals' on S, whose fibres are the unipotent rigid fundamental groups of the fibres of f. In the first chapter, I show exactly how to construct such an object by proving that a certain sequence of affine group schemes is split exact. This method involves choosing a base point on the curve S, however, a (rather modest) relative form of Tannakian duality gives a rephrasing of this construction in a base-point free way. This also enables me to construct path torsors entirely analogously to those used by Kim to define his period maps, in fact, the definition of the period maps if anything is easier than in the number field case. This is because the condition that the path torsos over the fundamental group have 'good reduction' at places over p is somewhat technical in the number field case, requiring a non-abelian p-adic Hodge theory comparison theorem, whereas in the function field setting, this 'good reduction' is built into the very construction of the path torsors.

In ordinary homotopy theory, at least for suitably 'nice' spaces X, the unipotent fundamental group $\pi_1(X, x)_{\mathbb{Q}}$ can be viewed as the fundamental group associated to a whole 'homotopy type' - the 'rational homotopy type' of X. In the second chapter, I ask to what extend the same is true for the unipotent rigid fundamental group. To this end, I offer several constructions of commutative differential graded algebras representing 'rigid rational homotopy types' - both in the absolute and in the relative case, and prove several comparison theorems between these construction, and between previous constructions made in special cases by Olsson and Kim/Hain. The main idea is that the rational homotopy type is obtained by simply remembering the multiplicative structure on the cohomology complex, and the comparison theorems are all effected by noting that the comparison theorems between the various p-adic cohomology theories respect this multiplicative structure. As a nice application of this construction, I prove that the rigid rational homotopy type of a variety over a finite field admits a weight filtration for the action of Frobenius. In particular this implies that the co-ordinate ring of the unipotent rigid fundamental group also admits a weight filtration - this extends results of Chiarellotto in the smooth case.

Ideally, I would like to connect these constructions of rigid rational homotopy types both to the standard construction of the unipotent rigid fundamental group (the absolute case) and to the construction of the first chapter in the relative case. Unfortunately, I am currently only able to effect this comparison in the absolute case, however, I do briefly discuss a 'homotopy section map' that, with the correct comparison theorem, should refine the period map defined in the first chapter.

If one takes a smooth and proper variety over a global function field K and spread out to some smooth and proper model $f : \mathcal{X} \to S$, it is possible to view the target of this homotopy section map as the set of rational homotopy sections with good reduction at all places of S. As in the case of the period map, the fact that these homotopy sections have 'good reduction' is built into the very definition of the relative rigid rational homotopy type, since they are, in a certain sense, sections of a 'sheaf of homotopy types' over the base S. In the third chapter I study the analogous situation for varieties with good reduction over a p-adic field K, that is schemes which arise as the generic fibre of the complement of a relative normal crossings divisor in a smooth and proper scheme $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$. In this case, the fact that the relative homotopy type is in fact some form of 'sheaf of rational homotopy types' is expressed by a certain form of Olsson's non-abelian p-adic

Hodge theory, relating the p-adic étale rational homotopy type of the generic fibre to the rigid rational homotopy type of the special fibre. I then use this comparison result to define a good notion of what it means for a section of this p-adic rational homotopy type to 'extend over the whole base $\mathrm{Spec}(\mathcal{O}_K)$'. More specifically, I show that this homotopy type admits a distinguished section over a certain localisation $\tilde{B}_{\mathrm{cr}}$ of Fontaine's ring of crystalline periods, and hence it makes sense to speak of a homotopy section 'trivialising' on base change to $\tilde{B}_{\mathrm{cr}}$. I thus get a notion of 'good reduction' for homotopy sections, entirely analogously to the corresponding notion for path torsors, and the fact that the sections coming from rational points all have good reduction then follows from functoriality of the comparison map. A comparison with the relative étale homotopy type of Barnea and Schlank then gives a refinement of their map from rational points to homotopy sections.

More detailed introductions are given at the start of each individual chapter.

Acknowledgements

First and foremost I would like to thank my advisor Dr. Pál for all that I have learned from him during my time as a PhD student. He has introduced me to much beautiful mathematics, and has guided me through the mathematical landscape with patience and an extraordinary depth of knowledge. It goes without saying that this thesis would not exist but for his help and encouragement.

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1 Relative fundamental groups

Let K be a number field and let C/K be a smooth, projective curve of genus $g > 1$, with Jacobian J. Then a famous theorem of Faltings states that the set $C(K)$ of K-rational points on C is finite. The group $J(K)$ is finitely generated, and under the assumption that its rank is strictly less than g, Chabauty in [16] was able to prove this theorem using elementary methods as follows. Let v be a place of K, of good reduction for C, and denote by C_v, J_v the base change to K_v. Then Chabauty defines a homomorphism

$$\log : J(K_v) \to H^0\left(J_v, \Omega^1_{J_v/K_v}\right) \tag{1.1}$$

and shows that there exists a non zero linear functional on $H^0(J_v, \Omega^1_{J_v/K_v})$ which vanishes on the image of $J(K)$. He then proves that pulling this back to $J(K_v)$ gives an analytic function on $J(K_v)$, which is not identically zero on $C(K_v)$, and which vanishes on $J(K)$. Hence $C(K) \subset C(K_v) \cap J(K)$ must be finite as it is contained in the zero set of a non-zero analytic function on $C(K_v)$.

In [34], Kim describes what he calls a 'non-abelian lift' of this method. Fix a point $p \in C(K)$. By considering the Tannakian category of integrable connections on C_v, one can define a 'de Rham fundamental group' $U^{\mathrm{dR}} = \pi_1^{\mathrm{dR}}(C_v, p)$, which is a pro-unipotent group scheme over K_v, as well as, for any other $x \in C(K_v)$, path torsors $P^{\mathrm{dR}}(x) = \pi_1^{\mathrm{dR}}(C_v, x, p)$ which are right torsors under U^{dR}. These group schemes and torsors come with extra structure, namely that of a Hodge filtration and, by comparison with the crystalline fundamental group of the reduction of C_v, a Frobenius action. He then shows that such torsors are classified by U^{dR}/F^0, and hence one can define 'period maps'

$$j_n : C(K_v) \to U_n^{\mathrm{dR}}/F^0 \tag{1.2}$$

where U_n^{dR} is the nth level nilpotent quotient of U^{dR}. If $n = 2$ then j_n is just the composition of the above log map with the inclusion $C(K_v) \to J(K_v)$. By analysing the image of this map, he is able to prove finiteness of $C(K)$ under certain conditions, namely if the dimension of U_n^{dR}/F^0 is greater than the dimension of the target of a global period map defined using the category of lisse étale sheaves on C. Moreover, when $n = 2$, this condition on dimensions is essentially Chabauty's condition that $\mathrm{rank}_{\mathbb{Z}} J(K) < \mathrm{genus}(C)$

(modulo the Tate-Shafarevich conjecture).

My interest lies in trying to develop a function field analogue of these ideas. The analogy between function fields in one variable over finite fields and number fields has been a fruitful one throughout modern number theory, and indeed the analogue of Mordell's conjecture was first proven for function fields by Grauert. In this chapter I discuss the problem of defining a good analogue of the global period map. This is defined in [34] using the Tannakian category of lisse $\mathbb{Q}_p$ sheaves on X, and this approach will not work in the function field setting. Neither p-adic nor ℓ-adic étale cohomology will give satisfactory answers, the first because, for example, the resulting fundamental group will be moduli dependent, i.e. will not be locally constant in families (see for example [50]), and the second because the ℓ-adic topology on the resulting target spaces for period maps will not be compatible with the p-adic topology on the source varieties. Instead I will work with the category of overconvergent F-isocrystals.

Let K be a finite extension of $\mathbb{F}_p(t)$, and let k be the field of constants of K, i.e. the algebraic closure of $\mathbb{F}_p$ inside K. Let $\overline{S}$ be the unique smooth projective, geometrically irreducible curve over k whose function field is K. If C/K is a smooth, projective, geometrically integral curve then one can choose a regular model for C. This is a regular, proper surface $\overline{X}/k$, equipped with a flat, proper morphism $f : \overline{X} \to \overline{S}$ whose generic fibre is C/K. Let $S \subset \overline{S}$ be the smooth locus of f, and denote by f also the pullback $f : X \to S$. The idea is to construct, for any section p of f, a 'non-abelian isocrystal' on S whose fibre at any closed point s 'is' the rigid fundamental group $\pi_1^{\mathrm{rig}}(X_s, p_s)$. The idea behind how to construct such an object is very simple.

Suppose that $f : X \to S$ is a Serre fibration of topological spaces, with connected base and fibres. If p is a section, then for any $s \in S$ the homomorphism $\pi_1(X, p(s)) \to \pi_1(S, s)$ is surjective, and $\pi_1(S, s)$ acts on the kernel via conjugation. This corresponds to a locally constant sheaf of groups on S, and the fibre over any point $s \in S$ is just the fundamental group of the fibre X_s. This approach makes sense for any fundamental group defined algebraically as the Tannaka dual of a category of 'locally constant' coefficients. So if $f : X \to S$ is a morphism of smooth varieties with section p, then $f_* : \pi_1^{\mathcal{C}_X}(X, x) \to \pi_1^{\mathcal{C}_S}(S, s)$ is surjective, and $\pi_1^{\mathcal{C}_S}(S, s)$ acts on the kernel. Here $\mathcal{C}_{(-)}$ is any appropriate category of coefficients, for example vector bundles with integrable connection, unipotent isocrystals etc., and e.g. $\pi_1^{\mathcal{C}_X}(X, x)$ is the Tannaka dual of this category with respect to the fibre functor x^*. This gives the kernel of f_* the structure of an 'affine group scheme over $\mathcal{C}_S$', and it makes sense to ask what the fibre is over any closed point $s \in S$. The main theorem of the first section is the following.

Theorem. *Suppose that $f : X \to S$ is a smooth morphism of smooth varieties over an algebraically closed field k of characteristic zero. Assume that both S and the fibres of f*

are connected, and that X is the complement of a relative normal crossings divisor in a smooth and proper S-scheme $\overline{X}$. Let $\mathcal{C}_S$ be the category of vector bundles with a regular integrable connection on S, and let $\mathcal{C}_X$ be the category of vector bundles with a regular integrable connection on X which are iterated extensions of those of the form $f^*\mathscr{E}$, with $\mathscr{E} \in \mathcal{C}_S$. Then the fibre of the corresponding affine group scheme over $\mathcal{C}_S$ at $s \in S$ is the de Rham fundamental group $\pi_1^{\mathrm{dR}}(X_s, p_s)$ of the fibre.

In Section 2, I discuss path torsors in the relative setting. I show in particular that for any other section q of f one can define an affine scheme $\pi_1^{\mathrm{dR}}(X/S, q, p)$ over $\mathcal{C}_S$ which is a right torsor under the relative de Rham fundamental group $\pi_1^{\mathrm{dR}}(X/S, p)$. The upshot of this is that there exists maps

$$j_n : X(S) \to H^1\left(S, \pi_1^{\mathrm{dR}}(X/S, p)_n\right) \tag{1.3}$$

which are a coarse characteristic zero function field analogue of Kim's global period maps. To investigate the characteristic zero picture further, I would want to define Hodge structures on these objects, and thus obtain finer period maps. However, my main interest lies in the positive characteristic case, and so I don't pursue these questions.

In Section 3, I define the relative rigid fundamental group in positive characteristic, mimicking the definition in characteristic zero. Instead of the category of vector bundles with regular integrable connections, I consider the category of overconvergent F-isocrystals (all varieties in Section 3 will be over a finite field, and Frobenius will always mean the *linear* Frobenius). I then proceed to use Caro's theory of cohomological operations for arithmetic $\mathcal{D}$-modules in order to prove the analogue of the above theorem in positive characteristic.

The upshot of this is that for a smooth and proper map $f : X \to S$ with geometrically connected fibres and smooth, geometrically connected base over a finite field k, and a section p of f, it is possible to define an affine group scheme $\pi_1^{\mathrm{rig}}(X/S, p)$ over the category of overconvergent F-isocrystals on S, which I call the relative fundamental group at p. The fibre of this over any point $s \in S$ is just the unipotent rigid fundamental group of the fibre X_s of f over s. As in the zero characteristic case, the general Tannakian formalism gives path torsos $\pi_1^{\mathrm{rig}}(X/S, p, q)$ for any other $q \in X(S)$, and hence there are period maps

$$X(S) \to H^1_{F,\mathrm{rig}}(S, \pi_1^{\mathrm{rig}}(X/S, p)) \tag{1.4}$$

where the RHS is a classifying set of F-torsors under $\pi_1^{\mathrm{rig}}(X/S, p)$, as well as finite level versions given by pushing out along the quotient map $\pi_1^{\mathrm{rig}}(X/S, p) \to \pi_1^{\mathrm{rig}}(X/S, p)_n$.

Finally, I study the targets of these period maps, and show that after replacing the set $H^1_{F,\mathrm{rig}}(S, \pi_1^{\mathrm{rig}}(X/S, p))$, classifying F-torsors, by $H^1_{\mathrm{rig}}(S, \pi_1^{\mathrm{rig}}(X/S, p))^{\phi=\mathrm{id}}$, the Frobenius invariant part of the set classifying torsors without F-structure, then under very restrictive hypotheses on the morphism $f : X \to S$, the target of this map has the structure of an algebraic variety. The argument here is simply a translation of the original argument of Kim, and what in my context are restrictive hypotheses are automatically satisfied for him.

There is still a long way to go to get a version of Kim's methods to work for function fields. There is still the question of how to define the analogue of the local period maps, and also to show that the domains of the period maps have the structure of varieties. Even then, it is very unclear what the correct analogue of the local integration theory will be in positive characteristic. There is still a very large amount of work to be done if such a project is to be completed.

1.1 Relative de Rham fundamental groups

Let $f : X \to S$ be a smooth morphism of smooth complex varieties, and suppose that f admits a good compactification, that is, there exists $\overline{X}$ smooth and proper over S, an open immersion $X \hookrightarrow \overline{X}$ over S, such that $D = \overline{X} \setminus X$ is a relative normal crossings divisor in $\overline{X}$. Let $p \in X(S)$ be a section. For every closed point $s \in S$ with fibre X_s, one can consider the topological fundamental group $G_s := \pi_1(X_s^{\mathrm{an}}, p(s))$, and as s varies, these fit together to give a locally constant sheaf $\pi_1(X/S, p)$ on S^{an}. Let

$$\hat{\mathcal{U}}(\mathrm{Lie}\ G_s) := \varprojlim \mathbb{C}[G_s]/\mathfrak{a}^n \tag{1.5}$$

denote the completed enveloping algebra of the Malcev Lie algebra of G_s, where $\mathfrak{a} \subset \mathbb{C}[G_s]$ is the augmentation ideal. According to Proposition 4.2 of [29], as s varies, these fit together to give a pro-local system on S^{an}, i.e. a pro-object $\hat{\mathcal{U}}_p^{\mathrm{top}}$ in the category of locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces on S^{an}. (Their theorem is a lot stronger than this, but this is all I need for now). According to Théorème 5.9 in Chapter II of [21], the pro-vector bundle with integrable connection $\hat{\mathcal{U}}_p^{\mathrm{top}} \otimes_{\mathbb{C}} \mathcal{O}_{S^{\mathrm{an}}}$ has a canonical algebraic structure. Thus given a smooth morphism $f : X \to S$ as above, with section p, one can construct a pro-vector bundle with connection $\hat{\mathcal{U}}_p$ on S, whose fibre at any closed point $s \in S$ is the completed enveloping algebra of the Malcev Lie algebra of $\pi_1(X_s^{\mathrm{an}}, p(s))$.

Denoting by $\mathfrak{g}_s$ the Malcev Lie algebra of $\pi_1(X_s^{\mathrm{an}}, p(s))$, $\hat{\mathcal{U}}(\mathfrak{g}_s) = (\hat{\mathcal{U}}_p)_s$ can be constructed algebraically, as $\mathfrak{g}_s$ is equal to Lie $\pi_1^{\mathrm{dR}}(X_s, p_s)$, the Lie algebra of the Tannaka dual of the category of unipotent vector bundles with integrable connection on X_s. This suggests the question of whether or not there is an algebraic construction of $\hat{\mathcal{U}}_p$?

I will not directly answer this question - instead I will construct the Lie algebra associated to $\hat{\mathcal{U}}_p$ - this is a pro-system $\mathscr{L}_p$ of Lie algebras with connection on S. The way I will do so is very simple, and is closely related to ideas used in [54] to study relatively unipotent mixed motivic sheaves.

Definition 1.1.1. To save saying the same thing over and over again, I will make the following definition. A 'good' morphism is a smooth morphism $f : X \to S$ of smooth varieties over a field k, with geometrically connected fibres and base, such that X is the complement of a relative normal crossings divisor in a smooth, proper S-scheme $\overline{X}$. Throughout this section I will assume that the ground field k is algebraically closed of characteristic 0.

I will assume that the reader is familiar with Tannakian categories, a good introductory reference is [38]. If $\mathcal{T}$ is a Tannakian category over a field k, and ω is a fibre functor on $\mathcal{T}$, in the sense of §1.9 of [23], I will denote the group scheme representing tensor automorphisms of ω by $G(\mathcal{T}, \omega)$. I will also use the rudiments of algebraic geometry in Tannakian categories, as explained in §5 of [22] - in particular I will talk about affine (group) schemes over Tannakian categories. I will denote the fundamental groupoid of a Tannakian category by $\pi(\mathcal{T})$, this is an affine group scheme over $\mathcal{T}$ which satisfies $\omega(\pi(\mathcal{T})) = G(\mathcal{T}, \omega)$ for every fibre functor ω (see for example 6.1 of [22]). If $\mathcal{T}$ is a Tannakian category over k, and k'/k is a finite extension, then I will denote the category of k'-modules in $\mathcal{T}$ by either $\mathcal{T} \otimes_k k'$, or $\mathcal{T}_{k'}$.

I will also assume familiarity with the theory of integrable connections and regular holonomic $\mathcal{D}$-modules on k-varieties, and will generally refer to [21] and [27] for details. A regular integrable connection on X is unipotent if it is a successive extension of the trivial connection, these form a Tannakian subcategory $\mathcal{N}\mathrm{IC}(X) \subset \mathrm{IC}(X)$ of the Tannakian category of regular integrable connections.

Definition 1.1.2. For X/k smooth and connected, the algebraic and de Rham fundamental groups of X at a closed point $x \in X$ are defined by

$$\pi_1^{\mathrm{alg}}(X, x) := x^*(\pi(\mathrm{IC}(X))) = G(\mathrm{IC}(X), x^*) \tag{1.6}$$

$$\pi_1^{\mathrm{dR}}(X, x) := x^*(\pi(\mathcal{N}\mathrm{IC}(X))) = G(\mathcal{N}\mathrm{IC}(X), x^*). \tag{1.7}$$

Remark 1.1.3. It follows from the Riemann-Hilbert correspondence that if $k = \mathbb{C}$, these affine group schemes are the pro-algebraic and pro-unipotent completions of $\pi_1(X^{\mathrm{an}}, x)$ respectively.

If $f : X \to Y$ is a morphism of smooth k-varieties, then vector bundles with integrable connection can be pulled back along f, this operation preserves regularity and is the usual pull-back on the underlying $\mathcal{O}_Y$-module. This induces a homomorphism $f_* : \pi_1^{\#}(X, x) \to \pi_1^{\#}(Y, f(x))$ for $\# = \mathrm{dR}, \mathrm{alg}$.

1.1.1 The relative fundamental group and its pro-nilpotent Lie algebra

Let $f : X \to S$ be a 'good' morphism. A regular integrable connection E on X is said to be relatively unipotent if there exists a filtration by horizontal sub-bundles, whose graded objects are all in the essential image of $f^* : \mathrm{IC}(S) \to \mathrm{IC}(X)$. I will denote the full subcategory of relatively unipotent objects in $\mathrm{IC}(X)$ by $\mathcal{N}_f\mathrm{IC}(X)$, which is a Tannakian subcategory. Suppose that $p \in X(S)$ is a section of f. There are functors of Tannakian categories

$$\mathcal{N}_f\mathrm{IC}(X) \underset{f^*}{\overset{p^*}{\rightleftarrows}} \mathrm{IC}(S) \tag{1.8}$$

and hence, after choosing a point $s \in S(k)$, homomorphisms

$$G\left(\mathcal{N}_f\mathrm{IC}(X), p(s)^*\right) \underset{p_*}{\overset{f_*}{\rightleftarrows}} G\left(\mathrm{IC}(S), s^*\right) \tag{1.9}$$

between their Tannaka duals. Let K_s denote the kernel of f_*. Then the splitting p_* induces an action of $\pi_1^{\mathrm{alg}}(S,s) = G(\mathrm{IC}(S), s^*)$ on K_s via conjugation. This corresponds to an affine group scheme over $\mathrm{IC}(S)$.

Lemma 1.1.4. *This affine group scheme is independent of s.*

Proof. Thanks to [22], §6.10, f_*, p_* above come from homomorphisms

$$p^*\left(\pi\left(\mathcal{N}_f\mathrm{IC}(X)\right)\right) \underset{p_*}{\overset{f_*}{\rightleftarrows}} \pi\left(\mathrm{IC}(S)\right) \tag{1.10}$$

of affine group schemes over $\mathrm{IC}(S)$. If $\mathcal{K}$ denotes the kernel of f_*, then $K_s = s^*(\mathcal{K})$. □

Definition 1.1.5. The relative de Rham fundamental group $\pi_1^{\mathrm{dR}}(X/S, p)$ of X/S at p is defined to be the affine group scheme $\mathcal{K}$ over $\mathrm{IC}(S)$.

Let $i_s : X_s \to X$ denote the inclusion of the fibre over s. Then there is a canonical functor $i_s^* : \mathcal{N}_f\mathrm{IC}(X) \to \mathcal{N}\mathrm{IC}(X_s)$. This induces a homomorphism $\pi_1^{\mathrm{dR}}(X_s, p_s) \to G(\mathcal{N}_f\mathrm{IC}(X), p_s^*)$ which is easily seen to factor through the fibre $\pi_1^{\mathrm{dR}}(X/S,p)_s := s^*(\mathcal{K}) = K_s$ of $\pi_1^{\mathrm{dR}}(X/S,p)$ over s.

Theorem 1.1.6. *Suppose that $k = \mathbb{C}$. Then $\phi : \pi_1^{\mathrm{dR}}(X_s, p_s) \to \pi_1^{\mathrm{dR}}(X/S,p)_s$ is an isomorphism.*

Proof. The point s gives fibre functors p_s^* on $\mathcal{N}\mathrm{IC}(X_s)$, $p(s)^*$ on $\mathcal{N}_f\mathrm{IC}(X)$ and s^* on $\mathrm{IC}(S)$. Write

$$\mathcal{K} = G\left(\mathcal{N}\mathrm{IC}(X_s), p_s^*\right), \quad \mathcal{G} = G\left(\mathcal{N}_f\mathrm{IC}(X), p(s)^*\right), \quad \mathcal{H} = G\left(\mathrm{IC}(S), s^*\right) \tag{1.11}$$

and also let

$$K = \pi_1\left(X_s^{\mathrm{an}}, p(s)\right), \quad G = \pi_1\left(X^{\mathrm{an}}, p(s)\right), \quad H = \pi_1\left(S^{\mathrm{an}}, s\right) \tag{1.12}$$

be the topological fundamental groups of X_s, X, S respectively. Then $\mathcal{K} = K^{\mathrm{un}}$, the pro-unipotent completion of K, and $\mathcal{H} = H^{\mathrm{alg}}$, the pro-algebraic completion of H. It needs to be shown that the sequence of affine group schemes

$$1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 1 \tag{1.13}$$

is exact, and I will use the equivalences of categories

$$\mathrm{IC}(X) \overset{\sim}{\to} \mathrm{Rep}_{\mathbb{C}}\left(\pi_1\left(X^{\mathrm{an}}, p(s)\right)\right), \quad \mathrm{IC}(S) \overset{\sim}{\to} \mathrm{Rep}_{\mathbb{C}}\left(\pi_1\left(S^{\mathrm{an}}, s\right)\right) \tag{1.14}$$

$$\mathrm{IC}(X_s) \overset{\sim}{\to} \mathrm{Rep}_{\mathbb{C}}\left(\pi_1\left(X_s^{\mathrm{an}}, p(s)\right)\right). \tag{1.15}$$

By Proposition 1.3 in Chapter I of [54], $\ker(\mathcal{G} \to \mathcal{H})$ is pro-unipotent. Hence according to Proposition 1.4 of *loc. cit.*, to show that ϕ is an isomorphism is equivalent to showing the following.

- If $E \in \mathcal{N}_f\mathrm{IC}(X)$ is such that $i_s^*(E)$ is trivial, then $E \cong f^*(F)$ for some F in $\mathrm{IC}(S)$.
- Let $E \in \mathcal{N}_f\mathrm{IC}(X)$, and let $F_0 \subset i_s^*(E)$ denote the largest trivial sub-object. Then there exists $E_0 \subset E$ such that $F_0 = i_s^*(E_0)$.
- There is a pro-action of $\mathcal{G}$ on $\hat{\mathcal{U}}(\mathrm{Lie}\,\mathcal{K})$ such that the corresponding action of $\mathrm{Lie}\,\mathcal{G}$ extends the left multiplication by $\mathrm{Lie}\,\mathcal{K}$.

The first is straightforward. Since f is topologically a fibration with section p, there is a split exact sequence

$$1 \to K \to G \leftrightarrows H \to 1 \tag{1.16}$$

and a representation V of G such that K acts trivially. I must show that V is the pullback of an H-representation - this is obvious! The second is no harder, I must show that if V is a G-representation, then V^K is a sub-G-module of V. But since K is normal in G, this is clear. For the third, note that $\hat{\mathcal{U}}(\mathrm{Lie}\,\mathcal{K}) = \hat{\mathcal{U}}(\mathrm{Lie}\,K) = \varprojlim \mathbb{C}[K]/\mathfrak{a}^n$, where $\mathfrak{a}$ is the augmentation ideal of $\mathbb{C}[K]$. Let H act on $\mathbb{C}[K]/\mathfrak{a}^n$ by conjugation and K by left multiplication. I claim that $\mathbb{C}[K]/\mathfrak{a}^n$ is finite dimensional, and unipotent as a K-representation.

Indeed, There are extensions of K-representations

$$0 \to \mathfrak{a}^n/\mathfrak{a}^{n+1} \to \mathbb{C}[K]/\mathfrak{a}^{n+1} \to \mathbb{C}[K]/\mathfrak{a}^n \to 0 \tag{1.17}$$

and hence, since the action of K on $\mathfrak{a}^n/\mathfrak{a}^{n+1}$ is trivial, it follows by induction that each $\mathbb{C}[K]/\mathfrak{a}^n$ is unipotent. There are also surjections

$$\left(\mathfrak{a}/\mathfrak{a}^2\right)^{\otimes n} \twoheadrightarrow \mathfrak{a}^n/\mathfrak{a}^{n+1} \tag{1.18}$$

for each n, and hence by induction, to show finite dimensionality it suffices to show that $\mathfrak{a}/\mathfrak{a}^2$ is finite dimensional. But $\mathfrak{a}/\mathfrak{a}^2 \cong K^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{C}$ is finite dimensional, as K is finitely generated.

Now, since $\mathbb{C}[K]/\mathfrak{a}^n$ is unipotent as a K-representation,it is relatively unipotent as a $G = K \rtimes H$-representation, hence $\mathbb{C}[K]/\mathfrak{a}^n$ is naturally an object in $\mathrm{Rep}_{\mathbb{C}}(\mathcal{G})$. Thus there is a pro-action of $\mathcal{G}$ on $\hat{\mathcal{U}}(\mathrm{Lie}\,\mathcal{K})$, and the action extends left multiplication by $\mathrm{Lie}\,\mathcal{K}$ as required. □

Remark 1.1.7. The co-ordinate algebra of $\pi_1^{\mathrm{dR}}(X/S,p)$ is an ind-object in the category of regular integrable connections on S. Hence it is possible to $\pi_1^{\mathrm{dR}}(X/S,p)$ as an affine group scheme over S in the usual sense, together with a regular integrable connection on the associated $\mathcal{O}_S$-Hopf algebra.

If $g : T \to S$ is any morphism of smooth varieties over k, then there is a homomorphism of fundamental groups

$$\pi_1^{\mathrm{dR}}(X_T/T, p_T) \to \pi_1^{\mathrm{dR}}(X/S,p) \times_S T := g^*(\pi_1^{\mathrm{dR}}(X/S,p)) \tag{1.19}$$

which corresponds to a horizontal morphism

$$\mathcal{O}_{\pi_1^{\mathrm{dR}}(X/S,p)} \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathcal{O}_{\pi_1^{\mathrm{dR}}(X_T/T,p_T)}. \tag{1.20}$$

Proposition 1.1.8. *If $k = \mathbb{C}$ then this is an isomorphism.*

Proof. The previous theorem implies that this induces an isomorphism on fibres over any point $t \in T(\mathbb{C})$. Hence by rigidity, it is an isomorphism. □

Write $G = \pi_1^{\mathrm{dR}}(X/S,p)$ and let G_n denote the quotient of G by the nth term in its lower central series. Let A_n denote the Hopf algebra of G_n, and $I_n \subset A_n$ the augmentation ideal. $L_n := \mathcal{H}om_{\mathcal{O}_S}\left(I_n/I_n^2, \mathcal{O}_S\right)$ is the Lie algebra of G_n. This is a coherent, nilpotent Lie algebra with connection, i.e the bracket $[\cdot,\cdot] : L_n \otimes L_n \to L_n$ is horizontal. There are natural morphisms $L_{n+1} \to L_n$, which form a pro-system of nilpotent Lie algebras with connection $\hat{L}_p$, whose universal enveloping algebra is the object $\hat{\mathcal{U}}_p$ considered in the introduction to this section.

1.1.2 Towards an algebraic proof of Theorem 1.1.6

Although I have a candidate for the relative fundamental group of a 'good' morphism $f : X \to S$ at a section p, I have only proved it is a good candidate when the ground field is the complex numbers. One might hope to be able to reduce to the case $k = \mathbb{C}$ via base change and finiteness arguments, but this approach will not work in a straightforward manner. Also, such an argument will not easily adapt to the case of positive characteristic, as in general one will not be able to lift a smooth proper family, even locally on the base. Instead, in this section I offer a more algebraic proof. Recall that the relative fundamental group $\pi_1^{\mathrm{dR}}(X/S, p)$ is an affine group scheme over $\mathrm{IC}(S)$, and there is a comparison morphism

$$\phi : \pi_1^{\mathrm{dR}}(X_s, p_s) \to \pi_1^{\mathrm{dR}}(X/S, p)_s \tag{1.21}$$

for any point $s \in S$. I want to show that this map is an isomorphism.

It follows from Proposition 1.4 in Chapter I of [54] and Appendix A of [26] that I need to prove the following:

- (Injectivity) Every $E \in \mathcal{N}\mathrm{IC}(X_s)$ is a sub-quotient of $i_s^*(F)$ for some $F \in \mathcal{N}_f\mathrm{IC}(X)$.
- (Surjectivity I) Suppose that $E \in \mathcal{N}_f\mathrm{IC}(X)$ is such that $i_s^*(E)$ is trivial. Then there exists $F \in \mathrm{IC}(S)$ such that $E \cong f^*(F)$.
- (Surjectivity II) Let $E \in \mathcal{N}_f\mathrm{IC}(X)$, and let $F_0 \subset i_s^*(E)$ denote the largest trivial sub-object. Then there exists $E_0 \subset E$ such that $F_0 = i_s^*(E_0)$.

To do so, I will need to use the language of algebraic $\mathcal{D}$-modules. Define the functor

$$f_*^{\mathrm{dR}} : \mathcal{N}_f\mathrm{IC}(X) \to \mathrm{IC}(S)$$

by $f_*^{\mathrm{dR}}(E) = \mathcal{H}^{-d}(f_+E)$ where f_+ is the usual push-forward for regular holonomic complexes of $\mathscr{D}$-modules, d is the relative dimension of $f : X \to S$, and I am considering a regular integrable connection on X as a $\mathcal{D}_X$-module in the usual way.

Lemma 1.1.9. *The functor f_*^{dR} lands in the category of regular integrable connections, and is a right adjoint to f^*.*

Proof. The content of the first claim is in the coherence of direct images in de Rham cohomology, using the comparison result 1.4 of [24], and the fact that a regular holonomic $\mathcal{D}_X$-module is a vector bundle iff it is coherent as an $\mathcal{O}_X$-module.

To see this coherence, first use adjointness of f_+ and f^+, together with the facts that $f^+\mathcal{O}_S = \mathcal{O}_X[-d]$ and $f_+\mathcal{O}_X$ is concentrated in degrees $\geq -d$, to get canonical adjunction morphism $f_*^{\mathrm{dR}}(\mathcal{O}_X) \to \mathcal{O}_S$ of regular holonomic $\mathcal{D}_X$-modules. This is an isomorphism by base changing to $\mathbb{C}$ and comparing with the usual topological push-forward of the constant sheaf $\mathbb{C}$. Hence $f_*^{\mathrm{dR}}\mathcal{O}_X$ is coherent, and via the projection formula, so is $f_*^{\mathrm{dR}}(f^*F)$ for any $F \in \mathrm{IC}(S)$. Hence using exact sequences in cohomology and induction on unipotence degree, $f_*^{\mathrm{dR}}E$ is coherent whenever E is relatively unipotent.

To prove to the second claim, just use that f^+ is adjoint to f_+, $f^+ = f^*[-d]$ on the subcategory of regular integrable connections, and f_+E is concentrated in degrees $\geq -d$ whenever E is a regular integrable connection. □

Remark 1.1.10. Although the Proposition is stated in [24] for $k = \mathbb{C}$, the same proof works for any algebraically closed field of characteristic zero.

Thus there is a canonical morphism $\varepsilon_E : f^*f_*^{\mathrm{dR}}E \to E$ which is the counit of the adjunction between f^* and f_*^{dR}.

Example 1.1.11. Suppose that $S = \mathrm{Spec}\,(k)$. Then

$$f_*^{\mathrm{dR}}E = H^0_{\mathrm{dR}}\,(X, E) = \mathrm{Hom}_{\mathcal{N}\mathrm{IC}(X)}\,(\mathcal{O}_X, E) \tag{1.22}$$

and the adjunction becomes the identification

$$\mathrm{Hom}_{\mathcal{N}\mathrm{IC}(X)}\,(V \otimes_k \mathcal{O}_X, E) = \mathrm{Hom}_{\mathrm{Vec}_k}\left(V, \mathrm{Hom}_{\mathcal{N}\mathrm{IC}(X)}\,(\mathcal{O}_X, E)\right). \tag{1.23}$$

Since f_*^{dR} takes objects in $\mathcal{N}_f\mathrm{IC}\,(X)$ to objects in $\mathrm{IC}\,(S)$, it commutes with base change and there is an isomorphism of functors

$$H^0_{\mathrm{dR}}\,(X_s, -) \circ i_s^* \cong s^* \circ f_*^{\mathrm{dR}} : \mathcal{N}_f\mathrm{IC}\,(X) \to \mathrm{Vec}_k \tag{1.24}$$

(see for example [32], Chapter III, Theorem 5.2).

Proposition 1.1.12. *Suppose that i_s^*E is trivial. Then the counit*

$$\varepsilon_E : f^*f_*^{\mathrm{dR}}E \to E \tag{1.25}$$

is an isomorphism.

Proof. Pulling back ε_E by i_s^*, and using base change, there is a morphism

$$\mathcal{O}_{X_s} \otimes_k H^0_{\mathrm{dR}}\,(X_s, i_s^*E) \to i_s^*E \tag{1.26}$$

which by the explicit description of 1.1.11 is seen to be an isomorphism (as i_s^*E is trivial). Hence by rigidity, ε_E must be an isomorphism. □

Proposition 1.1.13. *Let $E \in \mathcal{N}_f\mathrm{IC}(X)$, and let $F_0 \subset i_s^*(E)$ denote the largest trivial sub-object. Then there exists $E_0 \subset E$ such that $F_0 = i_s^*(E_0)$.*

Proof. Let $F = i_s^*(E)$. Since $H^0_{\mathrm{dR}}(X_s, F) = \mathrm{Hom}_{\mathrm{IC}(X_s)}(\mathcal{O}_{X_s}, F)$, it follows that $F_0 \cong \mathcal{O}_{X_s} \otimes_K H^0_{\mathrm{dR}}(X_s, F)$. Set $E_0 = f^* f_*^{\mathrm{dR}}(E)$, then the base change results proved above imply that $i_s^*(E_0) \cong F_0$, and that the natural map $E_0 \to E$ restricts to the inclusion $F_0 \to F$ on the fibre X_s. □

Corollary 1.1.14. *The map $\pi_1^{\mathrm{dR}}(X_s, p_s) \to \pi_1^{\mathrm{dR}}(X/S, p)_s$ is a surjection.*

I now turn to the proof of injectivity of the comparison map, borrowing heavily from ideas used in Section 2.1 of [28]. Define objects U_n of $\mathcal{N}\mathrm{IC}(X_s)$, the category of unipotent integrable connections on X_s inductively as follows. U_1 will just be $\mathcal{O}_{X_s}$, and U_{n+1} will be the extension of U_n by $\mathcal{O}_{X_s} \otimes_k H^1_{\mathrm{dR}}(X_s, U_n^\vee)^\vee$ corresponding to the identity under the isomorphisms

$$\begin{aligned}\mathrm{Ext}_{\mathrm{IC}(X_s)}\left(U_n, \mathcal{O}_{X_s} \otimes_k H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)^\vee\right) &\cong H^1_{\mathrm{dR}}\left(X_s, U_n^\vee \otimes_k H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)^\vee\right) \\ &\cong H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right) \otimes_k H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)^\vee \\ &\cong \mathrm{End}_k\left(H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)\right). \end{aligned} \tag{1.27}$$

Looking at the long exact sequence in de Rham cohomology associated to the short exact sequence $0 \to U_n^\vee \to U_{n+1}^\vee \to H^1_{\mathrm{dR}}(X_s, U_n^\vee) \otimes_k \mathcal{O}_{X_s} \to 0$ gives

$$\begin{aligned}0 \to H^0_{\mathrm{dR}}\left(X_s, U_n^\vee\right) \to H^0_{\mathrm{dR}}\left(X_s, U_{n+1}^\vee\right) &\to H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right) \\ &\xrightarrow{\delta} H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right) \to H^1_{\mathrm{dR}}\left(X_s, U_{n+1}^\vee\right). \end{aligned} \tag{1.28}$$

Lemma 1.1.15. *The connecting homomorphism δ is the identity.*

Proof. By dualising, the extension

$$0 \to U_n^\vee \to U_{n+1}^\vee \to \mathcal{O}_{X_s} \otimes_k H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right) \to 0 \tag{1.29}$$

corresponds to the identity under the isomorphism

$$\mathrm{Ext}_{\mathrm{IC}(X_s)}\left(\mathcal{O}_{X_s} \otimes_k H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right), U_n^\vee\right) \cong \mathrm{End}_k\left(H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)\right) \tag{1.30}$$

Now the lemma follows from the fact that for an extension $0 \to E \to F \to \mathcal{O}_{X_s} \otimes_k V \to 0$

of a trivial bundle by E, the class of the extension under the isomorphism

$$\begin{aligned}\operatorname{Ext}_{\mathrm{IC}(X_s)}\left(\mathcal{O}_{X_s}\otimes_k V, E\right) &\cong V^\vee \otimes H^1_{\mathrm{dR}}\left(X_s, E\right) \\ &\cong \operatorname{Hom}_k\left(V, H^1_{\mathrm{dR}}\left(X_s, E\right)\right)\end{aligned} \tag{1.31}$$

is just the connecting homomorphism for the long exact sequence

$$0 \to H^0_{\mathrm{dR}}\left(X_s, E\right) \to H^0_{\mathrm{dR}}\left(X_s, F\right) \to V \to H^1_{\mathrm{dR}}\left(X_s, E\right). \tag{1.32}$$

□

In particular, any extension of U_n by a trivial bundle $V \otimes_k \mathcal{O}_{X_s}$ is split after pulling back to U_{n+1}, and $H^0_{\mathrm{dR}}\left(X_s, U^\vee_{n+1}\right) \cong H^0_{\mathrm{dR}}\left(X_s, U^\vee_n\right)$. It then follows by induction that $H^0_{\mathrm{dR}}\left(X_s, U^\vee_n\right) \cong H^0_{\mathrm{dR}}\left(X_s, \mathcal{O}_{X_s}\right) \cong k$ for all n.

Definition 1.1.16. Define the unipotent class of an object $E \in \mathcal{N}\mathrm{IC}(X_s)$ inductively as follows. If E is trivial, then say E has unipotent class 1. If there exists an extension

$$0 \to V \otimes_k \mathcal{O}_{X_s} \to E \to E' \to 0 \tag{1.33}$$

with E' of unipotent class $\leq m-1$, then say that E has unipotent class $\leq m$.

Now let $x = p(s)$, $u_1 = 1 \in \left(U_1\right)_x \cong \mathcal{O}_{X_s,x} = k$, and choose a compatible system of elements $u_n \in \left(U_n\right)_x$ mapping to u_1.

Proposition 1.1.17. *Let $F \in \mathcal{N}\mathrm{IC}(X_s)$ be an object of unipotent class $\leq m$. Then for all $n \geq m$ and any $f \in F_x$ there exists a morphism $\alpha : U_n \to F$ such that $\alpha_x\left(u_n\right) = f$.*

Proof. I copy the proof of Proposition 2.1.6 of [28] and use induction on m. The case $m = 1$ is straightforward. For the inductive step, let F be of unipotent class m, and choose an exact sequence

$$0 \to E \overset{\psi}{\to} F \overset{\phi}{\to} G \to 0 \tag{1.34}$$

with E trivial and G of unipotent class $< m$. By induction there exists a morphism $\beta : U_{n-1} \to G$ such that $\phi_x\left(f\right) = \beta_x\left(u_{n-1}\right)$. Pulling back the extension (1.34) first by the morphism β and then by the natural surjection $U_n \to U_{n-1}$ gives an extension of U_n by

E, which must split, as observed above.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & F'' & \underset{\curvearrowleft}{\longrightarrow} & U_n & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & F' & \longrightarrow & U_{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \xrightarrow{\psi} & F & \xrightarrow{\phi} & G & \longrightarrow & 0
\end{array} \tag{1.35}$$

Let $\gamma : U_n \to F$ denote the induced morphism, then $\phi_x(\gamma_x(u_n) - f) = 0$. Hence there exists some $e \in E_x$ such that $\psi_x(e) = \gamma_x(u_n) - f$. Again by induction there exists γ' : $U_n \to E$ with $\gamma'_x(u_n) = e$. Finally let $\alpha = \gamma - \psi \circ \gamma'$, it is easily seen that $\alpha_x(u_n) = f$. □

Corollary 1.1.18. *Every E in $\mathcal{N}\mathrm{IC}(X_s)$ is a quotient of $U_m^{\oplus N}$ for some $m, N \in \mathbb{N}$.*

Proof. Suppose that E is of unipotent class $\leq m$. Let $e_1, \ldots, e_N$ be a basis for E_x. Then there is a morphism $\alpha : U_m^{\oplus N} \to E$ with every e_i in the image of the induced map on fibres. Thus α_x is surjective, and hence so is α. □

I can now inductively define relatively nilpotent integrable connections W_n. on X which restrict to the U_n on fibres. Define higher direct images in de Rham cohomology by $\mathbf{R}^i_{\mathrm{dR}} f_*(E) = \mathcal{H}^{i-d}(f_+ E)$, and begin the induction with $W_1 = \mathcal{O}_X$. As part of the induction I will assume that $\mathbf{R}^0_{\mathrm{dR}} f_*(W_n^\vee) \cong \mathbf{R}^0_{\mathrm{dR}} f_*(\mathcal{O}_X) = \mathcal{O}_S$, and that $\mathbf{R}^1_{\mathrm{dR}} f_*(W^\vee)$ and $\mathbf{R}^1_{\mathrm{dR}} f_*(W)$ are both coherent, i.e. regular integrable connections. I will define W_{n+1} to be an extension of W_n by the sheaf $f^* \mathbf{R}^1_{\mathrm{dR}} f_*(W_n^\vee)^\vee$, so consider the extension group

$$\mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^*\mathbf{R}^1_{\mathrm{dR}} f_*\left(W_n^\vee\right)^\vee\right) \cong H^1_{\mathrm{dR}}\left(X, W_n^\vee \otimes_{\mathcal{O}_X} f^*\mathbf{R}^1_{\mathrm{dR}} f_*\left(W_n^\vee\right)^\vee\right). \tag{1.36}$$

The Leray spectral sequence, together with the induction hypothesis and the projection formula, gives the 5-term exact sequence

$$\begin{aligned}
0 &\to H^1_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_*\left(W_n^\vee\right)^\vee\right) \to \mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^*\mathbf{R}^1_{\mathrm{dR}} f_*\left(W_n^\vee\right)^\vee\right) \\
&\to \mathrm{End}_{\mathrm{IC}(S)}\left(\mathbf{R}^1_{\mathrm{dR}} f_*\left(W_n^\vee\right)\right) \to H^2_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_*\left(W_n^\vee\right)^\vee\right) \\
&\to H^2_{\mathrm{dR}}(X, W_n^\vee \otimes_{\mathcal{O}_X} (\mathbf{R}^1_{\mathrm{dR}} f_* W_n^\vee)^\vee).
\end{aligned} \tag{1.37}$$

Now, the projection $p^* W_n^\vee \to \mathcal{O}_S$ induces a map

$$H^i_{\mathrm{dR}}\left(S, p^* W_n^\vee \otimes_{\mathcal{O}_X} (\mathbf{R}^1_{\mathrm{dR}} f_* W_n^\vee)^\vee\right) \to H^i_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_*\left(W_n^\vee\right)^\vee\right) \tag{1.38}$$

such that the composite (dotted) arrow

$$\begin{array}{ccc} H^i_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right) & \longrightarrow & H^i_{\mathrm{dR}}\left(X, W_n^\vee \otimes_{\mathcal{O}_X} \left(\mathbf{R}^1_{\mathrm{dR}} f_* W_n^\vee\right)^\vee\right) \\ \downarrow & & \downarrow \\ H^i_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right) & \longleftarrow & H^i_{\mathrm{dR}}\left(S, p^* W_n^\vee \otimes_{\mathcal{O}_X} \left(\mathbf{R}^1_{\mathrm{dR}} f_* W_n^\vee\right)^\vee\right) \end{array} \tag{1.39}$$

is as isomorphism, since it can be identified with the map induced by the composite arrow $\mathcal{O}_S \cong f_* W_n^\vee \cong p^* f^* f_* W_n^\vee \to p^* W_n^\vee \to \mathcal{O}_S$. Hence both the maps

$$\begin{aligned} H^1_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right) &\to \mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right) \\ H^2_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right) &\to H^2_{\mathrm{dR}}(X, W_n^\vee \otimes_{\mathcal{O}_X} \left(\mathbf{R}^1_{\mathrm{dR}} f_* W_n^\vee\right)^\vee) \end{aligned} \tag{1.40}$$

appearing in the 5-term exact sequence split. So there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right) & \longrightarrow & \mathrm{Ext}_{\mathrm{IC}(X_s)}\left(U_n, \mathcal{O}_{X_s} \otimes_k H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)^\vee\right) \\ \downarrow & & \| \\ \mathrm{End}_{\mathrm{IC}(S)}\left(\mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)\right) & \longrightarrow & \mathrm{End}_k\left(H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)\right) \end{array} \tag{1.41}$$

where the horizontal arrows are just restrictions to fibres, and the left hand vertical arrow is surjective. The identity morphism in $\mathrm{End}_k\left(H^1_{\mathrm{dR}}\left(X_s, U_n^\vee\right)\right)$ clearly lifts to the identity in $\mathrm{End}_{\mathrm{IC}(S)}\left(\mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)\right)$, and there exists a unique element of the extension group $\mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right)$ lifting the identity in $\mathrm{End}_{\mathrm{IC}(S)}\left(\mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)\right)$, and which maps to zero under the above splitting

$$\mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right) \to H^1_{\mathrm{dR}}\left(S, \mathbf{R}^1_{\mathrm{dR}} f_* \left(W_n^\vee\right)^\vee\right). \tag{1.42}$$

Let W_{n+1} be the corresponding extension.

Proposition 1.1.19. *Every object of* $\mathcal{N}\mathrm{IC}(X_s)$ *is a quotient of* $i_s^* E$ *for some* $E \in \mathcal{N}_f \mathrm{IC}(X)$.

Proof. To finish the induction step, I must show that

$$\mathbf{R}^0_{\mathrm{dR}} f_* \left(W_{n+1}^\vee\right) \cong \mathbf{R}^0_{\mathrm{dR}} f_* \left(W_n^\vee\right), \tag{1.43}$$

that $\mathbf{R}^1_{\mathrm{dR}} f_*(W_{n+1}^\vee)$ and $\mathbf{R}^1_{\mathrm{dR}} f_*(W_{n+1})$ are coherent, and that there exists a morphism $p^* W_{n+1}^\vee \to \mathcal{O}_S$ as in the induction hypothesis. For the first claim, looking at the long exact sequence of relative de Rham cohomology

$$0 \to \mathbf{R}^0_{\mathrm{dR}} f_* \left(W_n^\vee\right) \to \mathbf{R}^0_{\mathrm{dR}} f_* \left(W_{n+1}^\vee\right) \to \dots \tag{1.44}$$

it suffices to note that the given map restricts to an isomorphism on the fibre over s, and is hence an isomorphism. The scone claim follows from using the long exact sequence in cohomology and the inductive hypothesis for $\mathbf{R}^1_{\mathrm{dR}} f_* (W_n)$ and $\mathbf{R}^1_{\mathrm{dR}} f_* (W_n^\vee)$. For the third, note that it follows from the construction of W_{n+1} that the exact sequence

$$0 \to p^* W_n^\vee \to p^* W_{n+1}^\vee \to (\mathbf{R}^1_{\mathrm{dR}} f_* W_n^\vee)^\vee \to 0 \tag{1.45}$$

splits when pushed out via the map $p^* W_n^\vee \to \mathcal{O}_S$. This splitting induces a map $p^* W_{n+1}^\vee \to \mathcal{O}_S$ such that the diagram

$$\begin{array}{ccc} p^* W_n^\vee & \longrightarrow & p^* W_{n+1}^\vee \\ & \searrow & \downarrow \\ & & \mathcal{O}_S \end{array} \tag{1.46}$$

commutes. Now the fact that the diagram

$$\begin{array}{ccccccccc} & & f_* W_{n+1}^\vee & \longrightarrow & p^* f^* f_* W_{n+1}^\vee & \longrightarrow & p^* W_{n+1}^\vee & & \\ & \nearrow & \uparrow & & \uparrow & & \uparrow & \searrow & \\ \mathcal{O}_S & \longrightarrow & f_* W_n^\vee & \longrightarrow & p^* f^* f_* W_n^\vee & \longrightarrow & p^* W_n^\vee & \longrightarrow & \mathcal{O}_S \end{array} \tag{1.47}$$

commutes implies that the composite along the top row is an isomorphism, finishing the proof. □

Corollary 1.1.20. *Let $f : X \to S$ be a 'good' morphism over an algebraically closed field of characteristic zero, and p a section of f. Then the natural 'base change' map $\pi_1^{\mathrm{dR}}(X_s, p_s) \to \pi_1^{\mathrm{dR}}(X/S, p)_s$ is an isomorphism.*

Remark 1.1.21. It is possible to define a relative fundamental group when k is not necessarily algebraically closed (but still of characteristic 0) using identical methods. One can then show that the corresponding 'base change' question can be deduced from what I have proved in the algebraically closed case. Since this argument is rather fiddly, and not necessary in the context of this chapter, I have omitted it.

1.2 Path torsors, non-abelian crystals and period maps

If $\mathcal{T}$ is a Tannakian category over an arbitrary field k, and ω_i are fibre functors on $\mathcal{T}$, $i = 1, 2$, with values in the category of quasi-coherent sheaves on some k-scheme S, then the functor of isomorphisms $\omega_1 \to \omega_2$ is representable by an affine S-scheme, which is a $(G(\mathcal{T}, \omega_1), G(\mathcal{T}, \omega_2))$-bitorsor. This allows me to define path torsors under the algebraic and de Rham fundamental groups. In this section, I show how to do this in the relative case.

1.2.1 Torsors in Tannakian categories

Let $\mathcal{C}$ be a Tannakian category over a field k. A Tannakian $\mathcal{C}$-category is a Tannakian category $\mathcal{D}$ together with an exact, k-linear tensor functor $t : \mathcal{C} \to \mathcal{D}$. Say $\mathcal{D}$ is neutral over $\mathcal{C}$ if there exists an exact, faithful k-linear tensor functor $\omega : \mathcal{D} \to \mathcal{C}$ such that $\omega \circ t \cong \mathrm{id}$. Such functors will be called fibre functors. If such a functor ω is fixed, then $\mathcal{D}$ is said to be neutralised. Thanks to §6.10 of [22], there is a homomorphism

$$t^* : \pi(\mathcal{D}) \to t(\pi(\mathcal{C})) \tag{1.48}$$

of affine group schemes over $\mathcal{D}$. Applying ω induces a homomorphism

$$\omega(t^*) : \omega(\pi(\mathcal{D})) \to \pi(\mathcal{C}) \tag{1.49}$$

of affine group schemes over $\mathcal{C}$, and I define $G(\mathcal{D}, \omega) := \ker \omega(t^*)$.

For an affine group scheme G over $\mathcal{C}$, let $\mathcal{O}_G$ be its Hopf algebra, a representation of G is then defined to be an $\mathcal{O}_G$-comodule. That is an object $V \in \mathcal{C}$ together with a map $\delta : V \to \mathcal{O}_G \otimes V$ satisfying the usual axioms.

Definition 1.2.1. A torsor under G is a non-empty affine scheme $\mathrm{Sp}(\mathcal{O}_P)$ over $\mathcal{C}$, together with a $\mathcal{O}_G$-comodule structure on $\mathcal{O}_P$, such that the induced map $\mathcal{O}_P \otimes \mathcal{O}_P \to \mathcal{O}_P \otimes \mathcal{O}_G$ is an isomorphism.

Example 1.2.2. Suppose that $\mathcal{C} = \mathrm{Rep}_k(H)$, for some affine group scheme H over k. Then an affine group scheme G over $\mathcal{C}$ 'is' just an affine group scheme G_0 over k together with an action of H. A representation of G 'is' then just an H-equivariant representation of G_0, or in other words, a representation of the semi-direct product $G_0 \rtimes H$.

Representations have another interpretation. Suppose that V is an $\mathcal{O}_G$-comodule, and let R be a $\mathcal{C}$-algebra. A point $g \in G(R)$ is then a morphism $\mathcal{O}_G \to R$ of $\mathcal{C}$-algebras, and hence for any such g there is a morphism

$$V \to V \otimes R \tag{1.50}$$

which extends linearly to a morphism

$$V \otimes R \to V \otimes R. \tag{1.51}$$

This is an isomorphism, with inverse given by the map induced by g^{-1}. Hence there is an R-linear action of $G(R)$ on $V \otimes R$, for all $\mathcal{C}$-algebras R. The same proof as in the absolute case (Proposition 2.2 of [38]) shows that a representation of G (defined in terms of comodules) is equivalent to an R-linear action of $G(R)$ on $V \otimes R$, for all R.

For G an affine group scheme over $\mathcal{C}$, let $\mathrm{Rep}_{\mathcal{C}}(G)$ denote its category of representations, this is a Tannakian category over k. There are canonical functors

$$\mathcal{C} \underset{\omega}{\overset{t}{\rightleftarrows}} \mathrm{Rep}_{\mathcal{C}}(G) \tag{1.52}$$

given by 'trivial representation' and 'forget the representation'. This makes $\mathrm{Rep}_{\mathcal{C}}(G)$ neutral over $\mathcal{C}$. There is a natural homomorphism $G \to \omega(\pi(\mathrm{Rep}_{\mathcal{C}}(G)))$ which comes from the fact that by definition, G acts on $\omega(V)$ for all $V \in \mathrm{Rep}_{\mathcal{C}}(G)$. Since this action is trivial on everything of the form $t(W)$, $W \in \mathcal{C}$, again by definition, this homomorphism factors to give a homomorphism

$$G \to G(\mathrm{Rep}_{\mathcal{C}}(G), \omega). \tag{1.53}$$

Conversely, if $\mathcal{D}$ is neutral over $\mathcal{C}$, with fibre functor ω, then the action of $\omega(\pi(\mathcal{D}))$ on $\omega(V)$, for all $V \in \mathcal{D}$, induces an action of $G(\mathcal{D}, \omega)$ on $\omega(V)$, and hence a functor

$$\mathcal{D} \to \mathrm{Rep}_{\mathcal{C}}(G(\mathcal{D}, \omega)). \tag{1.54}$$

Proposition 1.2.3. *In the above situation, the homomorphism*

$$G \to G(\mathrm{Rep}_{\mathcal{C}}(G), \omega) \tag{1.55}$$

is an isomorphism, and the functor

$$\mathcal{D} \to \mathrm{Rep}_{\mathcal{C}}(G(\mathcal{D}, \omega)) \tag{1.56}$$

is an equivalence of categories.

Proof. Suppose first that $\mathcal{C} \cong \mathrm{Rep}_k(H)$ is neutral. In the first case, G can be identified with an affine group scheme G_0 over k together with an action of H, and the category $\mathrm{Rep}_{\mathcal{C}}(G)$ with the category of representations of the semi-direct product $G_0 \rtimes H$. The functor $\omega : \mathrm{Rep}_{\mathcal{C}}(G) \to \mathcal{C}$ can be identified with the forgetful functor from $G_0 \rtimes H$-representations to H-representations, and the morphism

$$\omega(\pi(\mathrm{Rep}_{\mathcal{C}}(G))) \to \pi(\mathcal{C}) \tag{1.57}$$

with the natural map

$$G_0 \rtimes H \to H \tag{1.58}$$

of affine group schemes with H-action. Thus the kernel of this map is identified with G_0 together with its given H-action. In other words,

$$G \to G(\mathrm{Rep}_{\mathcal{C}}(G), \omega) \tag{1.59}$$

is an isomorphism.

In the second case, $\mathcal{D}$ is also neutral, and corresponds to representations of some affine group scheme G. The functors t, ω give a surjection $G \to H$ and a splitting $H \to G$ which induces an action of H on $G_0 := \ker(G \to H)$ such that $G \cong G_0 \rtimes H$. Then $G(D, \omega)$ is identified with G_0 together with it's H-action, and $\mathcal{D} \to \mathrm{Rep}_{\mathcal{C}}(G(\mathcal{D}, \omega))$ with the natural functor from $G = G_0 \rtimes H$-representations to H-equivariant G_0-representations. It is thus an equivalence.

If $\mathcal{C}$ is not neutral, then choose a fibre functor with values in some k-scheme S, apply Théorème 1.12 of [23] and replace the affine group scheme H by a certain groupoid acting on a S (for more details see §3.3). The argument is then formally identical. □

Remark 1.2.4. The definition of the fundamental group $\pi_1^{\mathrm{dR}}(X/S, p)$ is then just the affine group scheme $G(\mathcal{N}_f \mathrm{IC}(X), p^*)$ over $\mathrm{IC}(S)$.

In order to define torsors of isomorphisms in the relative setting, first recall Deligne's construction in the absolute case, which uses the notion of a coend. So take categories $\mathcal{X}$ and $\mathcal{S}$, and a functor $F : \mathcal{X} \times \mathcal{X}^{\mathrm{op}} \to \mathcal{S}$. The coend of F is the universal pair (ζ, s) where s is an object of $\mathcal{S}$ and $\zeta : F \to s$ is a bi-natural transformation. Here s is the constant functor at $s \in \mathrm{Ob}(\mathcal{S})$, and bi-natural means that it is natural in both variables. If such an object exists, denote it by

$$\int^{\mathcal{X}} F(x, x). \tag{1.60}$$

If $\mathcal{S}$ is cocomplete then the coend always exists and is given concretely by the formula (see Chapter IX, Section 6 of [37])

$$\int^{\mathcal{X}} F(x, x) = \mathrm{colim}\left(\coprod_{f : x \to y \in \mathrm{Mor}(\mathcal{X})} F(x, y) \rightrightarrows \coprod_{x \in \mathrm{Ob}(\mathcal{X})} F(x, x) \right). \tag{1.61}$$

Suppose that $\mathcal{C}$ is a Tannakian category, and let $\omega_1, \omega_2 : \mathcal{C} \to \mathrm{QCoh}(S)$ be two fibre functors on $\mathcal{C}$. In [23], Deligne defines

$$L_S(\omega_1, \omega_2) = \int^{\mathcal{C}} \omega_1(V) \otimes \omega_2(V)^{\vee} \tag{1.62}$$

to be the coend of the bifunctor

$$\omega_1 \otimes \omega_2^\vee : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{QCoh}\,(S)\,, \tag{1.63}$$

and in §6 of *loc. cit.*, uses the tensor structure of $\mathcal{C}$ to define a multiplication on $L_S\,(\omega_1, \omega_2)$ which makes it into a quasi-coherent $\mathcal{O}_S$-algebra. He then proves that $\mathrm{Spec}\,(L_S\,(\omega_1, \omega_2))$ represents the functor of isomorphisms from ω_1 to ω_2.

Now let $\mathcal{C}$ be a Tannakian category, let $\mathcal{D}$ be neutral over $\mathcal{C}$, and suppose that $\omega_1, \omega_2 : \mathcal{D} \to \mathcal{C}$ are two fibre functors from $\mathcal{D}$ to $\mathcal{C}$. Define the coend

$$L_{\mathcal{C}}\,(\omega_1, \omega_2) := \int^{\mathcal{D}} \omega_1\,(V) \otimes \omega_2\,(V)^\vee \in \mathrm{Ind}\,(\mathcal{C})\,. \tag{1.64}$$

If $\eta : \mathcal{C} \to \mathrm{QCoh}\,(S)$ is a fibre functor, then η commutes with colimits, and hence $\eta\,(L_{\mathcal{C}}\,(\omega_1, \omega_2)) = L_S\,(\eta\omega_1, \eta\omega_2)$: this is a quasi-coherent $\mathcal{O}_S$-algebra, functorial in η. Since algebraic structures in Tannakian categories, such as commutative algebras, Hopf algebras, and so on, can be constructed 'functorially in fibre functors', (see for example §5.11 of [22]), it follows that there is a unique way of defining a $\mathcal{C}$-algebra structure on $L_{\mathcal{C}}\,(\omega_1, \omega_2)$ lifting the $\mathcal{O}_S$-algebra structure on each $\eta\,(L_{\mathcal{C}}\,(\omega_1, \omega_2))$. Moreover, since $\eta\,(\mathrm{Sp}\,(L_{\mathcal{C}}\,(\omega_1, \omega_2)))$ is a $(\eta\omega_1\,(\pi\,(\mathcal{D}))\,, \eta\omega_2\,(\pi\,(\mathcal{D})))$-bitorsor, functorially in η, the affine scheme

$$P_{\mathcal{C}}\,(\omega_1, \omega_2) := \mathrm{Sp}\,(L_{\mathcal{C}}\,(\omega_1, \omega_2)) \tag{1.65}$$

is a $(\omega_1\,(\pi\,(\mathcal{D}))\,, \omega_2\,(\pi\,(\mathcal{D})))$-bitorsor over $\mathcal{C}$.

What I actually want, however, is a $(G_{\mathcal{C}}\,(\mathcal{D}, \omega_2)\,, G_{\mathcal{C}}\,(\mathcal{D}, \omega_2))$-bitorsor. This is obtained as follows. Suppose that $V \in \mathcal{D}$, then by the definition of $L_{\mathcal{C}}\,(\omega_1, \omega_2)$ there is a morphism

$$\omega_1\,(V) \otimes \omega_2\,(V)^\vee \to L_{\mathcal{C}}\,(\omega_1, \omega_2) \tag{1.66}$$

which corresponds to a morphism

$$\omega_1\,(V) \to \omega_2\,(V) \otimes L_{\mathcal{C}}\,(\omega_1, \omega_2)\,. \tag{1.67}$$

Thus a morphism $L_{\mathcal{C}}\,(\omega_1, \omega_2) \to R$ for some $\mathcal{C}$-algebra R induces an R-linear morphism

$$\omega_1\,(V) \otimes R \to \omega_2\,(V) \otimes R \tag{1.68}$$

which is in fact an isomorphism, since it is so after applying any fibre functor.

Definition 1.2.5. Define $P_{\mathrm{triv}}\,(\omega_1, \omega_2)$ to be the sub-functor of $P_{\mathcal{C}}\,(\omega_1, \omega_2)$ which takes R to the set of all morphisms $L_{\mathcal{C}}\,(\omega_1, \omega_2) \to R$ such that for every V in the essential image of $t : \mathcal{C} \to \mathcal{D}$, the induced automorphism of $R \otimes \omega_1(V) = R \otimes \omega_2(V)$ is the identity.

Proposition 1.2.6. *The functor $P_{\text{triv}}(\omega_1, \omega_2)$ is representable by an affine scheme over $\mathcal{C}$, and is a $(G_{\mathcal{C}}(\mathcal{D}, \omega_1), G_{\mathcal{C}}(\mathcal{D}, \omega_2))$-bitorsor in the category of affine schemes over $\mathcal{C}$.*

Proof. First note that if $V \in \mathrm{Ob}(\mathcal{D})$, then $\omega_i(\pi(\mathcal{D}))$ acts on $\omega_i(V)$, and $G(\mathcal{D}, \omega_i)$ is the largest subgroup of $\omega_i(\pi(\mathcal{D}))$ whose action on $\omega_i(V)$ is trivial for all V in the essential image of t.

Now, if $p \in P_{\text{triv}}(\omega_1, \omega_2)(R)$ and $g \in G_{\mathcal{C}}(\mathcal{D}, \omega_1)(R)$ then $gp \in P_{\mathcal{C}}(\omega_1, \omega_2)(R)$ acts trivially on everything of the form $t(W)$, and hence lies in $P_{\text{triv}}(\omega_1, \omega_2)(R)$. Hence $G(\mathcal{D}, \omega_1)$ acts on $P_{\text{triv}}(\omega_1, \omega_2)$. For $p, p' \in P_{\text{triv}}(\omega_1, \omega_2)(R)$, $p^{-1}p'$ is an automorphism of $\omega_1(V) \otimes R$ which is trivial for all V in the essential image of t. Hence it must be an element of $G(\mathcal{D}, \omega_1)(R) \subset \omega(\pi_1(\mathcal{D}))(R)$. The same arguments work for $G_{\mathcal{C}}(\mathcal{D}, \omega_2)$.

Thus $P_{\text{triv}}(\omega_1, \omega_2)$ is a bi-pseudo-torsor, and to complete the proof, I must show that $P_{\text{triv}}(\omega_1, \omega_2)$ is represented by a non-empty affine scheme over $\mathcal{C}$. Now, consider the ind-object of $\mathcal{C}$

$$L_{\mathcal{C}}(\mathrm{id}, \mathrm{id}) = \operatorname{colim} \left(\coprod_{f:V \to W \in \mathrm{Mor}(\mathcal{C})} V \otimes W^{\vee} \rightrightarrows \coprod_{V \in \mathrm{Ob}(\mathcal{C})} V \otimes V^{\vee} \right). \tag{1.69}$$

For every fibre functor $\eta : \mathcal{C} \to \mathrm{QCoh}(S)$, there is a natural Hopf algebra structure on $\eta(L_{\mathcal{C}}(\mathrm{id}, \mathrm{id}) = L_S(\eta, \eta)$, and the Spec of this Hopf algebra is the affine group scheme of tensor automorphisms of η. Thus as before, $L_{\mathcal{C}}(\mathrm{id}, \mathrm{id})$ has a Hopf algebra structure, and it's formal Spec satisfies the defining property of the fundamental groupoid $\pi(\mathcal{C})$ of $\mathcal{C}$. Hence one can construct a morphism of affine $\mathcal{C}$-schemes

$$P_{\mathcal{C}}(\omega_1, \omega_2) \to \pi(\mathcal{C}) \tag{1.70}$$

as the formal Spec of the obvious morphism $L_{\mathcal{C}}(\mathrm{id}, \mathrm{id}) \to L_{\mathcal{C}}(\omega_1, \omega_2)$. Then $P_{\text{triv}}(\omega_1, \omega_2)$ is the fibre of $P_{\mathcal{C}}(\omega_1, \omega_2) \to \pi(\mathcal{C})$ over the identity section $\mathrm{Sp}(1) \to \pi(\mathcal{C})$. Hence it is the formal Spec of the algebra $L_{\text{triv}}(\omega_1, \omega_2)$ defined by the push-out diagram

$$\begin{array}{ccc} L_{\mathcal{C}}(\mathrm{id}, \mathrm{id}) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ L_{\mathcal{C}}(\omega_1, \omega_2) & \longrightarrow & L_{\text{triv}}(\omega_1, \omega_2) \end{array} \tag{1.71}$$

and is thus representable by an affine $\mathcal{C}$-scheme.

To prove that $P_{\text{triv}}(\omega_1, \omega_2) \neq \emptyset$, it suffices to show that $\eta(P_{\text{triv}}(\omega_1, \omega_2)) \neq \emptyset$ for any fibre functor $\eta : \mathcal{C} \to \mathrm{QCoh}(S)$. For any $f : T \to S$, $\eta(P_{\text{triv}}(\omega_1, \omega_2))(T)$ is the subset of

$\mathrm{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2)$ which maps to the identity under the natural map

$$r : \mathrm{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2) \to \mathrm{Isom}^{\otimes}(f^* \circ \eta\omega_1 t, f^* \circ \eta\omega_2 t) = \mathrm{Aut}^{\otimes}(f^* \circ \eta)\,. \tag{1.72}$$

Since $\eta\omega_1$ and $\eta\omega_2$ are fibre functors on $\mathcal{D}$, according to 1.13 of [23] the affine scheme $\underline{\mathrm{Isom}}^{\otimes}(\eta\omega_1, \eta\omega_2)$ of tensor isomorphisms from $\eta\omega_1$ to $\eta\omega_2$ is faithfully flat over S, and in particular is non-empty. Hence there is certainly some S-scheme $f : T \to S$ such that

$$\mathrm{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2) \tag{1.73}$$

is non-empty. Pick such a T, and pick some $p \in \mathrm{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2)$. Since the morphism $\omega_1(\pi(\mathcal{D})) \to \pi(\mathcal{C})$ admits a section, the induced homomorphism

$$\mathrm{Aut}^{\otimes}(f^* \circ \eta\omega_1) \to \mathrm{Aut}^{\otimes}(f^* \circ \eta) \tag{1.74}$$

is surjective, and there exists some $g \in \mathrm{Aut}^{\otimes}(f^* \circ \eta\omega_1)$ mapping to $r(p) \in \mathrm{Aut}^{\otimes}(f^* \circ \eta)$. Then $p' := g^{-1}p$ is an element of the set $\mathrm{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2)$ and $r(p') = \mathrm{id}$, thus $\eta(P_{\mathrm{triv}}(\omega_1, \omega_2))(T) \neq \emptyset$. □

Remark 1.2.7. This can be rephrased as follows. Consider the functors of $\mathcal{C}$ algebras

$$\begin{aligned} \underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2) : \mathcal{C}-\mathrm{alg} &\to (\mathrm{Set}) \\ R &\mapsto \mathrm{Isom}^{\otimes}(\omega_1(-) \otimes R, \omega_2(-) \otimes R)\,; \qquad (1.75) \\ \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}) : \mathcal{C}-\mathrm{alg} &\to (\mathrm{Set}) \\ R &\mapsto \mathrm{Aut}^{\otimes}((-) \otimes R)\,; \qquad (1.76) \end{aligned}$$

as well as the sub-functor $\underline{\mathrm{Isom}}^{\otimes}_{\mathcal{C}}(\omega_1, \omega_2)$, the 'functor of $\mathcal{C}$-isomorphisms $\omega_1 \to \omega_2$', defined to be the fibre over the identity of the natural morphism

$$\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2) \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id})\,. \tag{1.77}$$

Then the functor $\underline{\mathrm{Isom}}^{\otimes}_{\mathcal{C}}(\omega_1, \omega_2)$ is representable by the affine scheme $P_{\mathrm{triv}}(\omega_1, \omega_2)$ over $\mathcal{C}$, which is a $(G_{\mathcal{C}}(\mathcal{D}, \omega_1), G_{\mathcal{C}}(\mathcal{D}, \omega_2))$ bitorsor.

1.2.2 Path torsors under relative fundamental groups

Let k be an algebraically closed field of characteristic zero, S a connected, affine curve over k and $f : X \to S$ a 'good' morphism. Let p, x be sections of f. Applying the above methods gives an affine scheme over $\mathrm{IC}(S)$, the torsor of paths from x to p, which can be considered as an affine scheme $P(x) = \pi_1^{\mathrm{dR}}(X/S, x, p)$ over S, together with an integrable connection on $\mathcal{O}_{P(x)}$ (as a quasi-coherent $\mathcal{O}_S$ algebra). This is naturally a left torsor under $\pi_1^{\mathrm{dR}}(X/S,x)$ and a right torsor under $\pi_1^{\mathrm{dR}}(X/S,p) =: G$. Moreover, the action map $P(x) \times G \to P(x)$ is compatible with the connections, in the sense that the associated comodule structure

$$\mathcal{O}_{P(x)} \to \mathcal{O}_{P(x)} \otimes_{\mathcal{O}_S} \mathcal{O}_G \tag{1.78}$$

is horizontal, the RHS being given the tensor product connection. If G_n is the quotient of G by the nth term in its lower central series, denote the push-out torsor $P(x) \times^G G_n$ by $P(x)_n$. As before, the action map $P(x)_n \times G_n \to P(x)_n$ is compatible with the connections.

Definition 1.2.8. A ∇-torsor under G_n is a G_n-torsor P over S in the usual sense, together with a regular integrable connection on $\mathcal{O}_P$, such that the action map

$$\mathcal{O}_P \to \mathcal{O}_P \otimes \mathcal{O}_{G_n} \tag{1.79}$$

is horizontal. The set of isomorphism classes of ∇-torsors is denoted $H^1_\nabla(S, G_n)$.

Thus there are 'period maps'

$$X(S) \to H^1_\nabla(S, G_n) \tag{1.80}$$

which takes $x \in X(S)$ to the path torsor $P(x)_n$.

Remark 1.2.9.

1. This is not a good period map to study. For instance, if $k = \mathbb{C}$, then the relative fundamental group is not just an affine group scheme with connection. There are reasons to expect that one can put a 'non-abelian' variation of Hodge structure on this fundamental group. Similar considerations will apply to the path torsors, and the period maps should take these variations of Hodge structures into account.

2. Using the pro-nilpotent Lie algebra of $\pi_1^{\mathrm{dR}}(X/S,p)$ and the Campell-Hausdorff law, $\pi_1^{\mathrm{dR}}(X/S,p)$ can be viewed as a non-abelian sheaf of groups on the infinitesimal site of S/k. This interpretation can be used to give an alternative definition of the cohomology set $H^1_\nabla(S, G_n)$.

3. A natural question to ask is whether or not, as in the situation studied by Kim, the targets for the period maps have the structure of algebraic varieties. Since I am more interested in the positive characteristic case, I will not pursue this question here.

1.3 Crystalline fundamental groups of smooth families in char p

My goal in this section is to define the fundamental group of a smooth family $f : X \to S$ of varieties over a finite field. Many of the arguments are essentially the same as those given in Section 1.1.

I will assume that the reader is familiar with the theory of rigid cohomology and overconvergent (F-)isocrystals, a good reference is [3]. Assume that k is a finite field, of order $q = p^a$ and characteristic $p > 0$. Frobenius will always refer to linear Frobenius. If U/K is a variety, the category of overconvergent (F-)isocrystals on U/K is denoted (F-)$\mathrm{Isoc}^\dagger(U/K)$. These are Tannakian categories over K.

Define $\mathcal{N}\mathrm{Isoc}^\dagger(U/K)$ to be the full subcategory of $\mathrm{Isoc}^\dagger(U/K)$ on objects admitting a filtration whose graded pieces are constant. Chiarellotto and Le Stum in [18] define the rigid fundamental group $\pi_1^{\mathrm{rig}}(U,x)$ of U at a k-rational point x to be the Tannaka dual of $\mathcal{N}\mathrm{Isoc}^\dagger(U/K)$ with respect to the fibre functor x^*. This is a pro-unipotent group scheme over K.

Now suppose that $g : X \to S$ is a 'good', proper morphism over k, and let $p : S \to X$ be a section.

Definition 1.3.1. Say that $E \in F\text{-}\mathrm{Isoc}^\dagger(X/K)$ is relatively unipotent if there is a filtration of E, whose graded pieces are all in the essential image of $g^* : F\text{-}\mathrm{Isoc}^\dagger(S/K) \to F\text{-}\mathrm{Isoc}^\dagger(X/K)$. The full subcategory of relatively unipotent overconvergent F-isocrystals is denoted $\mathcal{N}_g F\text{-}\mathrm{Isoc}^\dagger(X/K)$.

The pair of functors

$$\mathcal{N}_g F\text{-}\mathrm{Isoc}^\dagger(X/K) \underset{g^*}{\overset{p^*}{\rightleftarrows}} F\text{-}\mathrm{Isoc}^\dagger(S/K) \tag{1.81}$$

makes $\mathcal{N}_g F\text{-}\mathrm{Isoc}^\dagger(X/K)$ neutral over $F\text{-}\mathrm{Isoc}^\dagger(S/K)$ in the sense of §2.1. This gives rise to an affine group scheme $G(\mathcal{N}_g F\text{-}\mathrm{Isoc}^\dagger(X/K), p^*)$ in $F\text{-}\mathrm{Isoc}^\dagger(S/K)$.

Definition 1.3.2. Define the relative fundamental group to be the affine group scheme $G(\mathcal{N}_g F\text{-}\mathrm{Isoc}^\dagger(X/K), p^*)$ in $F\text{-}\mathrm{Isoc}^\dagger(S/K)$.

For $s \in S$ a closed point, let $i_s : X_s \to X$ denote the inclusion of the fibre over s and let $g_s : X_s \to \mathrm{Spec}(k(s))$ denote the structure morphism. Let $K(s)$ denote the unique unramified extension of K with residue field $k(s)$. Let $\mathcal{V}(s)$ denote the ring of integers of $K(s)$. In keeping with notation of previous sections, let $\pi_1^{\mathrm{rig}}(X/S,p)_s$ denote the affine group scheme $s^*(\pi_1^{\mathrm{rig}}(X/S,p))$ over $K(s)$. The pull-back functor

$$i_s^* : \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \to \mathcal{N}\mathrm{Isoc}^\dagger(X_s/K(s)) \tag{1.82}$$

induces a homomorphism

$$\phi : \pi_1^{\mathrm{rig}}(X_s, p_s) \to \pi_1^{\mathrm{rig}}(X/S,p)_s \tag{1.83}$$

of affine group schemes over K. I would like to show again that this is an isomorphism. The question is whether or not the sequence of affine group schemes corresponding to the sequence of neutral Tannakian categories

$$\mathcal{N}\mathrm{Isoc}(X_s/K(s)) \leftarrow \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s) \leftarrow F\text{-Isoc}^\dagger(S/K) \otimes_K K(s) \tag{1.84}$$

is exact. Thus, as before, this boils down to the following three questions.

1. If $E \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$ is such that $i_s^* E$ is constant, is E of the form g^*F for some $F \in F\text{-Isoc}^\dagger(S/K) \otimes_K K(s)$?

2. If $E \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$, and $F_0 \subset i_s^* E$ denotes the largest constant sub-object, then does there exist $E_0 \subset E$ such that $F_0 = i_s^* E_0$?

3. Given $E \in \mathrm{Isoc}^\dagger(X_s/K(s))$, does there exist $F \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$ such that E is a quotient of $i_s^* F$?

Remark 1.3.3. Actually, these criteria only apply if the kernel of the homomorphism of group schemes corresponding to

$$\mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s) \leftarrow F\text{-Isoc}^\dagger(S/K) \otimes_K K(s) \tag{1.85}$$

is pro-unipotent, or using Lemma 1.3, Part I of [54], that every object E of the category $\mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$ has a non-zero subobject of the form f^*F for some $F \in F\text{-Isoc}^\dagger(S/K) \otimes_K K(s)$. To show this, let E_0 denote the largest relatively constant sub-object of E, considered in the category $\mathcal{N}_g F\text{-Isoc}^\dagger(X/K)$. Then functoriality of E_0 implies that a $K(s)$ module structure $K(s) \to \mathrm{End}(E)$ will induce one on E_0. Thus it suffices to show that an $K(s)$-module structure on f^*F induces one on F. But now just use the section p to get a homomorphism of rings $\mathrm{End}(f^*F) \to \mathrm{End}(F)$.

1.3.1 Base change

Hypotheses and notations will be as in the previous section, and I will make the following additional technical hypothesis.

Hypothesis 1.3.4. The morphism $g : X \to S$ is realisable, in the sense that there exists a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{P} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathfrak{Q} \end{array} \tag{1.86}$$

where $\mathfrak{P}, \mathfrak{Q}$ are smooth and proper formal $\mathcal{V}$-schemes, both horizontal arrows are locally closed immersions and $\mathfrak{P} \to \mathfrak{Q}$ is smooth around X.

Remark 1.3.5.

1. I should eventually be able to remove this technical hypothesis, using methods of 'recollement', but I do not worry about this for now.

2. One non-trivial example of such a g is given by a model for a smooth, proper, geometrically connected curve C over a function field K over a finite field. In this situation S' is the unique smooth, proper model for K, X' is a regular, flat, proper S'-scheme, whose generic fibre is C, $S \subset S'$ is an affine open subset of S' over which g is smooth, and X is the pre-image of S. Since X' is a regular, proper surface over a finite field, it is smooth, hence projective, and the above hypotheses really are satisfied.

In this section I will prove the following two theorems.

Theorem 1.3.6.

1. *Let $E \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$ and suppose that $i_s^* E$ is a constant isocrystal. Then there exists $E' \in F\text{-Isoc}^\dagger(S/K) \otimes_K K(s)$ such that $E \cong g^* E'$.*

2. *Let $E \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$, and let $F_0 \subset i_s^* E$ denote the largest constant subobject. Then there exists $E_0 \subset E$ such that $F_0 = i_s^* E_0$.*

Theorem 1.3.7. *Suppose that $E \in \mathcal{N}\text{Isoc}^\dagger(X_s/K(s))$. Then there exists some object $E' \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$ such that E is a quotient of $i_s^* E'$.*

Remark 1.3.8. The reason I have used categories of overconvergent F-isocrystals rather than overconvergent isocrystals without Frobenius is that the theory of 'six operations' has only fully been developed for overconvergent F-isocrystals. If six operations were to be resolved for overconvergent isocrystals in general, then I would be able to deduce results for smooth fibrations over any perfect field of positive characteristic, not just over finite fields where Frobenius can be linearised.

The method of proof will be entirely analogous to the proof in characteristic 0, replacing the algebraic $\mathcal{D}$-modules used there by their arithmetic counterparts, the theory of which was developed by Berthelot and Caro. It would be far too much of a detour to describe this theory in any depth, so instead I will just recall the notations and results needed, referring the reader to the series of articles [4–6] and [8–11, 13, 14] for details.

Let $F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{X/K})$ (resp. $F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{S/K})$) denote the category of overholonomic F-$\mathcal{D}$-modules on X (resp. S) as defined in Section 3 of [11]. There is a functor

$$\mathrm{sp}_{X,+} : F\text{-}\mathrm{Isoc}^{\dagger}(X/K) \to F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{X/K}) \tag{1.87}$$

which is an equivalence onto the full subcategory $F\text{-}\mathrm{Isoc}^{\dagger\dagger}(X/K)$ of overcoherent F-isocrystals (Theorem 2.3.16 of [15] and Théorème 2.3.1 of [10]) and compatible with the natural tensor products on both sides (Proposition 4.8 of [13]). The same also holds for S. Let

$$g_+ : F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{X/K}) \to F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{S/K}) \tag{1.88}$$

$$g^+ : F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{S/K}) \to F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{X/K}) \tag{1.89}$$

be the adjoint functors defined in Section 3 of [11]. By Théorème 4.2.12 of [14], for any $E \in F\text{-}\mathrm{Isoc}^{\dagger}(X/K)$, and any $i \in \mathbb{Z}$, $\mathcal{H}^i(g_+\mathrm{sp}_{X,+}(E)) \in F\text{-}\mathrm{Isoc}^{\dagger\dagger}(S/K)$ and define

$$g_* := sp^{-1}_{S,+}\mathcal{H}^{-d}(g_+\mathrm{sp}_{X,+}(-))(-d) : F\text{-}\mathrm{Isoc}^{\dagger}(X/K) \to F\text{-}\mathrm{Isoc}^{\dagger}(S/K) \tag{1.90}$$

where d is the relative dimension of X/S, and $(-d)$ denotes the Tate twist. Define as well the higher direct images

$$\mathbf{R}^i g_* := sp^{-1}_{S,+}\mathcal{H}^{-d+i}(g_+\mathrm{sp}_{X,+}(-))(-d) : F\text{-}\mathrm{Isoc}^{\dagger}(X/K) \to F\text{-}\mathrm{Isoc}^{\dagger}(S/K). \tag{1.91}$$

Let $s^! : F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{S/K}) \to F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{\mathrm{Spec}(k(s))/K(s)})$ denote the functor defined in Section 3 of [11].

Remark 1.3.9. Although Caro's functor $s^!$ lands in $F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{\mathrm{Spec}(k(s))/K})$ rather than $F\text{-}D^b_{\mathrm{surhol}}(\mathcal{D}_{\mathrm{Spec}(k(s))/K(s)})$, it can be easily adapted to land in the latter category. The base change result that I use below holds in this slightly altered context.

Now, if Y is any smooth, proper, realisable variety over k (i.e. Y can be embedded into a smooth and proper formal $\mathcal{V}$-scheme) with structure map $h : Y \to \mathrm{Spec}(k)$ then similarly define

$$\mathbf{R}^i h_* : F\text{-}\mathrm{Isoc}^{\dagger}(Y/K) \to F\text{-}\mathrm{Isoc}^{\dagger}(k/K) \tag{1.92}$$

where the target category is that of K-vector spaces equipped with an automorphism ϕ. There are also the rigid cohomology functors

$$H^i_{\mathrm{rig}}(Y, -) : F\text{-}\mathrm{Isoc}^{\dagger}(Y/K) \to F\text{-}\mathrm{Isoc}^{\dagger}(k/K) \tag{1.93}$$

where the Frobenius action on the rigid cohomology $H^i_{\mathrm{rig}}(Y,E)$ of an overconvergent F-isocrystal E is the one induced by functoriality.

Proposition 1.3.10. *For $h : Y \to \mathrm{Spec}(k)$ smooth and realisable there is an isomorphism of functors*

$$\mathbf{R}^i h_*(-) \cong H^i_{\mathrm{rig}}(Y,-) : F\text{-Isoc}^\dagger(Y/K) \to F\text{-Isoc}^\dagger(k/K). \tag{1.94}$$

Proof. Lemme 7.3.4 of [9] states that $\mathbf{R}^i h_*(-)(d) \cong H^i_{\mathrm{rig}}(Y,-)$, however, it is pointed out in Remark 3.15 (iii) of [1] that this is incorrect, and needs to be altered by a Tate twist to give the claimed result. □

Of course, a similar result holds over any finite extension of k.

Corollary 1.3.11. *Let $s \in S$ be a closed point. There is an isomorphism of functors $s^*\mathbf{R}^i g_*(-) \cong H^i_{\mathrm{rig}}(X_s, i_s^*(-)) : F\text{-Isoc}^\dagger(X/K) \to \mathrm{Vec}_{K(s)}$.*

Remark 1.3.12. I am deliberately ignoring Frobenius structure in the final target category of these two composite functors.

Proof. This follow from proper base change for arithmetic $\mathcal{D}$-modules (Théorème 4.4.2 of [14]), together with the identification $s^* = s^![\dim S]$ for overcoherent F-isocrystals on S (1.4.5 of [13]), and the previous proposition applied to $Y = X_s/k(s)$ (after forgetting the Frobenius structures). □

Proposition 1.3.13. *For $E \in F\text{-Isoc}^\dagger(X/K)$, $g_+\mathrm{sp}_{X,+}(E)$ is concentrated in degrees $\geq -d$.*

Proof. The complex of $\mathcal{D}^\dagger$-modules $g_+\mathrm{sp}_{X,+}(E)$ has overcoherent F-isocrystals for cohomology sheaves, and by the previous proposition, the fibre over s of $\mathcal{H}^i(g_+\mathrm{sp}_{X,+}(E))$ is zero for $i \leq -d$. Hence $\mathcal{H}^i(g_+\mathrm{sp}_{X,+}(E))$ is zero for $i \leq -d$. □

Proposition 1.3.14. *g_* is right adjoint to g^*.*

Proof. Since g_+ is right adjoint to g^+, this will follow from the previous proposition provided $g^+\mathrm{sp}_{S,+}(-)[d](d) = \mathrm{sp}_{X,+}g^*(-)$. To see this, note that it follows from Proposition 6.5.2, together with 6.4.3(b) and 6.4.4(c) of [9] that $g^!\mathrm{sp}_{S,+}(-)[-d] \cong \mathrm{sp}_{X,+}g^*(-)$, and from 5.4.8 of [12] that $g^![-d] \cong g^+[d](d)$ for overcoherent F-isocrystals on smooth k-varieties. □

Proof of Theorem 1.3.6. Because g_* and g^* are functorial, they extend to give adjoint functors

$$g^* : F\text{-Isoc}^\dagger(S/K) \otimes_K K(s) \rightleftarrows \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s) : g_* \tag{1.95}$$

such that (using the base change theorem as in the proof of Corollary 1.3.11) the counit $g^*g_*E \to E$ restricts to the counit of the adjunction

$$- \otimes_{K(s)} \mathcal{O}^\dagger_{X_s/K(s)} : \mathrm{Vec}_{K(s)} \rightleftarrows \mathrm{Isoc}^\dagger\left(X_s/K(s)\right) : H^0_{\mathrm{rig}}(X_s, -)\ . \tag{1.96}$$

on the fibre over s. Thus exactly as in the proof of Proposition 1.1.12, if i_s^*E is trivial, the counit $g^*g_*E \to E$ is an isomorphism on the fibre over s, and hence an isomorphism. Similarly, since $H^0_{\mathrm{rig}}(X_s, i_s^*E) \cong \mathrm{Hom}_{\mathrm{Isoc}^\dagger(X/K(s))}(\mathcal{O}_{X_s}, i_s^*E)$, (see Proposition 1.3.15 below) exactly the same argument as in Proposition 1.1.13 shows that in general $H^0_{\mathrm{rig}}(X_s, i_s^*E) \otimes_{K(s)} \mathcal{O}_{X_s}$ is the largest trivial subobject of i_s^*E. Hence setting $E_0 = g^*g_*E$, then $i_s^*E_0 \cong H^0_{\mathrm{rig}}(X_s, i_s^*E) \otimes_{K(s)} \mathcal{O}_{X_s}$ is the largest trivial sub-object of i_s^*E, proving (2), and if i_s^*E is trivial, then $E \cong E_0$, proving (1). □

I now turn to the proof of Theorem 1.3.7.

Proposition 1.3.15. *Suppose that $E, E' \in \mathrm{Isoc}^\dagger(X_s/K(s))$. Then there are canonical isomorphisms*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Isoc}^\dagger(X_s/K(s))}(E, E') &\cong H^0_{\mathrm{rig}}(X_s, \mathcal{H}\mathrm{om}(E, E')) \\ \mathrm{Ext}_{\mathrm{Isoc}^\dagger(X_s/K(s))}(E, E') &\cong H^1_{\mathrm{rig}}(X_s, \mathcal{H}\mathrm{om}(E, E')) \end{aligned} \tag{1.97}$$

and moreover if E, E' have Frobenius structures, this induces an isomorphism

$$\mathrm{Hom}_{F\text{-}\mathrm{Isoc}^\dagger(X_s/K(s))}(E, E') \cong H^0_{\mathrm{rig}}(X_s, \mathcal{H}\mathrm{om}(E, E'))^{\phi=1} \tag{1.98}$$

as well as a surjection

$$\mathrm{Ext}_{F\text{-}\mathrm{Isoc}^\dagger(X_s/K(s))}(E, E') \twoheadrightarrow H^1_{\mathrm{rig}}(X_s, \mathcal{H}\mathrm{om}(E, E'))^{\phi=1} \tag{1.99}$$

Proof. The first isomorphism is clear, and this immediately implies the third. The second is Proposition 1.3.1 of [19], from which the fourth is then easily deduced. □

Define the U_n inductively as follows. U_1 will just be $\mathcal{O}^\dagger_{X_s}$, and U_{n+1} will be the extension of U_n by $\mathcal{O}^\dagger_{X_s} \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right)^\vee$ corresponding to the identity under the isomorphisms

$$\begin{aligned} \mathrm{Ext}_{\mathrm{Isoc}^\dagger(X_s/K(s))}\left(U_n, \mathcal{O}^\dagger_{X_s} \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right)^\vee\right) \\ \cong H^1_{\mathrm{rig}}\left(X_s, U_n^\vee \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right)^\vee\right) \\ \cong H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right) \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right)^\vee \\ \cong \mathrm{End}_{K(s)}\left(H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right)\right). \end{aligned} \tag{1.100}$$

Looking at the long exact sequence in cohomology associated to the short exact sequence $0 \to U_n^\vee \to U_{n+1}^\vee \to \mathcal{O}_{X_s}^\dagger \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right) \to 0$ gives

$$\begin{aligned} 0 \to H^0_{\mathrm{rig}}\left(X_s, U_n^\vee\right) \to H^0_{\mathrm{rig}}\left(X_s, U_{n+1}^\vee\right) \to H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right) \\ \xrightarrow{\delta} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right) \to H^1_{\mathrm{rig}}\left(X_s, U_{n+1}^\vee\right). \end{aligned} \tag{1.101}$$

Lemma 1.3.16. *The connecting homomorphism δ is the identity.*

Proof. By dualising, the extension

$$0 \to U_n^\vee \to U_{n+1}^\vee \to \mathcal{O}_{X_s}^\dagger \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right) \to 0 \tag{1.102}$$

corresponds to the identity under the isomorphism

$$\mathrm{Ext}_{\mathrm{Isoc}^\dagger(X_s/K(s))}\left(\mathcal{O}_{X_s}^\dagger \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right), U_n^\vee\right) \cong \mathrm{End}_{K(s)}\left(H^1_{\mathrm{rig}}\left(X_s, U_n^\vee\right)\right) \tag{1.103}$$

Now the Lemma follows from the fact that, for an extension $0 \to E \to F \to \mathcal{O}_{X_s}^\dagger \otimes_{K(s)} V \to 0$ of a trivial bundle by E, the class of the extensions under the isomorphism

$$\begin{aligned} \mathrm{Ext}_{\mathrm{Isoc}^\dagger(X_s/K(s))}\left(\mathcal{O}_{X_s}^\dagger \otimes_K V, E\right) &\cong V^\vee \otimes_{K(s)} H^1_{\mathrm{rig}}\left(X_s, E\right) \\ &\cong \mathrm{Hom}_{K(s)}\left(V, H^1_{\mathrm{rig}}\left(X_s, E\right)\right) \end{aligned} \tag{1.104}$$

is just the connecting homomorphism for the long exact sequence

$$0 \to H^0_{\mathrm{rig}}\left(X_s, E\right) \to H^0_{\mathrm{rig}}\left(X_s, F\right) \to V \to H^1_{\mathrm{rig}}\left(X_s, E\right). \tag{1.105}$$

□

In particular, any extension of U_n by a trivial bundle $V \otimes_{K(s)} \mathcal{O}_{X_s}^\dagger$ is split after pulling back to U_{n+1}, and $H^0_{\mathrm{rig}}\left(X_s, U_{n+1}^\vee\right) \cong H^0_{\mathrm{dR}}\left(X_s, U_n^\vee\right)$. It then follows by induction that $H^0_{\mathrm{rig}}\left(X_s, U_n^\vee\right) \cong H^0_{\mathrm{rig}}(X_s, \mathcal{O}_{X_s}^\dagger) \cong K(s)$ for all n.

Remark 1.3.17. It can be shown inductively, using Proposition 1.3.15, that each U_{n+1} can be endowed with some Frobenius structure, since for any Frobenius structure on U_n, the identity in $\mathrm{End}_{K(s)}(H^1_{\mathrm{rig}}(X_s, U_n^\vee))$ will be Frobenius invariant, and hence will lift to some Frobenius structure on U_{n+1}.

Definition 1.3.18. Define the unipotent class of $E \in \mathcal{N}\mathrm{Isoc}^\dagger\left(X_s/K(s)\right)$ inductively as follows. If E is trivial, then say E has unipotent class 1. If there exists an extension

$$0 \to V \otimes_{K(s)} \mathcal{O}_{X_s}^\dagger \to E \to E' \to 0 \tag{1.106}$$

with E' of unipotent class $\leq m-1$, then say that E has unipotent class $\leq m$.

Now let $x = p(s)$, $u_1 = 1 \in x^*(U_1) = K(s)$, and choose a compatible system of elements $u_n \in x^*(U_n)$ mapping to u_1.

Proposition 1.3.19. *Let $F \in \mathcal{N}\mathrm{Isoc}^\dagger(X_s/K(s))$ be an object of unipotent class $\leq m$. Then for all $n \geq m$ and any $f \in x^*(F)$ there exists a homomorphism $\alpha : U_n \to F$ such that $(x^*\alpha)(u_n) = f$.*

Proof. As in the characteristic zero case, I copy the proof of Proposition 2.1.6 of [28] and use induction on m. The case $m = 1$ is straightforward. For the inductive step, let F be of unipotent class m, and choose an exact sequence

$$0 \to E \overset{\psi}{\to} F \overset{\phi}{\to} G \to 0 \tag{1.107}$$

with E trivial and G of unipotent class $< m$. By induction there exists a unique morphism $\beta : U_{n-1} \to G$ such that $(x^*\phi)(f) = (x^*\beta)(u_{n-1})$. Pulling back the extension (1.107) first by the morphism β and then by the natural surjection $U_n \to U_{n-1}$ gives an extension of U_n by E, which must split, as observed above.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & F'' & \longrightarrow & U_n & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & F' & \longrightarrow & U_{n-1} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \end{array} \tag{1.108}$$

Let $\gamma : U_n \to F$ denote the induced morphism, then $(x^*\phi)((x^*\gamma)(u_n) - f) = 0$. Hence there exists some $e \in x^*E$ such that $(x^*\psi)(e) = (x^*\gamma)(u_n) - f$. Again by induction there exists $\gamma' : U_n \to E$ with $(x^*\gamma')(u_n) = e$. Finally let $\alpha = \gamma - \psi \circ \gamma'$, it is easily seen that $(x^*\alpha)(u_n) = f$. □

Corollary 1.3.20. *Every E in $\mathcal{N}\mathrm{Isoc}^\dagger(X_s/K(s))$ is a quotient of $U_n^{\oplus m}$ for some $n, m \in \mathbb{N}$.*

Recall that there are the higher direct images $\mathbf{R}^i g_*(E)$ for any $E \in F\text{-}\mathrm{Isoc}^\dagger(X/K)$. Thanks to 2.1.4 of [8], and the compatibilities already noted between tensor products and pull-backs of arithmetic $\mathcal{D}$-modules and their counterparts for overconvergent F-isocrystals, these satisfy a projection formula

$$\mathbf{R}^i g_*(E \otimes g^*E') \cong \mathbf{R}^i g_*(E) \otimes E' \tag{1.109}$$

for any $E \in F\text{-Isoc}^\dagger(X/K)$ and $E' \in F\text{-Isoc}^\dagger(S/K)$. Letting h denote the structure morphism of S, the fact that $h_+ \circ g_+ = (h \circ g)_+$ implies that there is a Leray spectral sequence relating $\mathbf{R}^i h_*$, $\mathbf{R}^j g_*$ and $\mathbf{R}^{i+j}(h \circ g)_*$, the exact sequence of low degree terms of which reads

$$0 \to H^1_{\text{rig}}(S, g_*E) \to H^1_{\text{rig}}(X, E) \to H^0_{\text{rig}}(S, \mathbf{R}^1 g_* E) \to H^2_{\text{rig}}(S, g_*E) \to H^2_{\text{rig}}(X, E). \tag{1.110}$$

Let $W_1 = \mathcal{O}_X^\dagger$.

Theorem 1.3.21. *There exists an extension W_{n+1} of W_n by $g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee$ in the category $\mathcal{N}_g F\text{-Isoc}^\dagger(X/K)$ such that $i_s^* W_{n+1} = U_{n+1}$ and $g_* W_{n+1}^\vee \cong \mathcal{O}_S^\dagger$.*

Proof. The statement and its proof are by induction on n, and in order to prove it I will use the strengthened induction hypothesis stating in addition that there exists a morphism $p^* W_n^\vee \to \mathcal{O}_S^\dagger$ such that the composite morphism $\mathcal{O}_S^\dagger \cong g_* W_n^\vee \cong p^* g^* g_* W_n^\vee \to p^* W_n^\vee \to \mathcal{O}_S^\dagger$ is an isomorphism.

To check the base case it suffices to show that $g_*\mathcal{O}_X^\dagger \cong \mathcal{O}_S^\dagger$. By the results of the previous section, there is a natural morphism $\mathcal{O}_S^\dagger \to g_*\mathcal{O}_X^\dagger$ as the counit of the adjunction between g_* and g^*. By naturality, restricting this morphism to the fibre over s gives the counit $K(s) \to H^0_{\text{rig}}(X_s, \mathcal{O}_{X_s}^\dagger)$ of the adjunction between $H^0_{\text{rig}}(X_s, \cdot)$ and $\cdot \otimes_K \mathcal{O}_{X_s}^\dagger$, which is easily checked to be an isomorphism. Hence by rigidity, $\mathcal{O}_S^\dagger \to g_*\mathcal{O}_X^\dagger$ is an isomorphism.

So now suppose that W_n exists as claimed, and consider the extension group

$$\text{Ext}_{F\text{-Isoc}^\dagger(X/K)}(W_n, g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \twoheadrightarrow H^1_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee)^{\phi=1}. \tag{1.111}$$

The Leray spectral sequence, the projection formula above and the induction hypothesis that $g_* W_n^\vee \cong \mathcal{O}_S^\dagger$ gives an exact sequence

$$\begin{aligned} 0 \to H^1_{\text{rig}}(S, (\mathbf{R}^1 g_* W_n^\vee)^\vee) &\to H^1_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \\ &\to H^0_{\text{rig}}(S, \mathcal{E}\text{nd}(\mathbf{R}^1 g_* W_n^\vee)) \to H^2_{\text{rig}}(S, (\mathbf{R}^1 g_* W_n^\vee)^\vee) \\ &\to H^2_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee). \end{aligned} \tag{1.112}$$

Now, the projection $p^* W_n^\vee \to \mathcal{O}_S^\dagger$ induces a map

$$H^i_{\text{rig}}\left(S, p^* W_n^\vee \otimes (\mathbf{R}^1 g_* W_n^\vee)^\vee\right) \to H^i_{\text{rig}}\left(S, \mathbf{R}^1 g_* \left(W_n^\vee\right)^\vee\right) \tag{1.113}$$

for $i = 1, 2$ such that the composite (dotted) arrow

$$\begin{array}{ccc} H^i_{\text{rig}}\left(S, \mathbf{R}^1 g_* \left(W_n^\vee\right)^\vee\right) & \longrightarrow & H^i_{\text{rig}}\left(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee\right) \\ \downarrow & & \downarrow \\ H^i_{\text{rig}}\left(S, \mathbf{R}^1 g_* \left(W_n^\vee\right)^\vee\right) & \longleftarrow & H^i_{\text{rig}}\left(S, p^* W_n^\vee \otimes \mathbf{R}^1 g_* W_n^\vee)^\vee\right) \end{array} \tag{1.114}$$

is as isomorphism, since it can be identified with the map induced by the composite arrow $\mathcal{O}_S \cong f_* W_n^\vee \cong p^* f^* f_* W_n^\vee \to p^* W_n^\vee \to \mathcal{O}_S$. Hence both the maps

$$\begin{aligned} H^1_{\text{rig}}\left(S, \mathbf{R}^1 g_* \left(W_n^\vee\right)^\vee\right) &\to H^1_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \\ H^2_{\text{rig}}\left(S, \mathbf{R}^1 g_* \left(W_n^\vee\right)^\vee\right) &\to H^2_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \end{aligned} \tag{1.115}$$

appearing in the 5-term exact sequence split, compatibly with Frobenius actions. So there is a commutative diagram

$$\begin{array}{ccc} H^1_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) & \longrightarrow & H^1_{\text{rig}}(X_s, U_n^\vee \otimes_{K(s)} H^1_{\text{rig}}(X_s, U_n^\vee)^\vee) \\ \downarrow & & \| \\ H^0_{\text{rig}}(S, \mathcal{E}\text{nd}(\mathbf{R}^1 g_* W_n^\vee)) & \longrightarrow & \text{End}_{K(s)}(H^1_{\text{rig}}(X_s, U_n^\vee)) \end{array} \tag{1.116}$$

where the horizontal arrows are just restrictions to the fibre over s, and the left hand vertical arrow is surjective. The identity morphism in $\text{End}_{K(s)}(H^1_{\text{rig}}(X_s, U_n^\vee))$, which is Frobenius invariant (for some Frobenius structure on U_n, see Remark 1.3.17) and corresponds to the extension U_{n+1}, lifts to the identity in $H^0_{\text{rig}}(S, \mathcal{E}\text{nd}(\mathbf{R}^1 g_* W_n^\vee)) = \text{End}_{\text{Isoc}^\dagger(S/K)}(\mathbf{R}^1 g_* W_n^\vee))$, and this element is also Frobenius invariant. Thus there exists a unique Frobenius invariant element of $H^1_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee)$ lifting the identity in $\text{End}_{\text{Isoc}^\dagger(S/K)}(\mathbf{R}^1 g_* W_n^\vee))$, and which maps to zero under the above splitting

$$H^1_{\text{rig}}(X, W_n^\vee \otimes g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \to H^1_{\text{rig}}\left(S, \mathbf{R}^1 g_* \left(W_n^\vee\right)^\vee\right) \tag{1.117}$$

Let W'_{n+1} be any corresponding extension (the map from the extension group as F-isocrystals to the Frobenius invariant part of H^1 is surjective). Now, there is a natural map

$$\text{Ext}_{F\text{-Isoc}^\dagger(S/K)}(\mathcal{O}_S^\dagger, (\mathbf{R}^1 g_* W_n^\vee)^\vee) \xrightarrow{g^*} \text{Ext}_{F\text{-Isoc}^\dagger(X/K)}(W_n, g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \tag{1.118}$$

which has a section (denoted p^*) induced by the map $p^* W_n^\vee \to \mathcal{O}_S^\dagger$, and such that whole diagram

$$\begin{array}{ccc} H^1_{\mathrm{rig}}(S, (\mathbf{R}^1 g_* W_n^\vee)^\vee) & \rightleftarrows & H^1_{\mathrm{rig}}(X, W_n^\vee \otimes_{\mathcal{O}_X^\dagger} g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \\ \uparrow & & \uparrow \\ \mathrm{Ext}_{F\text{-}\mathrm{Isoc}^\dagger(S/K)}(\mathcal{O}_S^\dagger, (\mathbf{R}^1 g_* W_n^\vee)^\vee) & \rightleftarrows & \mathrm{Ext}_{F\text{-}\mathrm{Isoc}^\dagger(X/K)}(W_n, g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee) \end{array} \tag{1.119}$$

commutes. Let W_{n+1} be the extension corresponding to $[W'_{n+1}] - g^*p^*[W'_{n+1}]$ in the extension group $\mathrm{Ext}_{F\text{-}\mathrm{Isoc}^\dagger(X/K)}(W_n, g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee)$. Note that this splits when pulled back via p^* and then pushed out via $p^*W_n^\vee \to \mathcal{O}_S^\dagger$, and also has the same image as W'_{n+1} inside $H^1_{\mathrm{rig}}(X, W_n^\vee \otimes_{\mathcal{O}_X^\dagger} g^*(\mathbf{R}^1 g_* W_n^\vee)^\vee)$.

To complete the induction I need to show that $g_*W_{n+1}^\vee \cong \mathcal{O}_S^\dagger$, and that there exists a map $p_*W_{n+1}^\vee \to \mathcal{O}_S^\dagger$ as claimed. There is an exact sequence (using the projection formula and the fact that $g_*\mathcal{O}_X^\dagger \cong \mathcal{O}_S^\dagger$)

$$0 \to g_*W_n^\vee \to g_*W_{n+1}^\vee \to \mathbf{R}^1 g_* W_n^\vee \to \dots \tag{1.120}$$

and it follows from Lemma 1.3.16 together with base change that the arrow $g_*W_n^\vee \to g_*W_{n+1}^\vee$ restricts to an isomorphism on the fibre at s. Thus by rigidity it is an isomorphism. Finally, there is an exact sequence

$$0 \to p^*W_n^\vee \to p^*W_{n+1} \to (\mathbf{R}^1 g_* W_n^\vee)^\vee \to 0 \tag{1.121}$$

which splits when pushed out via the map $p^*W_n^\vee \to \mathcal{O}_S^\dagger$. This splitting induces a map $p^*W_{n+1}^\vee \to \mathcal{O}_S^\dagger$ such that the diagram

$$\begin{array}{ccc} p^*W_n^\vee & \longrightarrow & p^*W_{n+1}^\vee \\ & \searrow & \downarrow \\ & & \mathcal{O}_S^\dagger \end{array} \tag{1.122}$$

commutes. Now the fact that the diagram

$$\begin{array}{ccccccccc} & \nearrow & g_*W_{n+1}^\vee & \longrightarrow & p^*g^*g_*W_{n+1}^\vee & \longrightarrow & p^*W_{n+1}^\vee & \searrow & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \mathcal{O}_S^\dagger & \longrightarrow & g_*W_n^\vee & \longrightarrow & p^*g^*g_*W_n^\vee & \longrightarrow & p^*W_n^\vee & \longrightarrow & \mathcal{O}_S^\dagger \end{array} \tag{1.123}$$

commutes implies that the composite along the top row is an isomorphism, finishing the proof.

□

To complete the proof of Theorem 1.3.7, use the base extension functor

$$-\otimes_K K(s) : \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \to \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s), \tag{1.124}$$

which is defined on pages 155-156 of [38], to view the W_n as objects of the latter category.

1.3.2 Frobenius structures

The upshot of the previous section is that I now have an affine group scheme $\pi_1^{\text{rig}}(X/S,p)$ over the Tannakian category $F\text{-Isoc}^\dagger(S/K)$ whose fibre (ignoring Frobenius structures) over any closed point s is the usual rigid fundamental group $\pi_1^{\text{rig}}(X_s,p_s)$ as defined by Chiarellotto and le Stum in [18]. In Chapter II of [17], Chiarellotto defines a Frobenius isomorphism $F_* : \pi_1^{\text{rig}}(X_s,p_s) \overset{\sim}{\to} \pi_1^{\text{rig}}(X_s,p_s)$, by using the fact that Frobenius pullback induces an automorphism of the category $\mathcal{N}\text{Isoc}^\dagger(X_s/K)$. Since I have constructed $\pi_1^{\text{rig}}(X/S,p)$ as an affine group scheme over $F\text{-Isoc}^\dagger(S/K)$, it comes with a Frobenius structure that I can compare with Chiarellotto's. However, it is not obvious to me exactly what the relationship between these two Frobenius structures is, so instead I will endow $\pi_1^{\text{rig}}(X/S,p)$ with a different Frobenius, which I will be able to compare with the natural Frobenius on the fibres.

Remark 1.3.22. From now onward, I will consider $\pi_1^{\text{rig}}(X/S,p)$ as an affine group scheme over $\text{Isoc}^\dagger(S/K)$, via the forgetful functor.

Let $\sigma_S : S \to S$ denote the k-linear Frobenius, $X' = X \times_{S,\sigma_S} S$ the base change of X by σ_S, and $\sigma_{X/S} : X \to X'$ the relative Frobenius induced by the k-linear Frobenius σ_X of X. Let p' be the induced point of X', and $q = \sigma_{X/S} \circ p \in X'(S)$. Then by functoriality and base change there is a homomorphism

$$\pi_1^{\text{rig}}(X/S,p) \to \pi_1^{\text{rig}}(X'/S,q) \tag{1.125}$$

and an isomorphism

$$\pi_1^{\mathrm{rig}}(X'/S,p') \xrightarrow{\sim} \sigma_S^*\pi_1^{\mathrm{rig}}(X/S,p). \tag{1.126}$$

One can easily check that $p' = q \in X(S)$, and hence there is a natural morphism $\phi : \pi_1^{\mathrm{rig}}(X/S,p) \to \sigma_S^*\pi_1^{\mathrm{rig}}(X/S,p)$.

Lemma 1.3.23. *This is an isomorphism.*

Proof. Let $s \in S$ be a closed point, with residue field $k(s)$ of size q^a. The map induced by ϕ^a on the fibre $\pi_1^{\mathrm{rig}}(X_s,p_s)$ over s is the same as that induced by pulling back unipotent isocrystals on X_s by the $k(s)$-linear Frobenius on X_s. This is proved in Chapter II of [17] to be an isomorphism, thus ϕ^a is an isomorphism by rigidity. Hence ϕ is also an isomorphism. □

Now let $F_* : \sigma_S^*\pi_1^{\mathrm{rig}}(X/S,p) \xrightarrow{\sim} \pi_1^{\mathrm{rig}}(X/S,p)$ denote the inverse of ϕ, which by the proof of the previous lemma, reduces to the Frobenius structure as defined by Chiarellotto on closed fibres.

Definition 1.3.24. When I refer to 'the' Frobenius on $\pi_1^{\mathrm{rig}}(X/S,p)$, I will mean the isomorphism F_* just defined.

1.4 Cohomology and period maps

In this section I study the non-abelian cohomology of the unipotent quotients $\pi_1^{\mathrm{rig}}(X/S,p)_n$ of $\pi_1^{\mathrm{rig}}(X/S,p)$. Assumptions and notations will be exactly as in the previous two sections, so $g : X \to S$ will be a 'good', realisable morphism of varieties over a finite field, and p a section. Recall from Section 1.2.1 the notion of a torsor under an affine group scheme U over $\mathrm{Isoc}^\dagger(S/K)$.

Definition 1.4.1. Define $H^1_{\mathrm{rig}}(S,U)$ to be the pointed set of isomorphism classes of torsors under U.

Example 1.4.2. Suppose that U is the vector scheme associated to an overconvergent isocrystal E. Then Exemple 5.10 of [22] shows that there is a bijection $H^1_{\mathrm{rig}}(S,U) \xrightarrow{\sim} H^1_{\mathrm{rig}}(S,E)$.

If U has a Frobenius structure, that is an isomorphism $\phi : \sigma_S^*U \xrightarrow{\sim} U$, then define an F-torsor under U to be a U-torsor P, together with a Frobenius isomorphism $\phi_P : \sigma_S^*P \xrightarrow{\sim} P$ such that the action map $P \times U \to P$ is compatible with Frobenius.

Definition 1.4.3. Define $H^1_{F,\mathrm{rig}}(S,U)$ to be the set of isomorphism classes of F-torsors under U.

Given any torsor P under U, without F-structure, $\sigma_S^* P$ will be a torsor under $\sigma_S^* U$, and the isomorphism ϕ can be used to consider $\sigma_S^* P$ as a torsor under U. Hence there is a Frobenius action $\phi : H^1_{\mathrm{rig}}(S,U) \to H^1_{\mathrm{rig}}(S,U)$, and it is easy to see that the forgetful map

$$H^1_{F,\mathrm{rig}}(S,U) \to H^1_{\mathrm{rig}}(S,U) \tag{1.127}$$

is a surjection onto the subset $H^1_{\mathrm{rig}}(S,U)^{\phi=\mathrm{id}}$ fixed by the action of ϕ.

Given any point $x \in X(S)$, there are path torsors $P(x)$ under $\pi_1^{\mathrm{rig}}(X/S,p)$ as well as the finite level versions $P(x)_n$. Moreover, these come with Frobenius structures, and hence there compatible maps

$$\begin{array}{ccc} X(S) & \longrightarrow & H^1_{F,\mathrm{rig}}(S,\pi_1^{\mathrm{rig}}(X/S,p)_n) \\ & \searrow & \downarrow \\ & & H^1_{\mathrm{rig}}(S,\pi_1^{\mathrm{rig}}(X/S,p)_n)^{\phi=\mathrm{id}} \end{array} \tag{1.128}$$

for each $n \geq 1$.

In order to get a handle on this 'non-abelian' H^1, I first discuss the generalisation of Theorem 2.11 of [38] to non-neutral Tannakian categories via groupoids and their representations, following [23]. The reason for doing this is to obtain a generalisation of Example 1.2.2 giving a more explicit description of $H^1_{\mathrm{rig}}(S,U)$.

So let K be a field of characteristic 0, and Y a K-scheme.

Definition 1.4.4. A K-groupoid acting on Y is a K-scheme G, together with 'source' and 'target' morphisms $s,t : G \to Y$ and a 'law of composition' $\circ : G \times_{sYt} G \to G$ such that the diagram

$$\begin{array}{ccc} G \times_{sYt} G & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ Y & \xrightarrow{\Delta} & Y \times_K Y \end{array} \tag{1.129}$$

commutes, and such that the following condition holds: for any K-scheme T, the data of $Y(T)$, $G(T)$, $s,t,\circ$ forms a groupoid, where $Y(T)$ is the set of objects and $G(T)$ the set of morphisms.

Example 1.4.5. Suppose that $Y = \mathrm{Spec}(K)$. Then a K-groupoid acting on Y is just a group scheme over K.

Definition 1.4.6. If G is a K-groupoid acting on Y, then a representation of G is a quasi-coherent $\mathcal{O}_Y$-module V, together with a morphism $\rho(g) : s(g)^*V \to t(g)^*V$ for any K-scheme T and any point $g \in G(T)$. These morphisms must be compatible with base change $T' \to T$, as well as with the law of composition on G. Finally, if $\mathrm{id}_y \in G(T)$ is the 'identity morphism' corresponding to the 'object' $y \in Y(T)$, then the morphism $\rho(\mathrm{id}_y)$ is required to be the identity. A morphism of representations is defined in the obvious way, and the category of *coherent* representations is denoted by $\mathrm{Rep}(Y : G)$. Actions of G on any (group) scheme U over Y are defined similarly, by instead requiring morphisms $\rho(g) : U \times_{Y,s(g)} T \to U \times_{Y,t(g)} T$ of (group) schemes over T.

Example 1.4.7. If $Y = \mathrm{Spec}(K)$, then this just boils down to the usual definition of a representation of a group scheme over K.

Now suppose that $\mathcal{C}$ is a Tannakian category over K, which admits a fibre functor $\omega : \mathcal{C} \to \mathrm{Vec}_L$ taking values in some finite extension L/K. Let $\mathrm{pr}_i : \mathrm{Spec}(L \otimes_K L) \to \mathrm{Spec}(L)$ for $i = 1, 2$ denote the two projections. Thus there are two fibre functors $\mathrm{pr}_i^* \circ \omega : \mathcal{C} \to \mathrm{Mod}_{\mathrm{f.g.}}(L \otimes_K L)$ taking values in the category of finitely generated $L \otimes_K L$-modules, and the functor of isomorphisms $\underline{\mathrm{Isom}}^{\otimes}(\mathrm{pr}_1^* \circ \omega, \mathrm{pr}_2^* \circ \omega)$ is represented by an affine scheme $\underline{\mathrm{Aut}}_K^{\otimes}(\omega)$ over $L \otimes_K L$. The composite of the map $\underline{\mathrm{Aut}}_K^{\otimes}(\omega) \to \mathrm{Spec}(L \otimes_K L)$ with the two projections to $\mathrm{Spec}(L)$ makes $\underline{\mathrm{Aut}}_K^{\otimes}(\omega)$ into a K-groupoid acting on $\mathrm{Spec}(L)$. Moreover, if E is an object of $\mathcal{C}$, then $\omega(E)$ becomes a representation of $\underline{\mathrm{Aut}}_K^{\otimes}(\omega)$ in the obvious way. Thus there is a functor

$$\mathcal{C} \to \mathrm{Rep}(L : \underline{\mathrm{Aut}}_K^{\otimes}(\omega)) \tag{1.130}$$

and Théorème (1.12) of [23] states (in particular) the following.

Theorem 1.4.8. *The induced functor $\mathcal{C} \to \mathrm{Rep}(L : \underline{\mathrm{Aut}}_K^{\otimes}(\omega))$ is an equivalence of Tannakian categories.*

Finally, to get the generalisation of Example 1.2.2 that is needed, the following technical lemma is necessary.

Lemma 1.4.9. *([23], Corollaire 3.9). Let L/K be finite, and G a K-groupoid acting on $\mathrm{Spec}(L)$, affine and faithfully flat over over $L \otimes_K L$. Then any representation V of G is the colimit of its finite dimensional sub-representations.*

Corollary 1.4.10. *If $\mathcal{C}$ is a Tannakian category over K, ω a fibre functor with values in L, then an affine (group) scheme over $\mathcal{C}$ 'is' just an affine (group) scheme over L together with an action of $\underline{\mathrm{Aut}}_K^{\otimes}(\omega)$, and morphism of such objects 'are' just $\underline{\mathrm{Aut}}_K^{\otimes}(\omega)$-equivariant morphisms.*

Definition 1.4.11. Let G be a K-groupoid acting on $\mathrm{Spec}(L)$. If U is a group scheme over L with a G-action, denote by $H^1(G, U)$ the set of isomorphism classes of G-equivariant torsors under U.

Example 1.4.12.

- If V is a representation of G, then $\mathrm{Spec}(\mathrm{Sym}(V^\vee))$ naturally becomes a group scheme over L with a G-action. This latter object will be referred to as the vector scheme associated to V.

- If U is a unipotent affine group scheme over $\mathrm{Isoc}^\dagger(S/K)$ as above, then for any closed point $s \in S$, the unipotent group U_s over $K(s)$ attains an action of the K-groupoid $\underline{\mathrm{Aut}}_K^\otimes(s^*)$, and there is a natural bijection of sets

$$H^1_{\mathrm{rig}}(S,U) \xrightarrow{\sim} H^1(\underline{\mathrm{Aut}}_K^\otimes(s^*), U_s). \tag{1.131}$$

Suppose that $Y = \mathrm{Spec}(L)$, with L/K finite, and let G be a K groupoid acting on Y. Let U be a unipotent group over L, on which G acts.

Definition 1.4.13. A 1-cocyle for G with values in U is a map of K-schemes $\phi : G \to U$ such that

- The diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & U \\ & {\scriptstyle t}\searrow & \downarrow{\scriptstyle \text{canonical}} \\ & & \mathrm{Spec}(L) \end{array} \tag{1.132}$$

  commutes.

- For any K-scheme T, and points $g, h \in G(T)$ which are composable in the sense that $s(g) = t(h)$, $\phi(gh) = \phi(g) \cdot \rho(g)(\phi(h))$ holds. This equality needs some explanation. By the first condition above, $\phi(g)$ lands in the subset $\mathrm{Hom}_T(T, U \times_{L,t(g)} T)$ of $\mathrm{Hom}_K(T,U)$ which consists of those morphisms $T \to U$ which are such that the diagram

$$\begin{array}{ccc} T & \longrightarrow & U \\ & {\scriptstyle t(g)}\searrow & \downarrow{\scriptstyle \text{canonical}} \\ & & \mathrm{Spec}(L) \end{array} \tag{1.133}$$

  commutes. Similarly, $\phi(h) \in \mathrm{Hom}_T(T, U \times_{L,t(h)} T) = \mathrm{Hom}_T(T, U \times_{L,s(g)} T)$. Since U/L is a group scheme, $\mathrm{Hom}_T(T, U \times_{L,t(g)} T)$ is a group, and the action of G on U gives a homomorphism

$$\rho(g) : \mathrm{Hom}_T(T, U \times_{L,s(g)} T) \to \mathrm{Hom}_T(T, U \times_{L,t(g)} T). \tag{1.134}$$

  Hence the equality $\phi(gh) = \phi(g) \cdot \rho(g)(\phi(h))$ makes sense inside $\mathrm{Hom}_T(T, U \times_{L,t(g)} T)$.

The set of 1-cocycles with coefficients in U is denoted $Z^1(G,U)$. This set has a natural action of $U(L)$ via

$$(\phi * u)(g) = (t(g)^*u)^{-1} \cdot \phi(g) \cdot \rho(g)(s(g)^*(u)) \tag{1.135}$$

for any $g \in G(T)$, as above this makes sense inside the group $\mathrm{Hom}_T(T, U \times_{L,t(g)} T)$.

The point of introducing these definitions is the following.

Lemma 1.4.14. *Let G be a K-groupoid acting on* $\mathrm{Spec}(L)$*, and U a pro-unipotent group scheme over L with a G-action. Then there is a bijection between the non-abelian cohomology set $H^1(G,U)$ and the set of orbits of $Z^1(G,U)$ under the action of $U(L)$.*

Proof. Let P be a G-equivariant torsor under U. Since any torsor under a unipotent group scheme over an affine scheme is trivial, there is a point $p \in P(L)$. Now, for any $g \in G(T)$, consider the points $t(g)^*p$ and $s(g)^*p$ inside $\mathrm{Hom}_T(T, P \times_{L,t(g)} T)$ and $\mathrm{Hom}_T(T, P \times_{L,s(g)} T)$ respectively. There is a morphism $\rho(g) : P \times_{L,s(g)} T \to P \times_{L,t(g)} T$ and hence there exists a unique element $\phi(g) \in U \times_{L,t(g)} T(T)$ such that $t(g)^*p\phi(g) = \rho(g)s(g)^*p$. This gives rise to $\phi(g) \in U(T)$, and the map $g \mapsto \phi(g)$ is functorial, giving a map of schemes $\phi : G \to U$. The fact that $\phi(g) \in \mathrm{Hom}_T(T, U \times_{L,t(g)} T)$ means that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \phi\ } & U \\ & \searrow^{t} & \downarrow{\scriptstyle \text{canonical}} \\ & & \mathrm{Spec}(L) \end{array} \tag{1.136}$$

commutes, and one easily checks that ϕ satisfies the cocycle condition. A different choice of p differs by an element of $U(L)$, and one easily sees that this modifies ϕ exactly as in the action of $U(L)$ on $Z^1(G,U)$ defined above. Hence there is a well defined map

$$H^1(G,U) \to Z^1(G,U)/U(L). \tag{1.137}$$

Conversely, given a cocycle $\phi : G \to U$, define a torsor P as follows. The underlying scheme of P is just U, and the action of U on P is just the usual action of right multiplication. The cocycle ϕ can be used to twist the action of G as follows. If $g \in G(T)$, then define $\rho(g) : P \times_{L,s(g)} T \to P \times_{L,t(g)} T$ to be the unique map, compatible with the U action, taking the identity of $U \times_{L,s(g)} T = P \times_{L,s(g)} T$ to $\phi(g) \in U \times_{L,t(g)} T = P \times_{L,t(g)} T$. One easily checks that this descends to the quotient $Z^1(G,U)/U(L)$, and provides an inverse to the map defined above. $\square$

I now want to investigate more closely the case when U is a vector scheme, coming from some finite dimensional representation V of G. In this case define, for any $n \geq 0$

the space $C^n(G,V)$ of n-cochains of G in V as follows. Let $G^{(n)}$ denote the scheme of 'n-fold composable arrows in G', that is the sub-scheme of $G\times_K \ldots \times_K G$ (n copies), consisting of those points $(g_1,\ldots,g_n)$ such that $s(g_i)=t(g_{i+1})$ for all i, by convention set $G^{(0)}=\mathrm{Spec}(L)$. Then the space of n-cochains is simply the space of global sections of the coherent sheaf $(\delta_1^n)^*V$ on $G^{(n)}$, where $\delta_1^n : G^{(n)} \to \mathrm{Spec}(L)$ is defined to be the map $t\circ \mathrm{pr}_1$, where $\mathrm{pr}_1 : G^n \to G$ is projection onto the first factor. This can also be viewed as the set of morphisms $G^{(n)} \to \mathrm{Spec}(\mathrm{Sym}(V^\vee))$ making the diagram

$$\begin{array}{ccc} G^{(n)} & \longrightarrow & \mathrm{Spec}(\mathrm{Sym}(V^\vee)) \\ & \searrow^{t\circ\mathrm{pr}_1} & \downarrow \text{canonical} \\ & & \mathrm{Spec}(L) \end{array} \tag{1.138}$$

commute, and hence there are differentials $d^n : C^n(G,V) \to C^{n+1}(G,V)$ by

$$\begin{aligned}(d^n\phi)(g_1,\ldots,g_{n+1}) = \rho(g_1)\phi(g_2,\ldots,g_{n+1}) & \\ + \sum_{i=1}^{n}(-1)^i\phi(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) & \\ + (-1)^{n+1}\phi(g_1,\ldots,g_n) & \end{aligned} \tag{1.139}$$

for $n\geq 1$, where $g_1,\ldots,g_{n+1}$ are composable elements of $G(T)$, and all the summands on the RHS are global sections of the coherent sheaf $t(g_1)^*V$ on T. For $n=0$ define $(d^0\phi)(g)=\rho(g)\phi(s(g))-\phi(t(g))$. It is easily checked that these differentials make $C^\bullet(G,V)$ into a chain complex, and define the cohomology of G with coefficients in V to be the cohomology of this complex:

$$H^n(G,V) := H^n(C^\bullet(G,V)). \tag{1.140}$$

Lemma 1.4.15. *Let V be a representation of the groupoid G acting on* $\mathrm{Spec}(L)$*. Then there is a canonical bijection* $H^1(G,V) \xrightarrow{\sim} H^1(G,\mathrm{Spec}(\mathrm{Sym}(V^\vee)))$

Proof. Taking into account the description of the latter in terms of cocyles modulo the action of V, this is straightforward algebra. □

So far everything has been over a field K, however, exactly the same definitions make sense over any K-algebra R, and I similarly define the cohomology of an R-groupoid acting on $\mathrm{Spec}(R\otimes_K L)$. There is an obvious base extension functor, taking K-groupoids to R-groupoids, and cohomology functors

$$\underline{H}^n(G,V)(R) = H^n(G_R,V_R) \tag{1.141}$$

for any representation V of G. These are the cohomology groups of a natural complex-valued functor

$$\underline{C}^n(G,V)(R) = C^n(G_R,V_R). \tag{1.142}$$

Denote by $\underline{Z}^n(G,V)$ and $\underline{B}^n(G,V)$ the 'n-cocycle' and 'n-coboundary' functors respectively.

Proposition 1.4.16. *Suppose that $G = \mathrm{Spec}(A)$ is affine. Then for any K-algebra R there are canonical isomorphisms $H^n(G_R,V_R) \xrightarrow{\sim} H^n(G,V)\otimes_K R$ for all $n \geq 0$.*

Proof. In this case, there is an alternative algebraic description of the complex $C^\bullet(G,V)$. First of all, A is a commutative $L\otimes_K L$-algebra, hence A becomes an L-module in two different ways, using the two maps $L \to L\otimes_K L$. I will refer to these as the 'left' and 'right' structures, these two different L-module structures induce the same K-module structure. The groupoid structure corresponds to a morphism $\Delta : A \to A\otimes_L A$, using the two different L-module structures to form the tensor product.

The action of G on a representation V can be described by an L-linear map $\Delta_V : V \to V\otimes_{L,t} A$, where on the RHS the 'left' L-module structure on A is used to form the tensor product, and define the L-module structure on the result via the 'right' L-module structure on A. This map is required to satisfy axioms analogous to the comodule axioms for the description of a representation of an affine group scheme.

Hence the group $C^n(G,V)$ of n-cochains is simply the L-module $V\otimes_L A\otimes_L \ldots \otimes_L A$ (n copies of A). The boundary maps d^n can be described algebraically as well by

$$\begin{aligned} d^n(v\otimes a_1\otimes\ldots\otimes a_n) = {} & \Delta_V(v)\otimes a_1\otimes\ldots\otimes a_n \\ & + \sum_{i=1}^n v\otimes a_1\otimes\ldots\otimes\Delta(a_i)\otimes\ldots\otimes a_n \\ & + v\otimes a_1\otimes\ldots\otimes a_n\otimes 1. \end{aligned} \tag{1.143}$$

Exactly the same discussion applies over any K-algebra R, and one immediately sees that there is an isomorphism of complexes $C^\bullet(G_R,V_R)\cong C^\bullet(G,V)\otimes_K R$. Since any K-algebra is flat, the result follows. □

Remark 1.4.17. In other words, the cohomology functor $\underline{H}^n(G,V)$ is represented by the vector scheme associated to $H^n(G,V)$.

If U is a unipotent group scheme on which G acts, extend the sets $Z^1(G,U)$ and $H^1(G,U)$ to functors of K-algebras in the same way. Also define $H^0(G,U)$ to be the group of all $u\in U(L)$ such that $\rho(g)s(g)^*u = t(g)^*u$ for any $g\in G(T)$, and any K-scheme T. I will also need to make use of a truncated cochain complex functor for non-abelian cohomology. This is the complex

$$C^0(G,U) \to C^1(G,U) \to C^2(G,U) \tag{1.144}$$

where $C^n(G,U)$ is the pointed set of maps $G^{(n)} \to U$ commuting with the projection to $\mathrm{Spec}(L)$, $G^{(n)}$ being the scheme of n-fold composable arrows in G. Thus $C^0(G,U)$ can be identified with $U(L)$, and the differential $d^0 : U(L) \to C^1(G,U)$ is defined by $d^0u(g) = (\rho(g)s(g)^*u)(t(g)^*u)^{-1}$ for any $g \in G(T)$. The differential $d^1 : C^1(G,U) \to C^2(G,U)$ is defined by

$$d^1\phi(g,h) = \phi(g)(\rho(g)\phi(h))\phi(gh)^{-1}. \tag{1.145}$$

for $g \in G(T)$. Thus letting $e_1, e_2 \in C^1(G,U), C^2(G,U)$ denote the 'trivial' cochains, there are natural identifications $H^0(G,U) = (d^0)^{-1}(e_1)$ and $Z^1(G,U) = (d^1)^{-1}(e_2)$.

These constructions extend to functors $\underline{C}^\bullet(G,U)$, $\underline{H}^0(G,U)$ of K-algebras in the obvious way. It is straightforward to check that when V is a representation of G, the complex $\underline{C}^\bullet(G, \mathrm{Spec}(\mathrm{Sym}(V^\vee)))$ agrees with $\underline{C}^\bullet(G,V)$ in degrees ≤ 2.

Proposition 1.4.18. *Let $0 \to V \to U \to W \to 0$ be an exact sequence of unipotent group schemes over K, acted on by G, with V a vector scheme. Suppose that V is central in W. Then there is a sequence of cohomology functors*

$$\begin{aligned} 0 \to \underline{H}^0(G,V) \to \underline{H}^0(G,U) \to \underline{H}^0(G,W) \to \underline{H}^1(G,V) \\ \to \underline{H}^1(G,U) \to \underline{H}^1(G,W) \to \underline{H}^2(G,V) \end{aligned} \tag{1.146}$$

which is exact in the sense that for all K-algebras R, the induced sequence on R-points is exact as a sequence of pointed sets.

Proof. The only non-obvious part of the existence of the sequence is the construction and functoriality of the boundary maps $H^i(G_R, W_R) \to H^{i+1}(G_R, V_R)$, for $i = 0, 1$. So suppose that there is given a map $\phi : G_R^{(i)} \to W_R$ for $i = 0, 1$, which is a cocycle. Then lift this to a map $\tilde{\phi} : G_R^{(i)} \to U_R$, (since the surjection $U \to W$ always admits a section as a map of *schemes*) and thus obtain a map $d^i(\tilde{\phi}) : G_R^{(i)} \to V_R$ measuring the failure of this lifted cochain $\tilde{c}$ to be a cocycle. Then exactly as in Chapter VII, Appendix of [46], $d^i(\tilde{c})$ is actually a cocycle, and the class of $d^i(\tilde{\phi})$ in $H^{i+1}(G_R, V_R)$ does not depend on either the class of the cocycle ϕ or on the lift $\tilde{\phi}$. It is also easy to check functoriality of the induced map $H^i(G_R, W_R) \to H^{i+1}(G_R, V_R)$, since a lifting of $\phi : G_R^{(i)} \to W_R$ to a map $\tilde{\phi} : G_R^{(i)} \to U_R$ then induces a lifting of $\phi_S : G_S^{(i)} \to W_S$ to $\tilde{\phi}_S : G_S^{(i)} \to U_S$ for any R-algebra S. That the sequence is exact on R-points is more or less word for word the same as the argument as in Proposition 2, Appendix, Chapter VII of *loc. cit*, and consists of a series of fairly straightforward checks. □

Recall that if U is a unipotent group scheme, define U^n inductively by $U^1 = [U, U]$ and $U^n = [U^{n-1}, U]$ and U_n by $U_n = U/U^n$. Since U is unipotent over K, a field of characteristic zero, each U^n/U^{n+1} is a vector scheme, and $U = U_N$ for large enough N.

Theorem 1.4.19. *Let U be a unipotent group scheme acted on by G. Assume that G is affine, and for all $n \geq 1$, $H^0(G, U^n/U^{n+1}) = 0$. Then for all $n \geq 0$ the functor $\underline{H}^1(G, U_n)$ is represented by an affine scheme over K. In particular, the functor $\underline{H}^1(G, U)$ is represented by an affine scheme over K.*

Proof. Note that the hypotheses imply that $\underline{H}^0(G, U^n/U^{n+1})(R) = 0$ for all K-algebras R, and hence, by induction on n, that $\underline{H}^0(G, U_n)(R) = 0$ for all K-algebras R and all $n \geq 1$.

I will prove the theorem by induction on n, and the argument is almost word for word that given by Kim in the proof of Proposition 2, Section 1 of [33]. When $n = 1$, U_1 is just a vector scheme associated to a representation of G, and $\underline{H}^i(G, U_1)$ is representable for all i. For general $n \geq 1$, there is an exact sequence

$$1 \to U^{n+1}/U^{n+2} \to U_{n+1} \to U_n \to 1 \tag{1.147}$$

realising U_{n+1} as a central extension of U_n by the vector scheme U^{n+1}/U^{n+2}, and there is a splitting $s : U_n \to U_{n+1}$ of this exact sequence (just as a map of schemes). Looking at the long exact sequence in cohomology associated to this exact sequence, the boundary map $\underline{H}^1(G, U_n) \to \underline{H}^2(G, U^{n+1}/U^{n+2})$ map between representables (using the induction hypothesis for representability of $\underline{H}^1(G, U_n)$) and hence the pre-image of $0 \in \underline{H}^2(G, U^{n+1}/U^{n+2})$ is an (affine) closed sub-scheme of $\underline{H}^1(G, U_n)$, which will be denoted by $I(G, U_n)$. Thus there is a vector scheme $\underline{H}^1(G, U^{n+1}/U^{n+2})$, an affine scheme $I(G, U_n)$, and an exact sequence

$$1 \to \underline{H}^1(G, U^{n+1}/U^{n+2})(R) \to \underline{H}^1(G, U_{n+1})(R) \to I(G, U_n)(R) \to 1 \tag{1.148}$$

for all R, in the sense of Proposition 1.4.18. I claim that in fact this sequence is exact in the stronger sense that $\underline{H}^1(G, U^{n+1}/U^{n+2})(R)$ acts freely on $\underline{H}^1(G, U_{n+1})(R)$ (functorially in R), and the surjection $\underline{H}^1(G, U_{n+1})(R) \to I(G, U_n)(R)$ identifies $I(G, U_n)(R)$ with the set of orbits for this action. I will give the argument for K-points, since functoriality will be clear from the definition of the action, and the case of R-points is handled identically.

Since U^{n+1}/U^{n+2} is in the centre of U_{n+1}, there is an action of $Z^1(G, U^{n+1}/U^{n+2})$ on $Z^1(G, U_{n+1})$ by $(\sigma * \phi)(g) = \sigma(g)\phi(g)$ for $\phi \in Z^1(G, U^{n+1}/U^{n+2})$, $\sigma \in Z^1(G, U_{n+1})$ and $g \in G(T)$. It is straightforward to check that this descends to an action of $H^1(G, U^{n+1}/U^{n+2})$ on $H^1(G, U_{n+1})$. Suppose that $\sigma, \sigma' \in Z^1(G, U_{n+1})$ represent cohomology classes in the same orbit under this action. Thus there exists some $\phi \in Z^1(G, U^{n+1}/U^{n+2})$ and $u \in U_{n+1}(L)$ such that

$$\sigma(g)\phi(g) = (t(g)^*u)^{-1}\sigma'(g)\rho(g)(s(g)^*u) \tag{1.149}$$

for all $g \in G(T)$. Now projecting to U_n shows the cohomology classes of σ and σ' map to the same element of $H^1(G, U_n)$. This argument can be reversed, and so the orbits for this action can be exactly identified with $I(G, U_n)(K)$.

To show freeness of the action, suppose that the cohomology class of some cocycle $\phi \in Z^1(G, U^{n+1}/U^{n+2})$ stabilises the cohomology class of $\sigma \in Z^1(G, U_{n+1})$, so that

$$\sigma(g)\phi(g) = (t(g)^*u)^{-1}\sigma(g)\rho(g)(s(g)^*u) \tag{1.150}$$

for some $u \in U_{n+1}(L)$. Now, projecting to U_n gives

$$t(g)^*\bar{u} = \bar{\phi}(g)\rho(g)(s(g)^*\bar{u})\bar{\phi}(g)^{-1} \tag{1.151}$$

where I have written $\bar{u}$ (resp. $\bar{\phi}$) for the image of u (resp. ϕ) in U_n (resp. $Z^1(G, U_n)$). Now an easy induction using the fact that $H^0(G, U^m/U^{m+1}) = 0$ for all $m \geq 1$ implies that $\bar{u} = 1$, and thus, since U^{n+1}/U^{n+2} is central in U_{n+1}, that $\phi(g) = (t(g)^*u)^{-1}\rho(g)(s(g)^*u)$ is a coboundary.

Taking R to be the co-ordinate ring of $\underline{H}^1(G, U_n)$ and lifting the identity in $\underline{H}^1(G, U_n)(R)$ via the surjective map

$$\underline{Z}^1(G, U_n)(R) \to \underline{H}^1(G, U_n)(R) \tag{1.152}$$

gives a splitting

$$\underline{H}^1(G, U_n) \to \underline{Z}^1(G, U_n) \tag{1.153}$$

and composing with the map $s : \underline{C}^1(G, U_n) \to \underline{C}^1(G, U_{n+1})$ and the boundary map $d : \underline{C}^1(G, U_{n+1}) \to \underline{C}^2(G, U_{n+1})$ gives a map

$$dsi : \underline{H}^1(G, U_n) \to \underline{C}^2(G, U_{n+1}) \tag{1.154}$$

which factors through $\underline{Z}^2(G, U^{n+1}/U^{n+2})$, and is such that the induced map $\underline{H}^1(G, U_n) \to \underline{H}^2(G, U^{n+1}/U^{n+2})$ is the boundary map appearing in the long exact sequence of cohomology from Proposition 1.4.18. This implies

$$I(G, U_n) = (dsi)^{-1}(\underline{B}^2(G, U^{n+1}/U^{n+2})). \tag{1.155}$$

Now, the proof of Proposition 1.4.16 shows that the functors $\underline{C}^1(G, U^{n+1}/U^{n+2})$ and $\underline{B}^2(G, U^{n+1}/U^{n+2})$ are represented by the vector schemes associated to the K-vector spaces

$$\underline{C}^1(G, U^{n+1}/U^{n+2})(K) = C^1(G, U^{n+1}/U^{n+2}) \tag{1.156}$$
$$\underline{B}^2(G, U^{n+1}/U^{n+2})(K) = B^2(G, U^{n+1}/U^{n+2})$$

respectively, and hence there is a functorial splitting

$$a : \underline{B}^2(G, U^{n+1}/U^{n+2}) \to \underline{C}^1(G, U^{n+1}/U^{n+2}) \tag{1.157}$$

and can define a map $b : I(G, U_n) \to \underline{C}^1(G, U_{n+1})$ by $b(x) = (si)(x)((adsi)(x))^{-1}$. Then another explicit calculation shows that b factors throughout $\underline{Z}^1(G, U_{n+1})$, and the induced map $I(G, W) \to \underline{H}^1(G, U_{n+1})$ is a splitting of the natural surjection $\underline{H}^1(G, U_{n+1}) \to I(G, U_n)$.

Thus using the stronger sense in which the sequence (1.148) is exact, there is an isomorphism of functors

$$\underline{H}^1(G, U_{n+1}) \cong \underline{H}^1(G, U^{n+1}/U^{n+2}) \times I(G, U_n) \tag{1.158}$$

and hence $\underline{H}^1(G, U_{n+1})$ is represented by an affine scheme over K. □

Corollary 1.4.20. *With the assumptions as in the previous theorem, assume further that $H^1(G, U^i/U^{i+1})$ is finite dimensional for each n. Then $\underline{H}^1(G, U_n)$ is of finite type over K, of dimension at most $\sum_{i=1}^{n-1} \dim_K H^1(G, U^i/U^{i+1})$*

Recall that for a 'good' morphism $f : X \to S$ over a finite field satisfying Hypothesis 1.3.4, there are period maps

$$X(S) \to H^1_{\text{rig}}(S, \pi_1(X/S, p)_n) \tag{1.159}$$

taking a section to the corresponding path torsor. Choosing a closed point $s \in S$ means that this map can be interpreted as

$$X(S) \to H^1(\underline{\text{Aut}}^{\otimes}_K(s^*), \pi_1^{\text{rig}}(X_s, p(s))_n). \tag{1.160}$$

This latter set has the structure of an algebraic variety over K under the condition that

$$H^0_{\text{rig}}(S, \pi_1^{\text{rig}}(X/S, p)^n / \pi_1^{\text{rig}}(X/S, p)^{n+1}) \tag{1.161}$$

is zero for each n. If, for example, X is a model for a smooth projective curve C over a function field, then I expect this condition to be satisfied under certain non-isotriviality assumptions on the Jacobian of C.

2 Rigid rational homotopy types

Traditional rational homotopy theory is the study of homotopy theory 'tensored with $\mathbb{Q}$', and the traditional objects of study are the 'rationalisations' of topological spaces. This rationalisation is a functor $X \mapsto X_{\mathbb{Q}}$, defined for 'sufficiently nice' spaces X, such that

$$\pi_n(X_{\mathbb{Q}}, x) = \pi_n(X, x) \otimes_{\mathbb{Z}} \mathbb{Q} \; \forall n \geq 1. \tag{2.1}$$

(When $n = 1$, $\pi_1(X, x) \otimes_{\mathbb{Z}} \mathbb{Q}$ is interpreted as the pro-unipotent completion of $\pi_1(X, x)$ over $\mathbb{Q}$.) The beauty of this theory is that the homotopy type of $X_{\mathbb{Q}}$ can be represented by a commutative differential graded algebra (dga) over $\mathbb{Q}$, which when X is a simplicial complex is just given by a suitable model of the algebra of piecewise linear differentials on X. Thus, provided that one works with the whole cohomology complex $\mathbf{R}\Gamma(X, \mathbb{Q}) \in D^b(\mathbb{Q})$, there is a certain sense in which the rational homotopy theory of a space is encoded in the multiplicative structure on cohomology.

In the previous chapter, I studied unipotent fundamental groups, both for varieties over finite fields, and for certain varieties over function fields, by spreading out over a model for the base. In this chapter, I ask to what extent these unipotent groups can be viewed as the fundamental groups of certain rational homotopy types. These homotopy types, thanks to the above philosophy, should just come from 'remembering' the multiplicative structure on the cohomology complex. In other words, for a variety X/k, with k a finite field, and K a complete p-adic field with residue field k, then the p-adic (rigid) rational homotopy type should just be $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K, \mathcal{O}^\dagger_{X/K})$, provided that $\mathcal{O}^\dagger_{X/K}$ is interpreted as a dga rather than just as a complex.

In the first few sections of this chapter, based on the ideas just outlined, extend Olsson's and Kim/Hain's definitions of p-adic rational homotopy types (see [35, 42]) to define the rigid rational homotopy type of an arbitrary k-variety X, where k is a perfect field of characteristic $p > 0$. I do this in two different ways - firstly using embedding systems and overconvergent de Rham dga's, which is nothing more than an extension of Olsson's methods from the convergent to the overconvergent case, and secondly using Le Stum's overconvergent site. The main focus is on comparison results - comparisons with Olsson's and Kim/Hain's definitions are made, as well as comparisons between the two approaches. I also study Frobenius structures, and use these comparison theorems as well as Kim/Hain's result in the case of a good compactification to prove that the rigid rational homotopy type of a variety over a finite field is mixed. As a corollary of this I deduce that the higher rational homotopy groups of such varieties are mixed. I also use methods similar to Navarro-Aznar's in the Hodge theoretic context (see [39]) to discuss the uniqueness of the weight filtration for Frobenius on rational homotopy types.

I then turn to the relative rigid rational homotopy type, and again I give two definitions, one in terms of Le Stum's overconvergent site, and one in terms of framing systems and relative overconvergent de Rham complexes. The comparison between the two should then induce a Gauss–Manin connection on the latter, however, I have so far been unable to prove the required property of the former object, namely that it is 'crystalline' in the sense of derived categories. What I can show is that this would follow from a certain 'generic coherence' result for Le Stum's relative overconvergent cohomology, of which there are analogues in other versions of p-adic cohomology such as the theory of arithmetic $\mathcal{D}$-modules or relative rigid cohomology. Here my approach is strongly influenced again by Navarro-Aznar in his paper [40] on relative de Rham rational homotopy theory.

2.1 Differential graded algebras and affine stacks

In this section I quickly recall some of the tools used by Olsson in [42] to define homotopy types of varieties in positive characteristic, that is Toën's theory of affine stacks. Although later I will mainly be focusing on the theory of differential graded algebras, I include this material to emphasise the fact that what I am doing is an extension of a particular case of Olsson's work. I will also need it to prove a comparison theorem between different constructions of unipotent fundamental groups.

Let K be a field of characteristic 0. Denote by dga_K the category of unital, differential graded algebras over K, concentrated in non-negative degrees, which are assumed to be (graded) commutative. (In this thesis, dga's will always be commutative and non-negatively graded, unless otherwise mentioned). Denote by Δ the simplicial category, that is the category whose objects are ordered sets $[n] = \{0, \ldots, n\}$ and morphisms order preserving maps, and denote by Alg_K^Δ the category of cosimplicial K-algebras, that is the category of functors $\Delta \to \mathrm{Alg}_K$. Aff_K will denote the category of affine schemes over K, that is the opposite category of Alg_K, which will be endowed with the fpqc topology unless otherwise mentioned. Denote by $\mathrm{Pr}(K)$ (resp. $\mathrm{Sh}(K)$) the category of presheaves (resp. sheaves) on Aff_K. $\mathrm{SPr}(K)$ will denote the category of simplicial presheaves on Aff_K, that is the category of functors $\mathrm{Aff}_K \to \mathrm{Set}^\Delta$ into simplicial sets. There are functors

$$D : \mathrm{dga}_K \to \mathrm{Alg}_K^\Delta \tag{2.2}$$
$$\mathrm{Spec} : (\mathrm{Alg}_K^\Delta)^\circ \to \mathrm{SPr}(K) \tag{2.3}$$

where D is the Dold-Kan de-normalisation functor (see Chapter 8.4 of [53]).

Suppose that $F \in \mathrm{SPr}(K)$, and $x \in F_0(R)$ for some $R \in \mathrm{Aff}_K$. Then for all $n \geq 1$ there is a presheaf of groups $\pi_n^{\mathrm{pr}}(F, x) : \mathrm{Aff}_K/R \to (\mathrm{Groups})$ which takes $S \to R$ to $\pi_n(|F(S)|, x)$ (here $|\cdot|$ is the geometric realisation functor). Define $\pi_n(F, x)$ to be the sheafification of this presheaf. Also define $\pi_0(F)$ to be the sheafification of the presheaf $R \mapsto \pi_0(|F(R)|)$.

Definition 2.1.1. A morphism $A^* \to B^*$ in dga_K is said to be a:

- weak equivalence if it induces K-module isomorphisms on cohomology;
- fibration if it is surjective in each degree;
- cofibration if it satisfies the left lifting property with respect to trivial fibrations.

Definition 2.1.2. A morphism $A^\bullet \to B^\bullet$ in Alg_K^Δ is said to be a:

- weak equivalence if the induced map $H^*(N(A^\bullet)) \to H^*(N(B^\bullet))$ on the cohomology of the normalised complex of the underlying cosimplicial K-module is an isomorphism;
- fibration if it is level-wise surjective;
- cofibration if it satisfies the left lifting property with respect to trivial fibrations.

Definition 2.1.3. A morphism $F \to G$ in $\mathrm{SPr}(K)$ is said to be a:

- weak equivalence if it induces isomorphisms on all homotopy groups;
- cofibration if for every $R \in \mathrm{Aff}_K$, $F(R) \to G(R)$ is a cofibration in SSet;
- fibration if it satisfies the right lifting property with respect to trivial cofibrations.

Then D is an equivalence of model categories, and Spec is right Quillen (Proposition 2.2.2 of [51]). Thus these descend to functors D, $\mathbf{R}\mathrm{Spec}$ on the level of homotopy categories.

Many of the methods used will rely on 'cohomological descent' - in many situations it will be necessary to know that the rational homotopy type of a variety X can be deduced from the cosimplicial rational homotopy type of an appropriate Zariski (étale, smooth, proper) hyper-cover of X. The way to do this is by using the functor of Thom-Sullivan cochains, this is a functor

$$\mathrm{Th} : \mathrm{dga}_K^{\Delta} \to \mathrm{dga}_K \tag{2.4}$$

which essentially is a multiplicative version of the functor taking a double complex to it's total complex. It is defined as follows (see §2.11-2.14 of [43]). Let $R_p = \mathcal{O}(\Delta_K^p)$ denote the K-algebra of functions on the 'algebraic p-simplex' Δ_K^p, that is $R_p = K[t_0, \ldots, t_p]/(\sum_i t_i = 1)$, which is made into a simplicial K-algebra $R_\bullet$ in the obvious way. Let $\Omega^*_{\Delta_K^\bullet}$ be the de Rham algebra of this simplicial ring, this is a simplicial dga over K.

Let $\mathscr{M}_\Delta$ denote the category where object are morphisms $[m] \to [n]$ in Δ, and where a morphism from $[m] \to [n]$ to $[m'] \to [n']$ is a commutative square

$$\begin{array}{ccc} [m] & \longrightarrow & [n] \\ \uparrow & & \downarrow \\ [m'] & \longrightarrow & [n']. \end{array} \tag{2.5}$$

Given any $A^{*,\bullet} \in \mathrm{dga}_K^{\Delta}$, there is a functor

$$\Omega^*_{\Delta_K^\bullet} \otimes A^{*,\bullet} : \mathscr{M}_\Delta \to \mathrm{dga}_K \tag{2.6}$$

$$([m] \to [n]) \mapsto \Omega^*_{\Delta_K^m} \otimes_K A^{*,n} \tag{2.7}$$

and $\mathrm{Th}(A^{*,\bullet})$ is defined to be $\varprojlim(\Omega^*_{\Delta_K^\bullet} \otimes A^{*,\bullet})$.

Proposition 2.1.4 ([43], Theorem 2.12)**.** *Let $C_K^{\geq 0}$ denote the category of non-negatively graded chain complexes of K-modules. There is a natural transformation of functors*

$$\begin{array}{ccc} \mathrm{dga}_K^{\Delta} & \xrightarrow{\mathrm{Th}} & \mathrm{dga}_K \\ {\scriptstyle \mathrm{forget}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{forget}} \\ (C_K^{\geq 0})^{\Delta} & \xrightarrow[\mathrm{Tot}_N]{} & C_K^{\geq 0} \end{array} \tag{2.8}$$

where Tot_N is the functor which takes a cosimplicial chain complex $C^{*,\bullet}$ to the total complex of the normalised double complex $N(C^{*,\bullet})$. Moreover, this natural transformation is a quasi-isomorphism when evaluated on objects of dga_K^{Δ}.

In the introduction to this chapter, I explained how rational homotopy types are more or less given by $\mathbf{R}\Gamma(X, K)$, where K is a suitable 'constant' dga. I now briefly recall how to make sense of this 'derived push-forward for dga's. If $(\mathcal{T}, \mathcal{O})$ is a ringed topos, with $\mathcal{O}$ a $\mathbb{Q}$-algebra, then the category $\mathrm{dga}(\mathcal{T}, \mathcal{O})$ of $\mathcal{O}$-dga's is a model category, with weak equivalences/fibrations defined to be those morphism which are weak equivalences/fibrations of the underlying complexes, and cofibrations defined using a lifting property. If $f : (\mathcal{T}, \mathcal{O}) \to (\mathcal{T}', \mathcal{O}')$ is a morphism of ringed topoi, and both $\mathcal{O}, \mathcal{O}'$ are $\mathbb{Q}$-algebras, with $f^{-1}\mathcal{O}' \to \mathcal{O}$ flat, then f_* is right Quillen, and hence can be derived to give a functor $\mathbf{R}f_*$ between homotopy categories of dga's. By the definition of the model category structure on $\mathrm{dga}(\mathcal{T}, \mathcal{O})$, taking $\mathbf{R}f_*$ commutes with passing to the underling complex. When there is no likelihood of confusion, I will often write $\mathrm{dga}(\mathcal{O})$ instead of $\mathrm{dga}(\mathcal{T}, \mathcal{O})$. I will also write dga_R when $\mathcal{T}$ is the punctual topos and R is a $\mathbb{Q}$-algebra.

If k is a perfect field of positive characteristic, I will construct homotopy types by considering dga's on cosimplicial lifts to characteristic zero; thus I will want to consider cosimplicial 'spaces' $V_\bullet$ over a field K of characteristic 0. In this situation, the functor $\mathrm{Th} \circ \Gamma : \mathrm{dga}(V_\bullet, K) \to \mathrm{dga}_K$ is right Quillen, and hence derives to give

$$\mathbf{R}\Gamma_{\mathrm{Th}} : \mathrm{dga}(V_\bullet, K) \to \mathrm{dga}_K \tag{2.9}$$

which I will sometimes abusively denote $\mathrm{Th} \circ \mathbf{R}\Gamma$.

2.2 Rational homotopy types of varieties

Let k be a perfect field of characteristic $p > 0$, and K a complete, discretely valued field with residue field k. Denote by $\mathcal{V}$ the ring of integers of K, and by ϖ a uniformiser. In this section I will define, for any variety X/k (variety = separated scheme of finite type) a stack $(X/K)_{\mathrm{rig}} \in \mathrm{Ho}(\mathrm{SPr}(K))$ which will compute both the unipotent fundamental group of X/K as well as the cohomology of unipotent overconvergent isocrystals. I essentially use Olsson's methods from [42], but replacing 'embedding systems' by 'framing systems'. This allows me to extend the definition of crystalline (unipotent) schematic homotopy types to non-smooth and non-proper k-varieties.

2.2.1 The definition of rigid homotopy types

Throughout, formal $\mathcal{V}$-schemes will be assumed to be ϖ-adic, topologically of finite type over $\mathcal{V}$, and separated. A frame over $\mathcal{V}$, as defined by Berthelot, consists of a triple $(U, \overline{U}, \mathscr{U})$ where $U \subset \overline{U}$ is an open embedding of k-varieties, and $\overline{U} \subset \mathscr{U}$ is a closed immersion of formal $\mathcal{V}$-schemes (considering $\overline{U}$ as a formal $\mathcal{V}$-scheme via its k-variety structure). Say that a frame is smooth if the structure morphism $\mathscr{U} \to \mathrm{Spf}(\mathcal{V})$ is smooth in some neighbourhood of U, and proper if $\mathscr{U}$ is proper over $\mathcal{V}$. Denote the generic fibre if $\mathscr{U}$ in the sense of rigid analytic spaces by $\mathscr{U}_{K0}$, the reason for this being that later on I will want to consider Berkovich spaces, and I will need a way to distinguish the two. Let

X/k be a variety over k.

Definition 2.2.1. A framing system for X/K by definition consists of a simplicial frame $\mathfrak{U}_\bullet = (U_\bullet, \overline{U}_\bullet, \mathscr{U}_\bullet)$ such that:

- $U_\bullet \to X$ is a Zariski hyper-covering (or an étale or proper hyper-covering);
- for each n, $(U_n, \overline{U}_n, \mathscr{U}_n)$ is a smooth and proper frame.

Proposition 2.2.2. *Every pair X/K as above admits a framing system.*

Proof. Let $\{U_i\}$ be a finite open affine covering for X. Then there exists an embedding $U_i \to \mathbb{P}^{n_i}_k$ for some n_i; let $\overline{U}_i$ be the closure of U_i in $\mathbb{P}^{n_i}_k$. So there is a frame $(U, \overline{U}, \mathscr{U})$ where $U = \coprod_i U_i$, $\overline{U} = \coprod_i \overline{U}_i$ and $\mathscr{U} = \coprod_i \widehat{\mathbb{P}}^{n_i}_{\mathcal{V}}$. Now define $U_n = U \times_X \ldots \times_X U$, with n copies of U, and similarly define $Y_n = \overline{U} \times_k \ldots \times_k \overline{U}$ and $\mathscr{U}_n = \mathscr{U} \times_{\mathcal{V}} \ldots \times_{\mathcal{V}} \mathscr{U}$, fibre product in the category of formal $\mathcal{V}$-schemes. Then there is a simplicial triple $(U_\bullet, Y_\bullet, \mathscr{U}_\bullet)$, and a framing system $(U_\bullet, \overline{U}_\bullet, \mathscr{P}_\bullet)$ for X is obtained by taking $\overline{U}_n$ to be the closure of U_n in Y_n. □

Given a framing system $\mathfrak{U}_\bullet$ for X/K, consider the simplicial rigid analytic space $V_0(\mathfrak{U}_\bullet) :=]\overline{U}_\bullet[_{\mathscr{U}_\bullet 0}$ over K (here the 0 refers to the fact that generic fibres are taken in the sense of rigid spaces, rather than Berkovich spaces), as well as a sheaf of K-dga's $j^\dagger\Omega^*_{]\overline{U}_\bullet[_{\mathscr{U}_\bullet 0}}$ on this simplicial space. Here $j^\dagger$ is Berthelot's functor of overconvergent sections.

Definition 2.2.3. The rational homotopy type of X/K is by definition

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) := \mathrm{Th}(\mathbf{R}\Gamma(j^\dagger\Omega^*_{]\overline{U}_\bullet[_{\mathscr{U}_\bullet 0}})) \in \mathrm{Ho}(\mathrm{dga}_K). \tag{2.10}$$

Denote by $(X/K)_{\mathrm{rig}}$ the affine stack $\mathbf{R}\mathrm{Spec}(D(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))))$; I may sometimes refer to $(X/K)_{\mathrm{rig}}$ as the rational homotopy type of X, and will try to keep any confusion this might cause to a minimum.

Remark 2.2.4. As a rational homotopy type, this definition only captures unipotent information about the fundamental group. In [42], Olsson defines a pointed homotopy type that captures the whole pro-algebraic theory of a geometrically connected, smooth and proper k-variety. It would not be hard to mimic his methods to give a general definition for an arbitrary geometrically connected k-variety, but to do so would involve a choice of base-point. For the most part I want to avoid doing this, which is why I only consider rational homotopy types.

Of course, I need to prove that the definition is independent of the framing system $\mathfrak{U}_\bullet$ chosen. The first step is to show that the rigid cohomology of X/K can be recovered from $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$.

Lemma 2.2.5. *Consider the forgetful functor $\varphi : \mathrm{Ho}(\mathrm{dga}_K) \to \mathrm{Ho}(C_K^{\geq 0})$. Then*

$$\varphi(\mathbf{R}_{\mathrm{Th}}\Gamma(\Omega^*(\mathcal{O}^\dagger_{X/K}))) \cong \mathbf{R}\Gamma_{\mathrm{rig}}(X/K) \tag{2.11}$$

the latter being the rigid cohomology of X/K.

Proof. Using Proposition 2.1.4, this just follows from cohomological descent for rigid cohomology, see e.g. Theorem 7.1.2 of [52]. □

Corollary 2.2.6. *The object $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ is independent of the framing system chosen.*

Proof. The proof is exactly as in [42], Section 2.24. If $\mathfrak{V}_\bullet = (V_\bullet, \overline{V}_\bullet, \mathscr{V}_\bullet)$ and $\mathfrak{U}_\bullet = (U_\bullet, \overline{U}_\bullet, \mathscr{U}_\bullet)$ are two framing systems for X/K, then their product $(U_\bullet \times_X V_\bullet, \overline{U}_\bullet \times_k \overline{V}_\bullet, \mathscr{U}_\bullet \times_{\mathcal{V}} \mathscr{V}_\bullet)$, maps to both $\mathfrak{U}_\bullet$ and $\mathfrak{V}_\bullet$, and after replacing $\overline{U}_\bullet \times_k \overline{V}_\bullet$ by the closure of $U_\bullet \times_X V_\bullet$, this framing system is smooth and proper. Hence I may assume that there is a map $\mathfrak{V}_\bullet \to \mathfrak{U}_\bullet$. This induces a map $\mathrm{Th}(\mathbf{R}\Gamma(j^\dagger\Omega^*_{V_0(\mathfrak{U}_\bullet)})) \to \mathrm{Th}(\mathbf{R}\Gamma(j^\dagger\Omega^*_{V_0(\mathfrak{V}_\bullet)}))$ in $\mathrm{Ho}(\mathrm{dga}_K)$ and to check that it is an isomorphism, I may forget the algebra structure and prove that it is an isomorphism in $\mathrm{Ho}(C_K^{\geq 0})$. But this is true because (after forgetting the algebra structure) both sides compute the rigid cohomology of X/K. □

2.2.2 Comparison with Navarro-Aznar's construction of homotopy types

Suppose that the variety X/k is 'suitably nice', in that it admits an embedding into a smooth and proper frame $\mathfrak{X} = (X, \overline{X}, \mathscr{X})$. Then from the work of Navarro-Aznar in [40], it would seem that there is another way of computing the homotopy type of X/k. One considers the sheaf of dga's $j^\dagger\Omega^*_{]\overline{X}[_{\mathscr{X}0}}$ on $]\overline{X}[_{\mathscr{X}0}$, and then simply defines the rational homotopy type of X/k to be $\mathbf{R}\Gamma(j^\dagger\Omega^*_{]\overline{X}[_{\mathscr{X}0}})$. That this agrees with the above definition follows from the fact that if $A^\bullet \in \mathrm{Ho}(\mathrm{dga}_K)^\Delta$ is the constant cosimplicial object on A, then $\mathrm{Th}(A^\bullet) \cong A$.

2.2.3 Comparison with Olsson's homotopy types

Now suppose that X is geometrically connected, smooth and proper, and that $K = \mathrm{Frac}(W(k))$ is the fraction field of the Witt vectors of k. Then Olsson has define a pointed stack $X_{\mathscr{C}} \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ associated to the category $\mathscr{C}$ of unipotent convergent isocrystals on X. In this section I would like to compare $(X/K)_{\mathrm{rig}}$ with $X_{\mathscr{C}}$.

I must therefore review Olsson's construction of $X_{\mathscr{C}}$. He considers an embedding system for X, that is an étale hyper-covering $U_\bullet$ of X, together with an embedding of $U_\bullet$ into a simplicial p-adic formal scheme $\mathscr{P}_\bullet$, which is formally smooth over $W = W(k)$. He then considers the p-adic completion $D_\bullet$ of the divided power envelope of $U_\bullet$ in $\mathscr{P}_\bullet$, and considers the sheaf of K-dga's $\Omega^*_{D_\bullet} \otimes_W K$ on $\mathscr{P}_\bullet$. He then defines $X_{\mathscr{C}}$ as the stack $\mathbf{R}\mathrm{Spec}(D(\mathrm{Th}(\mathbf{R}\Gamma(\Omega^*_{D_\bullet} \otimes_W K))))$. If $x \in X(k)$, then $x : \mathrm{Spec}(k) \to X$ induces a morphism $\mathbf{R}\Gamma(\Omega^*_{D_\bullet} \otimes_W K) \to K$ and hence makes $X_{\mathscr{C}}$ naturally into a pointed stack.

Now, there exists a framing system $\mathfrak{U}_\bullet = (U_\bullet, \overline{U}_\bullet, \mathscr{U}_\bullet)$ for X such that $(U_\bullet, \mathscr{U}_\bullet)$ is an embedding system for X, for example any framing system constructed as in Proposition 2.2.2 will do. Letting $D_\bullet$ be the p-adic completion of the divided power envelope of $U_\bullet$ in $\mathscr{U}_\bullet$, the canonical map $(D_\bullet)_{K0} \to \mathscr{U}_{K0}$ factors through $]U_\bullet[_{\mathscr{U}_\bullet 0}$ and hence there is a natural morphism

$$\mathbf{R}\Gamma(j^\dagger \Omega^*_{]\overline{U}_\bullet[_{\mathscr{U}_\bullet 0}}) \to \mathbf{R}\Gamma(\Omega^*_{D_\bullet} \otimes_W K) \tag{2.12}$$

in $\mathrm{Ho}(\mathrm{dga}_K^\Delta)$. I claim that it becomes an isomorphism after applying Th. Indeed, I may forget the algebra structure and prove that it is an isomorphism in $\mathrm{Ho}(\mathrm{Ch}_K^{\geq 0})$. But the the LHS computes the rigid cohomology of X/K, and the RHS the convergent cohomology of X/K. Since X is proper, they coincide.

2.2.4 Functoriality and Frobenius structures

f

In this section I discuss the functoriality of the rational homotopy type, as well as how to put a Frobenius structure on the rational homotopy type of a k-variety X.

So suppose that $f : X \to Y$ is a morphism of k-varieties, $\mathfrak{U}_\bullet = (U_\bullet, \overline{U}_\bullet, \mathscr{U}_\bullet)$ is a framing system for X, $\mathfrak{V}_\bullet = (V_\bullet, \overline{V}_\bullet, \mathscr{V}_\bullet)$ is a framing system for Y, and $\mathfrak{f} : \mathfrak{U}_\bullet \to \mathfrak{V}_\bullet$ is a morphism covering $f : X \to Y$. Note that given $f : X \to Y$ such a set-up always exists. This induces a morphism

$$\mathfrak{f}^*_{K0} : j^\dagger \Omega^*_{V_0(\mathfrak{V}_\bullet)} \to j^\dagger \Omega^*_{V_0(\mathfrak{U}_\bullet)} \tag{2.13}$$

in $\mathrm{dga}(V_0(\mathfrak{U}_\bullet), K)$ and hence a morphism

$$\mathfrak{f}^* : \mathbf{R}\Gamma_{\mathrm{Th}}(j^\dagger \Omega^*_{V_0(\mathfrak{V}_\bullet)}) \to \mathbf{R}\Gamma_{\mathrm{Th}}(j^\dagger \Omega^*_{V_0(\mathfrak{U}_\bullet)}) \tag{2.14}$$

in $\mathrm{Ho}(\mathrm{dga}_K)$.

Of course, it needs to be checked that this is independent of the choice of $\mathfrak{f}$, I will not do this here but wait until §3 when there will be given an alternative construction of the rational homotopy type which is clearly functorial. I will, however, still speak of the induced morphism

$$f^* : \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y/K})) \to \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \tag{2.15}$$

in $\mathrm{Ho}(\mathrm{dga}_K)$. Similar ideas can be used to define Frobenius structures.

Definition 2.2.7. An F-framing of X is a framing $\mathfrak{U}_\bullet = (U_\bullet, \overline{U}_\bullet, \mathscr{U}_\bullet)$ as above, together with a lifting $F_\bullet : \mathscr{U}_\bullet \to \mathscr{U}_\bullet$ of Frobenius compatible with the Frobenius on K.

Such an $F_\bullet$ induces a quasi-isomorphism

$$\phi : j^\dagger \Omega^*_{V_0(\mathfrak{U}_\bullet)} \otimes_{K,\sigma} K \to j^\dagger \Omega^*_{V_0(\mathfrak{U}_\bullet)} \tag{2.16}$$

in $\mathrm{dga}(V_0(\mathfrak{U}_\bullet), K)$ and hence a isomorphism

$$\phi : \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \otimes_{K,\sigma} K \to \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \tag{2.17}$$

in $\mathrm{Ho}(\mathrm{dga}_K)$. Again, this seemingly depended on the choice of Frobenius $F_\bullet : \mathscr{U}_\bullet \to \mathscr{U}_\bullet$, and I will prove in §3 that it does not. Moreover, if $X \to Y$ is a morphism of k-varieties, I will show that the induced morphism

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y/K})) \to \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \tag{2.18}$$

is compatible with Frobenius, in the sense that there a commutative diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y/K})) \otimes_{K,\sigma} K & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \otimes_{K,\sigma} K \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y/K})) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \end{array} \tag{2.19}$$

in $\mathrm{Ho}(\mathrm{dga}_K)$. Thus $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ can be viewed as a functor taking values in the category F-$\mathrm{Ho}(\mathrm{dga}_K)$ of objects A in $\mathrm{Ho}(\mathrm{dga}_K)$ together with a Frobenius isomorphism $A \otimes_{K,\sigma} K \overset{\sim}{\to} A$.

2.2.5 Mixedness for homotopy types

In this section, I will suppose that $k = \mathbb{F}_q$ is a finite field, and that K is the fraction field of the Witt vectors $W = W(k)$ of k. Frobenius will mean the q-power Frobenius. In §6 of [35], Kim and Hain define mixedness for a Frobenius dga, and prove that if X/k is a geometrically connected, smooth k-variety, with good compactification, then the F-dga that they define to represent the rational homotopy type of X is mixed. I wish to extend their results to show that the rigid rational homotopy type of any k-variety X is mixed, and the proof is in three steps:

- a comparison between my rigid homotopy type and their crystalline homotopy type, when both are defined;

- a descent result for rigid homotopy types, which will follow easily from the corresponding theorem in cohomology;
- a result stating that mixedness is preserved under this descent operation.

So let (Y,M) be a geometrically connected, log-smooth and proper k-variety, such that the log structure M comes from a strict normal crossings divisor $D\subset Y$. The reader is referred to *loc. cit.* for the definition of the crystalline rational homotopy type $A_{(Y,M)}$ of (Y,M); this is a K-dga with a Frobenius structure. Let $Y^\circ = Y\setminus D$ be the complement of D. .

Proposition 2.2.8. *There is a quasi-isomorphism* $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y^\circ/K}))\cong A_{(Y,M)}$.

Proof. Choose a finite open affine covering $\{U_i\}$ of Y, let $Y_0=\coprod_i U_i$ and let $Y_\bullet\to Y$ be the associated Čech hyper-covering. The pullback to $Y_\bullet$ of the log structure on Y is defined by some strict normal crossings divisor $H_\bullet\subset Y_\bullet$. Since everything is affine, both $Y_\bullet$ and the divisor defining the log structure lift to characteristic zero, so there exists an exact closed immersion $Y_\bullet\to Z_\bullet$ into a smooth simplicial log scheme over W. Let $D_\bullet$ be the divided power envelope of $Y_\bullet$ in $Z_\bullet$, and $\widehat{D}_\bullet$ its p-adic completion. There is thus a diagram of cosimplicial dga's

$$\begin{array}{ccc}\mathbf{R}\Gamma(]Y_\bullet[_{\widehat{Z}_\bullet 0},j^\dagger\Omega^*_{]Y_\bullet[_{\widehat{Z}_\bullet 0}}) & \longleftarrow & \mathbf{R}\Gamma(]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0},\omega^*_{]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}}) \\ & & \downarrow \\ \mathbf{R}\Gamma(D_{\bullet K},\omega^*_{D_\bullet}\otimes_W K) & \longrightarrow & \mathbf{R}\Gamma(\widehat{D}_{\bullet K},\omega^*_{\widehat{D}_\bullet}\otimes_W K)\end{array} \tag{2.20}$$

where:

- the rigid space $]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}$ together with its logarithmic de Rham complex is defined as in §2.2 of [47];
- the top horizontal arrow comes from the natural morphism $]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}\to]Y_\bullet[_{\widehat{Z}_\bullet 0}$;
- the right hand vertical arrow comes from the fact that writing $\omega^*_{\widehat{Z}_\bullet}$ for the logarithmic de Rham complex on $\widehat{Z}_\bullet$,

$$\omega^*_{]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}}\cong(\omega^*_{\widehat{Z}_\bullet}\otimes_W K)|_{]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}} \tag{2.21}$$

$$\omega^*_{\widehat{D}_\bullet}\cong\omega^*_{\widehat{Z}_\bullet}\otimes_{\mathcal{O}_{\widehat{Z}_\bullet}}\mathcal{O}_{\widehat{D}_\bullet} \tag{2.22}$$

  and the natural map $\widehat{D}_{\bullet K}\to\widehat{Z}_{\bullet K}$ factors though $]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}$;

- the bottom horizontal arrow is given by p-adic completion.

Now apply the functor Th to obtain the diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y^\circ/K})) & \longleftarrow & \mathrm{Th}\left(\mathbf{R}\Gamma(]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}, \omega^*_{]Y_\bullet[^{\log}_{\widehat{Z}_\bullet 0}})\right) \\ & & \downarrow \\ A_{(Y,M)} & \longrightarrow & \mathrm{Th}\left(\mathbf{R}\Gamma(\widehat{D}_{\bullet K}, \omega^*_{\widehat{D}_\bullet} \otimes_W K)\right) \end{array} \tag{2.23}$$

where the isomorphism

$$\mathrm{Th}\left(\mathbf{R}\Gamma(]Y_\bullet[_{\widehat{Z}_\bullet 0}, j^\dagger\Omega^*_{]Y_\bullet[_{\widehat{Z}_\bullet 0}})\right) \cong \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y^\circ/K})) \tag{2.24}$$

comes from using cohomological descent for partially overconvergent cohomology and the isomorphism

$$\mathrm{Th}\left(\mathbf{R}\Gamma(D_{\bullet K}, \omega^*_{D_\bullet} \otimes_W K)\right) \cong A_{(Y,M)} \tag{2.25}$$

is in §4 of [35].

I claim that all these morphisms are in fact quasi-isomorphisms. Indeed, the cohomology groups of the top left dga are rigid cohomology groups of Y°, those of the top right are the log-analytic cohomology groups of (Y, M) in the sense of Chapter 2 of [47], those of the bottom right are log-convergent cohomology groups of (Y, M), and those of the bottom left are log-crystalline cohomology groups of (Y, M), tensored with K.

On cohomology, the top horizontal and right vertical arrows are the comparison maps between rigid and log-analytic cohomology and log-analytic and log-convergent cohomology defined in §2.4 and §2.3 of *loc. cit.* respectively, where they are proved to be isomorphisms. The bottom horizontal arrow is the comparison map between log crystalline and log convergent cohomology, which is proved to be an isomorphism in *loc. cit.* □

Now let X be a k-variety, and $Y_\bullet \to X$ a simplicial k-variety mapping to X. Then there is an augmented cosimplicial object

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \to \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y_\bullet/K})) \tag{2.26}$$

in F-Ho(dga$_K$), which induces a morphism

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \to \mathrm{Th}(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y_\bullet/K}))). \tag{2.27}$$

The descent theorem I will need is the following.

Proposition 2.2.9. *Suppose that $Y_\bullet \to X$ is a proper hyper-covering. Then*

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \to \mathrm{Th}(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y_\bullet/K}))) \tag{2.28}$$

is an isomorphism in F-$\mathrm{Ho}(\mathrm{dga}_K)$.

Proof. I may obviously ignore both the F-structure, and the algebra structure. But now it follows from Proposition 2.1.4 together with cohomological descent for rigid cohomology that the induced morphism on cohomology is an isomorphism. □

Remark 2.2.10. The reader might object that $\mathrm{Th}(-)$ does not make sense as a functor on $\mathrm{Ho}(\mathrm{dga}_K)^\Delta$. However, this does not matter since in the only place where I wish to apply this result (namely Theorem 2.2.13) below, there a specific object of dga_K^Δ representing $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y_\bullet/K})) \in \mathrm{Ho}(\mathrm{dga}_K)^\Delta$.

I now recall Kim and Hain's definition of mixedness for an F-dga over K.

Definition 2.2.11. Say that $A \in F$-dga_K is mixed if there exists a quasi-isomorphism $A \simeq B$ in F-dga_K and a multiplicative filtration $W_\bullet B$ of B such that $H^{p-q}(\mathrm{Gr}^W_p(B))$ is pure of weight q for all p, q. Say A is strongly mixed if the filtration can be chosen on A itself.

Lemma 2.2.12. *Let $A^\bullet$ be a cosimplicial K-dga with Frobenius action, such that each A^n is strongly mixed. Assume moreover that the cosimplicial structure is compatible with the filtrations. Then $\mathrm{Th}(A^\bullet)$ is mixed.*

Proof. First forget the algebra structure on $A^\bullet$, and treat it as just a cosimplicial complex of K-modules. There are then two filtrations on $A^\bullet$ - one coming from the weight filtration W on each A^n, and the other coming from the filtration by simplicial degree. This induces two filtrations W and D on $\mathrm{Tot}_N(A^\bullet) := \mathrm{Tot}(N(A^\bullet))$; define F to be the convolution $D*W$ of these filtrations. Similarly define the filtration F on the un-normalised total complex $\mathrm{Tot}(A^\bullet)$ (where the chain maps in one direction are the alternating sums of the coface maps), and there is a filtered quasi-isomorphism

$$\mathrm{Tot}(A^\bullet) \simeq \mathrm{Tot}_N(A^\bullet) \tag{2.29}$$

arising from the usual comparison of Tot and Tot_N. Now

$$\begin{aligned} H^{p-q}(\mathrm{Gr}^F_p \mathrm{Tot}_N(A^\bullet)) &= H^{p-q}(\mathrm{Gr}^F_p \mathrm{Tot}(A^\bullet)) && (2.30)\\ &= H^{p-q}(\mathrm{Tot}(\mathrm{Gr}^F_p A^\bullet)) && (2.31)\\ &= \bigoplus_{i+j=p} H^{p-q}(\mathrm{Tot}(\mathrm{Gr}^D_i \mathrm{Gr}^W_j A^\bullet)) && (2.32)\\ &= \bigoplus_i H^{p-i-q}(\mathrm{Gr}^W_{p-i} A^i) && (2.33)\end{aligned}$$

which is pure of weight q. To take account of the multiplicative structure on $A^\bullet$, simply use Lemme 6.4 of [39], which says that the complex $\mathrm{Tot}_N(A^\bullet)$ considered above, with the filtration $D * W$, is filtered quasi-isomorphic (as a filtered complex) to $\mathrm{Th}(A^\bullet)$ with a certain naturally defined multiplicative filtration. □

The proof that the rigid rational homotopy type is mixed is now straightforward.

Theorem 2.2.13. *Let k be a finite field, and $K = \mathrm{Frac}(W(k))$. Let X be a geometrically connected k-variety. Then the rational homotopy type $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ is mixed.*

Proof. By de Jong's theorem on alterations, there exists a proper hyper-covering $X_\bullet \to X$ such that $X_\bullet$ admits a good compactification, that is an embedding $X_\bullet \to Y_\bullet$ into a smooth and proper simplicial k-scheme with complement a strict normal crossings divisor on each level Y_n. Let M_n be the log structure associated to this divisor. By Propositions 2.2.8 and 2.2.9 there a quasi-isomorphism

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \cong \mathrm{Th}\left(A_{(Y_\bullet, M_\bullet)}\right) \tag{2.34}$$

of dga's with Frobenius. Let $\mathrm{Spec}(k)^\circ$ denote the scheme $\mathrm{Spec}(k)$ with the log structure of the punctured point, and let $(Y_\bullet, M_\bullet^\circ)$ denote the pullback of $(Y_\bullet, M_\bullet)$ via the natural morphism $\mathrm{Spec}(k)^\circ \to \mathrm{Spec}(k)$. Since log crystalline cohomology in [35] is calculated relative to the log structure induced on $\mathrm{Spec}(W(k))$ via the Teichmüller lift from that on $\mathrm{Spec}(k)$, it follows that there is a Frobenius invariant, level-wise quasi-isomorphism

$$A_{(Y_\bullet, M_\bullet^\circ)} \cong A_{(Y_\bullet, M_\bullet)} \tag{2.35}$$

as cosimplicial dga's. Hence there is also a quasi-isomorphism

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \cong \mathrm{Th}\left(A_{(Y_\bullet, M_\bullet^\circ)}\right) \tag{2.36}$$

of dga's with Frobenius. Now, although each $A_{(Y_n, M_n^\circ)}$ is not strongly mixed, each is quasi-isomorphic to one that is, call it $\tilde{A}_{(Y_n, M_n^\circ)}$ (this is the dga $TW(W\tilde{\omega}[u])$ in the notation of *loc. cit.* - note that since Y is assumed to be smooth, I can work with the dga $W\tilde{\omega}[u]$ rather than $C(W\tilde{\omega}[u]))$. This dga is functorial in (Y, M) in exact the same manner as $A_{(Y,M^\circ)}$. Moreover, the weight filtrations on these dga's are also functorial, and hence the result now follows from Lemma 2.2.12 and the corresponding result in the log-smooth and proper case, which is Theorem 3 of *loc. cit.* □

Remark 2.2.14. In what follows I will generally replace $A_{(Y,M)}$ by this quasi-isomorphic strongly mixed complex $\tilde{A}_{(Y,M^\circ)}$ - since the latter is functorial in $(\overline{Y}, M)$ this will not cause any problems.

Remark 2.2.15. Strictly speaking, Kim and Hain's definition cannot be applied to the object $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ since the Frobenius action is only in the homotopy category. However, Theorem 3.47 of [42] allows this action to be lifted to the category dga_K, uniquely up to quasi-isomorphism. Alternatively, since there is (up to functorial quasi-isomorphism) a Frobenius action on each dga $A_{(\overline{Y}_n,M_n)}$, this can be used to put a Frobenius action on $\mathrm{Th}(A_{(\overline{Y}_\bullet,M_\bullet)})$. The comparison theorem then says that after applying the functor $F\text{-}\mathrm{dga}_K \to F\text{-}\mathrm{Ho}(\mathrm{dga}_K)$ this is isomorphic to $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$.

If $x \in X(k)$ is a point, then I can use similar methods to the previous section to define an object $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}), x)$ in the homotopy category of augmented F-dga's over K, where the augmentation comes from 'pulling back' to the point x. All the above comparison isomorphisms go through in this augmented situation, as does the definition of mixedness. Thus as in §6 of [35], if X/k is geometrically connected, then the bar complex $B(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}), x))$ associated to the augmented F-dga $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}), x)$ is mixed.

Recall that the homotopy groups of X/k are defined by

$$\pi_1^{\mathrm{rig}}(X, x) = \mathrm{Spec}(H^0(B(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}), x)))) \tag{2.37}$$

$$\pi_n^{\mathrm{rig}}(X, x) = (QH^{n-1}(B(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}), x))))^\vee, \quad n \geq 2. \tag{2.38}$$

where Q is the functor of indecomposable cohomology classes.

Corollary 2.2.16. *Let X/k be a geometrically connected variety, and $x \in X(k)$. Then the rational homotopy groups $\pi_n^{\mathrm{rig}}(X, x)$ are mixed for all $n \geq 1$.*

Remark 2.2.17. For $n = 1$ this means that $H^0(B(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}), x)))$ is mixed.

Although I have proved that there is a mixed structure on the rational homotopy type of a k-variety X, in order to define such a structure, I needed to choose a log smooth and proper resolution $(\overline{Y}_\bullet, M_\bullet) \to X$ of X. Hence *a priori* the filtration on $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ depends on this resolution. Thus the question remains of how 'independent' this structure is of the resolution chosen. In order to answer this question, I will need to talk about the different notions of equivalence for filtered dga's, as well as tidying up the slightly sloppy definition of the mixed structure on $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ given above.

Suppose that $f : A \to B$ is a filtered morphism between filtered dga's. That is A and B are equipped with multiplicative filtrations, and f is compatible with the filtrations. Thus f defines a morphism

$$E_r^{\bullet,\bullet}(f) : E_r^{\bullet,\bullet}(A) \to E_r^{\bullet,\bullet}(B) \tag{2.39}$$

between the spectral sequences associated to the filtrations on A and B.

Definition 2.2.18. Say that f is an E_r quasi-isomorphism if $E_{r+1}^{p,q}(f)$ is an isomorphism for all p, q.

Remark 2.2.19. The notion of filtered quasi-isomorphism of filtered complexes used above exactly corresponds to an E_0-quasi-isomorphism. It is also worth noting that filtered dga's do not form a model category.

I want to consider the following categories, as well as the obvious augmented versions:

- F-Ho(dga$_K$), the category of F-objects in Ho(dga$_K$). This is where the rational homotopy type $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ lives;
- $F^{\mathcal{M}}$-dga$_K$ - the category of mixed Frobenius dga's over K - i.e. Frobenius dga's A^* with a filtration $P_\bullet$ which is regular ($P_iA^* = 0$ for $i \ll 0$), exhaustive ($\cup_i P_iA^* = A^*$) and is such that $H^{q-p}(\mathrm{Gr}_p^P(A^*))$ is pure of weight q. Thanks to the work of Kim and Hain, for $(\overline{Y}, D)$ a smooth and proper k-variety with strict normal crossings divisor D, the rational homotopy type $A_{(\overline{Y},D)}$ can be viewed functorially as an object in this category;
- for each $r \geq 0$, the category Ho$_r$($F^{\mathcal{M}}$-dga$_K$) which is the localisation of $F^{\mathcal{M}}$-dga$_K$ with respect to E_r-quasi-isomorphisms. Note that due to the results of [30] this category is locally small, i.e. morphisms between any two objects form a set, rather than just a class.

An E_r-quasi-isomorphism of regular, exhausted filtered complexes is a quasi-isomorphism (Théorème 2.1 of *loc. cit.*), hence there are obvious forgetful functors

$$\mathrm{Ho}_r(F^{\mathcal{M}}\text{-dga}_K) \to F\text{-Ho}(\mathrm{dga}_K) \tag{2.40}$$

for each r. Choosing a resolution $(\overline{Y}_\bullet, D_\bullet) \to X$ of a k-variety X gives an isomorphism

$$\mathrm{Th}(A_{(\overline{Y}_\bullet,D_\bullet)}) \cong \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \tag{2.41}$$

in F-Ho(dga$_K$). The question then remains, in what sense is $\mathrm{Th}(A_{(\overline{Y}_\bullet,D_\bullet)})$ independent of the resolution chosen?

Lemma 2.2.20. *The object* $\mathrm{Th}(A_{(\overline{Y}_\bullet, D_\bullet)}) \in \mathrm{Ho}_1(F^{\mathcal{M}}\text{-dga}_K)$ *in the category of mixed complexes localised at* E_1*-quasi-isomorphisms depends only on* X*.*

Remark 2.2.21. Hence $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ may be viewed canonically as an object of the category $\mathrm{Ho}_1(F^{\mathcal{M}}\text{-dga}_K)$.

Proof. Since for any two resolutions there is a third mapping to both, it suffices to prove that any quasi-isomorphism between mixed complexes is in fact an E_1-quasi-isomorphism. But this follows easily from the fact that the spectral sequence degenerates at the E_2-page. □

Of course, in the same manner, for any rational point $x \in X(k)$ the augmented dga $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}), x)$ can be viewed as an object in the category

$$\mathrm{Ho}_1(F^{\mathcal{M}}\text{-dga}^*_K) \tag{2.42}$$

where the $*$ refers to the fact that dga's considered are augmented.

Let DGA_K denote the category of commutative dga's over K which are not necessarily concentrated in non-negative degrees, similar notation will be used for unbounded mixed Frobenius dga's. As proved in §6 of [35], the bar construction for dga's can be extended to a functor

$$B : F^{\mathcal{M}}\text{-dga}^*_K \to F^{\mathcal{M}}\text{-DGA}_K \tag{2.43}$$

or in other words, the bar complex of a mixed, augmented Frobenius dga is a mixed Frobenius dga (they do not explicitly state that the filtration is regular, but it follows easily from the proof of Lemma 10 of *loc. cit.*). Let

$$H^* : F^{\mathcal{M}}\text{-dga}_K \to F^{\mathcal{M}}\text{-dga}_K \tag{2.44}$$

denote the cohomology functor, as well as the corresponding version for unbounded dga's. Denote by $\mathrm{Ho}_1^{\mathrm{con}}(F^{\mathcal{M}}\text{-dga}^*_K)$ the localised category of augmented dga's with connected cohomology.

Lemma 2.2.22. *There are factorisations*

$$\begin{gathered} H^* \circ B : \mathrm{Ho}_1^{\mathrm{con}}(F^{\mathcal{M}}\text{-dga}^*_K) \to F^{\mathcal{M}}\text{-DGA}_K \\ H^* : \mathrm{Ho}_1(F^{\mathcal{M}}\text{-dga}_K) \to F^{\mathcal{M}}\text{-dga}_K \end{gathered} \tag{2.45}$$

Proof. By the proof of the previous lemma (any quasi-isomorphism between mixed complexes is an E_1-quasi-isomorphism - this applies to unbounded dga's as well), the first factorisation follows from the fact that the bar complex sends quasi-isomorphisms between dga's with connected cohomology to quasi-isomorphisms. The seconds is easier, and just uses the fact that E_1-quasi-isomorphisms are quasi-isomorphisms. □

Corollary 2.2.23. *Let X/k be geometrically connected. Then the mixed structures on the cohomology ring $H^*_{\mathrm{rig}}(X/K) \in F^{\mathcal{M}}\text{-dga}_K$ and the homotopy groups $\pi_n^{\mathrm{rig}}(X,x)$, $n \geq 1$ for any $x \in X(k)$, are independent of the resolution chosen.*

Remark 2.2.24. It should be noted that the $\pi_1^{\mathrm{rig}}(X,x)$ referred to in the above Corollary is not, *a priori*, the same as that constructed using the Tannakian category of unipotent overconvergent isocrystals, and hence this does no immediately give a weight filtration on the co-ordinate ring of the latter object. However, I will prove later on (§2.5) that they do coincide, thus proving the existence of a weight filtration on the (co-ordinate ring of the) 'usual' unipotent rigid fundamental group.

2.2.6 Homotopy obstructions

I now briefly discuss a crystalline homotopy obstruction to the existence of maps between varieties, and of sections of maps between k-varieties, which is nothing more than an application of the functoriality of the previous section. For any variety X/k, $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ is an object of $F\text{-Ho}(\mathrm{dga}_K)$, functorial in X, and so there is a map

$$\mathrm{Mor}_{\mathrm{Sch}/k}(Y,X) \to [\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})), \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y/K}))]_{F\text{-Ho}(\mathrm{dga}_K)}. \tag{2.46}$$

where $[\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})), \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y/K}))]_{F\text{-Ho}(\mathrm{dga}_K)}$ denotes the set of all morphisms $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})) \to \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{Y/K}))$ in $F\text{-Ho}(\mathrm{dga}_K)$. This can be used to study the set of maps from Y to X, or the set of sections of a given map $X \to Y$

I will not pursue this idea, since I actually wish to develop a more refined homotopical approach to studying sections. To motivate why this better approach is needed, consider the morphism

$$\mathbb{A}^1_k \to \mathbb{A}^1_k, \quad x \mapsto x^2 \tag{2.47}$$

which clearly does not have a section. However, this cannot be detected on the level of rational homotopy types, since $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{\mathbb{A}^1_k/K})) = K$. Instead, I will develop a relative rational homotopy type which will associate to any morphism $X \to Y$ a dga on Y (in a sense that will be made clear later) in a functorial manner. Before I do so, however, I will first give an alternative perspective on the rigid rational homotopy type.

2.3 Overconvergent sheaves and homotopy types

In this section I wish to describe a different way to construct the rational homotopy type of a k-variety X, using the theory of modules on a certain 'overconvergent' site attached to X/K, the definition of which is due to Le Stum. To motivate this slightly altered perspective, it may be helpful to discuss the analogous situation in characteristic zero. So let $X/\mathbb{C}$ be a smooth, proper algebraic variety - then the rational homotopy type of X is defined to be $\mathbf{R}\Gamma_{\mathrm{Zar}}(\Omega^*_X)$, using similar methods to those seen already. Why does this give the 'right' answer?

The reason is that after passing to the analytic topology of X, Ω^*_X is quasi-isomorphic, as a dga, to the constant sheaf of dga's $\underline{\mathbb{C}}$, and standard theorems comparing Zariski and analytic cohomology of coherent sheaves will then give an isomorphism $\mathbf{R}\Gamma_{\mathrm{an}}(\underline{\mathbb{C}}) \cong \mathbf{R}\Gamma_{\mathrm{Zar}}(\Omega^*_X)$ in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{C}})$. The former is then the 'correct' rational homotopy type of X, essentially because of Théorème 5.5 of [40]. So there is a functor $\mathbf{R}\Gamma_{\mathrm{an}} : \mathrm{Ho}(\mathrm{dga}(X^{\mathrm{an}}, \mathbb{C})) \to \mathrm{Ho}(\mathrm{dga}_{\mathbb{C}})$, and $\mathbf{R}\Gamma_{\mathrm{Zar}}(\Omega^*_X)$ gives a way of *computing* $\mathbf{R}\Gamma_{\mathrm{an}}(\underline{\mathbb{C}})$ in an algebraic fashion.

There is now an obvious third candidate for defining the rational homotopy type of X - simply consider the constant crystal $\mathcal{O}_{X/\mathbb{C}}$ on the infinitesimal site of $X/\mathbb{C}$, and take $\mathbf{R}\Gamma_{\mathrm{inf}}(\mathcal{O}_{X/\mathbb{C}})$ in the sense of dga's, rather than complexes. Tracing through Grothendieck's comparison theorems will then show that this is naturally isomorphic to $\mathbf{R}\Gamma_{\mathrm{Zar}}(\Omega^*_X)$ in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{C}})$. This can now be easily transposed into positive characteristic, since the infinitesimal site has a good analogue in rigid cohomology - the overconvergent site of Le Stum. This leads to the second definition of the rigid rational homotopy type, namely as $\mathbf{R}\Gamma(\mathcal{O}^\dagger_{X/K})$, where $\mathcal{O}^\dagger_{X/K}$ is the constant crystals on the overconvergent site, and $\mathbf{R}\Gamma$ is taken in the sense of dga's.

2.3.1 The overconvergent site

I first recall the definition of Le Stum's overconvergent site, and give a new definition of the rational homotopy type. The main reference is [49]. I will systematically consider analytic spaces in the sense of Berkovich, and will call an *analytic variety* over K a locally Hausdorff, good, strictly K-analytic space. If V is an analytic variety, denote by V_0 the underlying rigid space, and $\pi_V : V_0 \to V$ the natural map. If $\mathscr{P}$ is a formal $\mathcal{V}$-scheme, $\mathscr{P}_K$ will denote its Berkovich generic fibre, and $\mathscr{P}_{K0}$ its rigid generic fibre. (This is the reason for putting 0's everywhere in the previous section). Recall that a Berkovich space is called good if every point has an affinoid neighbourhood.

Definition 2.3.1. An overconvergent variety over $\mathcal{V}$ consists of the data of a k-variety X, a formal $\mathcal{V}$-scheme $\mathscr{P}$ and an analytic K-variety V, together with an embedding $X \subset \mathscr{P}$ of formal $\mathcal{V}$-schemes, and a morphism $\lambda : V \to \mathscr{P}_K$ of Berkovich spaces. An overconvergent variety will often be denoted $(X \subset \mathscr{P} \leftarrow V)$. A morphism of overconvergent varieties is a commutative diagram

$$\begin{array}{ccccccc} X' & \hookrightarrow & \mathscr{P}' & \xleftarrow{\mathrm{sp}} & \mathscr{P}'_K & \xleftarrow{\lambda'} & V' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_K} & & \downarrow{\scriptstyle u} \\ X & \hookrightarrow & \mathscr{P} & \xleftarrow{\mathrm{sp}} & \mathscr{P}_K & \xleftarrow{\lambda} & V \end{array} \tag{2.48}$$

and the category of overconvergent varieties over $\mathcal{V}$ is denoted $\mathrm{An}(\mathcal{V})$.

For $(X \subset \mathscr{P} \leftarrow V)$ an overconvergent variety, define the tube $]X[_V = (\mathrm{sp} \circ \lambda)^{-1}(X) \subset V$. Denote by $i_{X,V} :]X[_V \to V$ the natural inclusion. (I will often write i_X instead.) A morphism of overconvergent varieties is called a strict neighbourhood if $X = X'$, $\mathscr{P} = \mathscr{P}'$ $u : V' \to V$ is the inclusion of an open neighbourhood of $]X[_V$ in V, and $]X[_{V'} =]X[_V$. In *loc. cit.* Le Stum proves that the category $\mathrm{An}(\mathcal{V})$ admits calculus of right fractions with respect to strict neighbourhoods, and denotes by $\mathrm{An}^\dagger(\mathcal{V})$ the localised category.

The category $\mathrm{An}(\mathcal{V})$ admits a topology coming from the analytic topology of V, and this induces a topology on $\mathrm{An}^\dagger(\mathcal{V})$, called the analytic topology. Since the formal scheme $\mathscr{P}$ plays less of a role in the category $\mathrm{An}^\dagger(\mathcal{V})$, objects are usually denoted by e.g. (X, V). The functor $(X, V) \mapsto \Gamma(]X[_V, i_{X,V}^{-1}\mathcal{O}_V)$ is then well defined, and is a sheaf of $\mathrm{An}^\dagger(\mathcal{V})$, denoted $\mathcal{O}^\dagger_{\mathcal{V}}$ and called the sheaf of overconvergent functions. .

If $u : (Y, W) \to (X, V)$ is a morphism in $\mathrm{An}^\dagger(\mathcal{V})$, denote by $u^\dagger$ the canonical pullback functor from $i_{X,V}^{-1}\mathcal{O}_V$-modules to $i_{Y,W}^{-1}\mathcal{O}_W$-modules. Thus in the usual way a $\mathcal{O}^\dagger_{\mathcal{V}}$-module E can be described as a collection $\{E_{(X,V)}\}$ of $i_{X,V}^{-1}\mathcal{O}_V$-modules, one for each overconvergent variety (X, V), together with morphisms $u^\dagger E_{(X,V)} \to E_{(Y,W)}$ for each morphism $u : (Y, W) \to (X, V)$, satisfying the obvious compatibilities. The various $i_X^{-1}\mathcal{O}_V$-modules $E_{(X,V)}$ will be referred to as the realisations of E.

Fix some object (C, O) of $\mathrm{An}^\dagger(\mathcal{V})$, and consider the restricted category $\mathrm{An}^\dagger(C, O)$ of all objects of $\mathrm{An}^\dagger(\mathcal{V})$ over (C, O). Denote by $j_{C,O} : \mathrm{An}^\dagger(C, O) \to \mathrm{An}^\dagger(\mathcal{V})$ the corresponding morphism of sites. Le Stum defines a morphism of sites $I_{C,O} : \mathrm{Sch}(C) \to \mathrm{An}^\dagger(C, O)$ (where $\mathrm{Sch}(C)$ is given the coarse topology), given by $I_{C,O}^{-1}(X, V) = X$.

Definition 2.3.2. Let X be an algebraic variety over C, let $\underline{X}$ be the corresponding representable sheaf on the site $\mathrm{Sch}(C)$. Then the sheaf of overconvergent varieties over X above (C, O) is by definition $X/O := j_{C,O!} I_{C,O*}\underline{X}$. The site $\mathrm{An}^\dagger(X/O)$ is the restricted site of objects of $\mathrm{An}^\dagger(\mathcal{V})$ over X/O, and the corresponding topos is denoted $(X/O)_{\mathrm{An}^\dagger}$. The restriction of $\mathcal{O}^\dagger_{\mathcal{V}}$ to $\mathrm{An}^\dagger(X/O)$ will be denoted $\mathcal{O}^\dagger_{X/O}$.

For any morphism of C-varieties $f : X \to Y$ there is a morphism of sheaves $X/O \to Y/O$ and hence a morphism of topoi $f_{\mathrm{An}^\dagger} : (X/O)_{\mathrm{An}^\dagger} \to (Y/O)_{\mathrm{An}^\dagger}$. Since $f_{\mathrm{An}^\dagger}^{-1}(\mathcal{O}^\dagger_{Y/O}) = \mathcal{O}^\dagger_{X/O}$, this naturally becomes a morphism of ringed topoi. Letting $p : X \to C$ denote the structure morphism of a C-variety X, the functor $O' \mapsto (C, O')$ defines a morphism of sites $\mathrm{An}^\dagger(C, O) \to \mathrm{Open}(]C[_O)$. Consider the composite morphism of topoi $p_{X/O} : (X/O)_{\mathrm{An}^\dagger} \to (C, O)_{\mathrm{An}^\dagger} \to]C[^{\mathrm{an}}_O$.

If (C, O) is an overconvergent variety, denote by $(-)^{\mathrm{an}}$ the derived push-forward functor $\mathbf{R}\pi_*$ for the morphism of ringed spaces

$$\pi : (]\overline{C}[_{O_0}, j^\dagger \mathcal{O}_{]\overline{C}[_{O_0}}) \to (]\overline{C}[_O, i_{C*} i_C^{-1} \mathcal{O}_{]\overline{C}[_O}) \tag{2.49}$$

where $\overline{C}$ denote the closure of C inside the 'unmentioned' formal scheme of (C, O). The main results of *loc. cit.* are the following.

Theorem 2.3.3. *([49], Theorem 3.6.7). Let $\mathscr{S}$ be a good formal $\mathcal{V}$-scheme, and consider the object $(\mathscr{S}_k, \mathscr{S}_K)$ of* $\mathrm{An}^\dagger(\mathcal{V})$*. Let X be an algebraic variety over $\mathscr{S}_k$, with structure morphism p.*

1. *There is a canonical equivalence between the category of finitely presented $\mathcal{O}^\dagger_{X/\mathscr{S}_K}$-modules and the category of overconvergent isocrystals on $X/\mathscr{S}$.*

2. *For any overconvergent isocrystal E on $X/\mathscr{S}$, there is an functorial isomorphism $(\mathbf{R}p_{X/\mathscr{S},\mathrm{rig}*}E)^{\mathrm{an}} \cong \mathbf{R}p_{X/\mathscr{S}_K*}E$.*

When $\mathscr{S} = \mathrm{Spf}(\mathcal{V})$, I will often write Γ instead of $p_{X/K*}$, thus for a finitely presented $\mathcal{O}^\dagger_{X/K}$-module E, the above result becomes an isomorphism

$$H^i_{\mathrm{rig}}(X/K, E) \cong \mathbf{R}^i\Gamma((X/O)_{\mathrm{An}^\dagger}, E). \tag{2.50}$$

This result is not quite enough for my purposes - I want to be able to take any smooth triple $(S, \overline{S}, \mathscr{S})$, which is not accounted for in Le Stum's comparison theorem. However, this extension is straightforward.

Theorem 2.3.4. *Let $(S, \overline{S}, \mathscr{S})$ be a smooth triple, with $\mathscr{S}$ good, and $p : X \to S$ a morphism of k-varieties. Let E be an overconvergent isocrystal on $(X/\mathscr{S})$. Let $i_S :]S[_{\mathscr{S}} \to]\overline{S}[_{\mathscr{S}}$ denote the (closed) inclusion. Then there is a quasi-isomorphism*

$$i_S^{-1}(\mathbf{R}p_{X/\mathscr{S},\mathrm{rig}*}E)^{\mathrm{an}} \cong (\mathbf{R}p_*E)_{(S,\mathscr{S}_K)}. \tag{2.51}$$

Proof. It suffices to show that $(\mathbf{R}p_{X/\mathscr{S},\mathrm{rig}*}E)^{\mathrm{an}} \cong i_{S*}(\mathbf{R}p_*E)_{(S,\mathscr{S}_K)}$, and the proof of this is virtually word for word the same as in the proof of Theorem 3.6.7 of [49], taking care that in the proof of Proposition 3.5.8 of *loc. cit.* one must replace the analytic space $\mathscr{S}_K$ by $]\overline{S}[_{\mathscr{S}}$ (and similarly in the rigid case) and the equality $\mathbf{R}v_{K*} \circ i_{X*} = \mathbf{R}v_{K*}$ by the equality $\mathbf{R}v_{K*} \circ i_{X*} = i_{S*} \circ \mathbf{R}v_{K*}$ □

Let $\mathrm{dga}(\mathcal{O}^{\dagger}_{X/K})$ denote the category of sheaves of $\mathcal{O}^{\dagger}_{X/K}$-dga's. If $f : X \to Y$ is a morphism of k-varieties, then since $f^{-1}(\mathcal{O}^{\dagger}_{Y/K}) = \mathcal{O}^{\dagger}_{X/K}$, the functor $f_* : \mathrm{dga}(\mathcal{O}^{\dagger}_{X/K}) \to \mathrm{dga}(\mathcal{O}^{\dagger}_{Y/K})$ is right Quillen, and thus derives to give

$$\mathbf{R}f_* : \mathrm{Ho}(\mathrm{dga}(\mathcal{O}^{\dagger}_{X/K})) \to \mathrm{Ho}(\mathrm{dga}(\mathcal{O}^{\dagger}_{Y/K})). \tag{2.52}$$

There is also the absolute version

$$\mathbf{R}\Gamma : \mathrm{Ho}(\mathrm{dga}(\mathcal{O}^{\dagger}_{X/K})) \to \mathrm{Ho}(\mathrm{dga}_K). \tag{2.53}$$

The definition of the rational homotopy type of a k-variety is now straightforward.

Definition 2.3.5. Define the rational homotopy type of X to be the dga $\mathbf{R}\Gamma(\mathcal{O}^{\dagger}_{X/K}) \in \mathrm{Ho}(\mathrm{dga}_K)$. If $f : X \to Y$ is a morphism of k-varieties, then the relative rational homotopy type of X over Y is $\mathbf{R}f_*(\mathcal{O}^{\dagger}_{X/K}) \in \mathrm{Ho}(\mathrm{dga}(\mathcal{O}^{\dagger}_{Y/K}))$.

It is easy to check that the rational homotopy type is functorial, and so there is a map

$$\mathrm{Mor}_{\mathrm{Sch}/k}(X,Y) \to [\mathbf{R}\Gamma(\mathcal{O}^{\dagger}_{Y/K}), \mathbf{R}\Gamma(\mathcal{O}^{\dagger}_{X/K})]_{\mathrm{Ho}(\mathrm{dga}_K)}. \tag{2.54}$$

Similarly, for every morphism $f : X \to Y$ there is a map from the set of sections of f to the set of sections (taking care with contravariance!) of the induced map $\mathbf{R}\Gamma(\mathcal{O}^{\dagger}_{Y/K}) \to \mathbf{R}\Gamma(\mathcal{O}^{\dagger}_{X/K})$ in $\mathrm{Ho}(\mathrm{dga}_K)$. There is also the obvious relative version of this.

2.3.2 A comparison theorem

In this section, I prove the following comparison result.

Theorem 2.3.6. *There is an isomorphism*

$$\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^{\dagger}_{X/K})) \cong \mathbf{R}\Gamma(\mathcal{O}^{\dagger}_{X/K}) \tag{2.55}$$

in $\mathrm{Ho}(\mathrm{dga}_K)$.

The idea is that after using simplicial methods to (essentially) reduce to the case where there exists an embedding of X into a smooth formal $\mathcal{V}$-scheme, it only really needs observing that Le Stum's comparison of rigid cohomology and cohomology of the overconvergent site respects multiplicative structures.

So suppose that $\mathfrak{U}_\bullet = (U_\bullet, \overline{U}_\bullet, \mathscr{U}_\bullet)$ is a framing system for X, with $U_\bullet \to X$ a Zariski hyper-covering. Define the category $\mathrm{dga}(\mathcal{O}^{\dagger}_{U_\bullet/K})$ of dga's on the simplicial ringed topos $(U_\bullet/K)_{\mathrm{An}^\dagger}$ in the standard way. As before, consider the functor of Thom-Whitney global sections

$$\mathbf{R}\Gamma_{\mathrm{Th}} = \mathrm{Th} \circ \mathbf{R}\Gamma : \mathrm{Ho}(\mathrm{dga}(\mathcal{O}^{\dagger}_{U_\bullet/K})) \to \mathrm{Ho}(\mathrm{dga}_K). \tag{2.56}$$

There is also an obvious restriction functor $(-)|_{U_\bullet} : \mathrm{dga}(\mathcal{O}^{\dagger}_{X/K}) \to \mathrm{dga}(\mathcal{O}^{\dagger}_{U_\bullet/K})$, thus giving two functors

$$\mathrm{Ho}(\mathrm{dga}(\mathcal{O}^{\dagger}_{X/K})) \underset{\mathbf{R}\Gamma}{\overset{\mathbf{R}\Gamma_{\mathrm{Th}}\circ(-)|_{U_\bullet}}{\rightrightarrows}} \mathrm{Ho}(\mathrm{dga}_K). \tag{2.57}$$

Note that there is an obvious natural transformation $\mathbf{R}\Gamma \Rightarrow \mathbf{R}\Gamma_{\mathrm{Th}} \circ (-)|_{U_\bullet}$.

Proposition 2.3.7. *This natural transformation is an isomorphism when evaluated on* $\mathcal{O}^{\dagger}_{X/K}$.

Proof. As usual, it suffices to show that it induces an isomorphism on cohomology. But this just follows from cohomological descent for overconvergent cohomology, see Section 3.6 of [49]. □

I now want to extend Le Stum's overconvergent version of 'linearisation of differential operators' to deal both with dga's and with simplicial Berkovich spaces. To start with, consider the following diagram of simplicial ringed topoi

$$\begin{array}{ccc} (U_\bullet,]U_\bullet[_{\mathscr{U}_\bullet})_{\mathrm{An}^\dagger} & \xrightarrow{\varphi_{\mathfrak{U}_\bullet}} &]U_\bullet[_{\mathscr{U}_\bullet} \\ \Big\downarrow{\scriptstyle j_{\mathfrak{U}_\bullet}} & & \\ (U_\bullet/K)_{\mathrm{An}^\dagger} & & \end{array} \tag{2.58}$$

where:

- $(U_\bullet/K)_{\mathrm{An}^\dagger}$ is as described above, $]U_\bullet[_{\mathscr{U}_\bullet}$ is the tube associated to $(U_\bullet, \mathscr{U}_\bullet)$ and $(U_\bullet,]U_\bullet[_{\mathscr{U}_\bullet})_{\mathrm{An}^\dagger}$ is the simplicial topos of sheaves on $\mathrm{An}^\dagger(\mathcal{V})$ over the representable simplicial sheaf associated to $(U_\bullet,]U_\bullet[_{\mathscr{U}_\bullet})$;
- $j_{\mathfrak{U}_\bullet}$ arises from the natural morphism $(U_\bullet,]U_\bullet[_{\mathscr{U}_\bullet}) \to (U_\bullet/K)$ of simplicial sheaves, and $\varphi_{\mathfrak{U}_\bullet}$ is the 'realisation map'. For more details, see §1.4 and §2.1 of *loc. cit.*

The induced maps $j_{\mathfrak{U}_\bullet}^{-1}(\mathcal{O}^{\dagger}_{U_\bullet/K}) \to \mathcal{O}^{\dagger}_{U_\bullet/]U_\bullet[_{\mathscr{U}_\bullet}}$ and $\varphi_{\mathfrak{U}_\bullet}^{-1}(K) \to \mathcal{O}^{\dagger}_{U_\bullet/]U_\bullet[_{\mathscr{U}_\bullet}/K}$ are both flat, and so there derived linearisation of the overconvergent de Rham dga

$$\mathbf{R}L(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K}) := \mathbf{R}j_{\mathfrak{U}_\bullet *}\varphi_{\mathfrak{U}_\bullet}^*(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K}) \tag{2.59}$$

as in Chapter 3 of *loc. cit.*, exists as an object of $\mathrm{Ho}(\mathrm{dga}(\mathcal{O}^{\dagger}_{U_\bullet/K}))$.

Proposition 2.3.8. *There is an isomorphism*

$$\mathcal{O}^\dagger_{U_\bullet/K} \to \mathbf{R}L(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K}) \tag{2.60}$$

in $\mathrm{Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{U_\bullet/K}))$.

Proof. To define the morphism, it suffices to define a morphism

$$\mathcal{O}^\dagger_{U_\bullet/K} \to j_{\mathfrak{U}_\bullet *}\varphi^*_{\mathfrak{U}_\bullet}(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K}) \tag{2.61}$$

in $\mathrm{dga}(\mathcal{O}^\dagger_{U_\bullet/K})$, or equivalently a map $\mathcal{O}^\dagger_{U_n/K} \to j_{\mathfrak{U}_n *}\varphi^*_{\mathfrak{U}_n}(i_{U_n}^{-1}\mathcal{O}_{(\mathscr{U}_n)_K})$ of K-algebras, functorially in n, such that the composite map $\mathcal{O}^\dagger_{U_n/K} \to j_{\mathfrak{U}_n *}\varphi^*_{\mathfrak{U}_n}(i_{U_n}^{-1}\Omega^1_{(\mathscr{U}_n)_K})$ is zero. But exactly as in Proposition 3.3.10 of [49], since $\mathcal{O}^\dagger_{U_n/K}$ is a crystal, $j_{\mathfrak{U}_\bullet}^{-1}\mathcal{O}^\dagger_{U_\bullet/K} \cong \varphi^*_{\mathfrak{U}_\bullet}(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K})$, and hence this map arises via the adjunction between $j_{\mathfrak{U}_\bullet *}$ and $j_{\mathfrak{U}_\bullet}^{-1}$. To prove that the induced map $\mathcal{O}^\dagger_{U_\bullet/K} \to \mathbf{R}L(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K})$ is a quasi-isomorphism, I may forget the algebra structure, and prove that it is a quasi-isomorphism of complexes in each simplicial degree. But by definition, the map $\mathcal{O}^\dagger_{U_n/K} \to \mathbf{R}L(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K})_n$ is exactly the augmentation map that Le Stum constructs. That this is a quasi-isomorphism is then Proposition 3.5.4. of *loc. cit.* □

Proposition 2.3.9. *There is an isomorphism*

$$\mathbf{R}\Gamma(\mathbf{R}L(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K})) \cong \mathbf{R}\Gamma(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K})) \tag{2.62}$$

in $\mathrm{Ho}(\mathrm{dga}_K)^\Delta$.

Proof. Just note that the proof of Proposition 3.3.9 of *loc. cit.* carries over *mutatis mutandis* to the simplicial/dga situation. □

Combining these two results, in order to complete the proof of Theorem 2.3.6, I just need to verify that there is a canonical isomorphism

$$\mathbf{R}\Gamma(j^\dagger\Omega^*_{V_0(\mathfrak{U})_\bullet}) \cong \mathbf{R}\Gamma(i_{U_\bullet}^{-1}\Omega^*_{(\mathscr{U}_\bullet)_K}) \tag{2.63}$$

in $\mathrm{Ho}(\mathrm{dga}_K)^\Delta$. I may work level-wise, where there is a natural map

$$\mathbf{R}\Gamma(j^\dagger\Omega^*_{V_0(\mathfrak{U})_n}) \to \mathbf{R}\Gamma(i_{U_n}^{-1}\Omega^*_{(\mathscr{U}_n)_K}) \tag{2.64}$$

which comes from the map of topoi $V_0(\mathfrak{U})_n \to (\mathscr{U}_n)_K$ and the comparison between $j^\dagger\Omega^*_{]\overline{U}_n[_{\mathscr{U}_{n0}}}$ and $i_{U_n}^{-1}\Omega^*_{(\mathscr{U}_n)_K}$, as in Proposition 3.4.3 of [49]. To show that it is an isomorphism is I may forget the algebra structure, and invoke Le Stum's results from §3 of *loc. cit.*

2.3.3 Frobenius structures

I am now in a position to prove that the rigid rational homotopy type is functorial. Indeed, it is clear that the definition in terms of the overconvergent site is functorial, and it is also not too difficult to see by functoriality of the comparison morphism that the map $f^* : \mathbb{R}\Gamma(\mathcal{O}^\dagger_{Y/K}) \to \mathbb{R}\Gamma(\mathcal{O}^\dagger_{X/K})$ induced by any morphism $f : X \to Y$ is the same as that induced by any lift of f to a map $\mathfrak{f}$ between framing systems for X and Y. In particular, this latter map is independent of the lift $\mathfrak{f}$.

In order to put Frobenius structures on the dga's obtained from the overconvergent site, Le Stum's base change morphism 1.4.6 of [49] will need to be examined slightly more closely. So suppose that $\alpha : K \to K'$ is a finite extension of complete, discretely valued fields, and let $\mathcal{V} \to \mathcal{V}'$ (resp. $k \to k'$) be the induced finite extension of rings of integers (resp. residue fields). Then there is a morphism of sites

$$\alpha : \mathrm{An}^\dagger(\mathcal{V}') \to \mathrm{An}^\dagger(\mathcal{V}) \tag{2.65}$$

which is induced by $(X \subset \mathscr{P} \leftarrow V) \mapsto (X_{k'} \subset \mathscr{P}_{\mathcal{V}'} \leftarrow V_{K'})$. This base extension functor has an adjoint, which considers an overconvergent variety (Y, W) over K' as one over K - note that this holds only if the extension $K \to K'$ is finite. Hence the pull-back morphism α^{-1} on presheaves has a simple description - namely $(\alpha^{-1}\mathcal{F})(Y, W) = \mathcal{F}(Y, W)$ where on the LHS (Y, W) is considered as an overconvergent variety over K', and on the RHS as one over K. In particular, $\alpha^{-1}(\mathcal{O}^\dagger_{\mathcal{V}}) = \mathcal{O}^\dagger_{\mathcal{V}'}$, and α extends to a morphism of ringed sites.

Now suppose that X is a k-variety, and consider the sheaf (X/K) on $\mathrm{An}^\dagger(\mathcal{V})$, which is the sheafification of the presheaf $(C, O) \mapsto \mathrm{Mor}_k(C, X)$. By the above comments, $\alpha^{-1}(X/K)$ is the sheafification of the presheaf $(C', O') \mapsto \mathrm{Mor}_k(C', X) = \mathrm{Mor}_{k'}(C', X_{k'})$. Thus $\alpha^{-1}(X/K) = (X_{k'}/K')$, and hence there is a morphism of ringed topoi

$$(X_{k'}/K')_{\mathrm{An}^\dagger} \to (X/K)_{\mathrm{An}^\dagger}. \tag{2.66}$$

More generally, exploiting functoriality of $(Y/K')_{\mathrm{An}^\dagger}$ in Y as a k'-variety, for any k' variety Y and any commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k) \end{array} \tag{2.67}$$

there is an induced morphism of ringed topoi

$$f : (Y/K')_{\mathrm{An}^\dagger} \to (X/K)_{\mathrm{An}^\dagger} \tag{2.68}$$

such that that $f^{-1}(\mathcal{O}^\dagger_{X/K}) = \mathcal{O}^\dagger_{Y/K'}$. The situation I am interested in is when $\sigma : K \to K$ is a lifting of the absolute Frobenius on k, and $F_X : X \to X$ is the absolute Frobenius on X. Then there is a morphism

$$F_X : (X/K)_{\mathrm{An}^\dagger} \to (X/K)_{\mathrm{An}^\dagger} \tag{2.69}$$

of ringed topoi, and if $f : X \to Y$ is a morphism of k-varieties, then there is a commutative square

$$\begin{array}{ccc} (X/K)_{\mathrm{An}^\dagger} & \xrightarrow{F_X} & (X/K)_{\mathrm{An}^\dagger} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (Y/K)_{\mathrm{An}^\dagger} & \xrightarrow{F_Y} & (Y/K)_{\mathrm{An}^\dagger}. \end{array} \tag{2.70}$$

Hence there is a base change map

$$\Phi_{X/Y} : F_Y^{-1}\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K}) \to \mathbf{R}f_*(\mathcal{O}^\dagger_{X/K}) \tag{2.71}$$

in $\mathrm{Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{Y/K}))$.

Proposition 2.3.10. *If $Y = \mathrm{Spec}(k)$ is a point, then $\Phi_{X/k}$ is a quasi-isomorphism.*

Proof. This is a straightforward application of the comparison theorem. It is not too difficult to check that this is compatible base change, and hence the morphism induced by $\Phi_{X/k}$ on cohomology is the usual Frobenius on rigid cohomology, which is an isomorphism. □

Remark 2.3.11. Similarly to the problem of functoriality, the map $\Phi_{X/k}$ is the same as the map induced by a lift of the absolute Frobenius to a framing system for X. Again, this implies that the latter is independent of the choice of this lift.

Remark 2.3.12. I do not know whether or not $\Phi_{X/Y}$ is a quasi-isomorphism in general. It would follow, for example, if it Frobenius was known to be bijective on relative overconvergent cohomology.

2.4 Relative crystalline homotopy types

In this section, I define relative rational homotopy types, and again there will be two approaches - one via rigid cohomology and cohomological descent and one via the overconvergent site of Le Stum.

In rigid cohomology, the relative theory is expressed with respect to a base frame. I will also systematically work with pairs of varieties over k, that is I will work in the category consisting of open immersions $S \to \overline{S}$ of k-varieties, and where morphisms are commutative diagrams. The reason I do this is to more easily apply the results of [20] on cohomological descent.

Fix a base frame $\mathfrak{S} = (S, \overline{S}, \mathscr{S})$, which I assume to be smooth and proper over $\mathcal{V}$. Although most of what follows will work in greater generality, I will be mainly interested in the case where S is a smooth, geometrically connected curve over k, $\overline{S}$ is its unique compactification, and $\mathscr{S}$ is a lifting of $\overline{S}$ to a smooth formal curve over $\mathcal{V}$.

Definition 2.4.1. Say that a frame $\mathfrak{U} = (U, \overline{U}, \mathscr{U})$ over $\mathfrak{S}$ is smooth if $\mathscr{U} \to \mathscr{S}$ is smooth in a neighbourhood of U, and proper if $\mathscr{U} \to \mathscr{S}$ is.

Definition 2.4.2. Let $(X, \overline{X})$ be a pair of varieties over k, that is an open immersion of separated k-schemes of finite type. Let $f : (X, \overline{X}) \to (S, \overline{S})$ be a morphism of pairs. Then an $(X, \overline{X})$-frame over $\mathfrak{S}$ is a frame $\mathfrak{Y} = (Y, \overline{Y}, \mathscr{Y})$ over $\mathfrak{S}$ together with a morphism $(Y, \overline{Y}) \to (X, \overline{X})$ such that the diagram

$$\begin{array}{ccc} (Y, \overline{Y}) & \longrightarrow & (X, \overline{X}) \\ & \searrow & \downarrow \\ & & (S, \overline{S}) \end{array} \tag{2.72}$$

commutes.

Definition 2.4.3. Let $f : (X, \overline{X}) \to (S, \overline{S})$ be as above. Then define a framing system for f to be simplicial $(X, \overline{X})$-frame $\mathfrak{Y}_\bullet = (Y_\bullet, \overline{Y}_\bullet, \mathscr{Y}_\bullet)$ over $\mathfrak{S}$, such that each $\mathfrak{Y}_n$ is smooth over $\mathfrak{S}$, and which is universally de Rham descendable, in the sense of [20], Definition 10.1.3.

Of course, the definition is rigged exactly to apply Chiarellotto and Tsuzuki's theory of cohomological descent for relative rigid cohomology. Since I am really interested in the case of a morphism $X \to S$, I need to check that I am not unduly restricting the scope of the theory.

Proposition 2.4.4. *Suppose that $X \to S$ is a morphism of k-varieties. Then there exists a a pair $(X, \overline{X})$ and a morphism of pairs $f : (X, \overline{X}) \to (S, \overline{S})$ such that $\overline{X}$ is proper over S and f admits a framing system.*

Proof. That there exists a proper $\overline{S}$-scheme $\overline{X}$ and a morphism of pairs $f : (X, \overline{X}) \to (S, \overline{S})$ as claimed is Nagata's compactification theorem.

By Example 6.1.3, (1) of [52], it suffices to show that there exists a Zariski covering of $(X,\overline{X})$ over $\mathfrak{S}$, that is an $(X,\overline{X})$ frame $\mathfrak{U}=(U,\overline{U},\mathscr{U})$ which is smooth over $\mathfrak{S}$, such that $\overline{u}:\overline{U}\to X$ is an open covering and $U=\overline{u}^{-1}(X)$. Now, since $\overline{X}$ is separated and of finite type over $\mathrm{Spec}(k)$, there exists an open affine cover $\overline{U}_i$ of $\overline{X}$, and a closed embedding $\overline{U}_i\hookrightarrow\mathbb{A}_k^{n_i}$ into some affine space over k. Now define $\overline{U}=\coprod_i\overline{U}_i$, U to be the pull-back of $\overline{U}\to\overline{X}$ to X. Since $\overline{X}\to\overline{S}$ is proper, it is an open mapping onto its (closed) image, and hence there exists an open subset $\mathscr{S}_i$ of $\mathscr{S}$ such that for each i induced map $\overline{U}_i\to\widehat{\mathbb{A}}_{\mathcal{V}}^{n_i}\times_{\mathcal{V}}\mathscr{S}_i$ is a closed immersion. Thus setting $\mathscr{U}=\coprod_i\widehat{\mathbb{A}}_{\mathcal{V}}^{n_i}\times_{\mathcal{V}}\mathscr{S}_i$ gives the required Zariski cover $(U,\overline{U},\mathscr{U})$ of $(X,\overline{X})$ over $\mathfrak{S}$. □

Now proceed exactly as in the previous section, simply replacing the frame $\mathrm{Sp}(K)=(\mathrm{Spec}(k),\mathrm{Spec}(k),\mathrm{Spf}(\mathcal{V}))$ everywhere by $\mathfrak{S}$. If $f:(X,\overline{X})\to(S,\overline{S})$ is a morphism of pairs and $\mathfrak{f}:\mathfrak{Y}_\bullet\to\mathfrak{S}$ is a framing system for f, then $V_0(\mathfrak{Y}_\bullet):=]\overline{Y}_\bullet[_{\mathscr{Y}_\bullet 0}$ is a simplicial rigid space over $]\overline{S}[_{\mathscr{S}0}$. Let $\mathrm{dga}(V_0(\mathfrak{Y}_\bullet),j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}})$ denote the category of sheaves of $j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}$-dga's on the simplicial space $V_0(\mathfrak{Y}_\bullet)$.

Exactly as in the absolute case, consider the derived push-forward functor

$$\mathbf{R}_{\mathrm{Th}}\mathfrak{f}_{K0*}:\mathrm{Ho}(\mathrm{dga}(V_0(\mathfrak{Y}_\bullet),j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}))\to\mathrm{Ho}(\mathrm{dga}(]\overline{S}[_{\mathscr{S}0},j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}))\tag{2.73}$$

as well as for each n the sheaf of $j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}$-dga's $j^\dagger\Omega^*_{]\overline{Y}_n[_{\mathscr{Y}_n0}/]\overline{S}[_{\mathscr{S}0}}$ which fit together to gives a sheaf of $j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}$-dga's $j^\dagger\Omega^*_{]\overline{Y}_\bullet[_{\mathscr{Y}_\bullet 0}/]\overline{S}[_{\mathscr{S}0}}$ on $V_0(\mathfrak{Y}_\bullet)$.

Definition 2.4.5. Define the relative rigid rational homotopy type to be

$$\mathbf{R}_{\mathrm{Th}}f_*(\Omega^*(\mathcal{O}^\dagger_{X/S})):=\mathbf{R}_{\mathrm{Th}}\mathfrak{f}_{K0*}(j^\dagger\Omega^*_{]\overline{Y}_\bullet[_{\mathscr{Y}_\bullet 0}/]\overline{S}[_{\mathscr{S}0}})\in\mathrm{Ho}(\mathrm{dga}(]\overline{S}[_{\mathscr{S}0},j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}})).\tag{2.74}$$

As noted above, the relative rational homotopy type may also be defined using the functoriality of the overconvergent site. A morphism $f:X\to S$ of varieties induces a functor

$$\mathbf{R}f_*:\mathrm{Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{X/K}))\to\mathrm{Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{S/K}));\tag{2.75}$$

define the relative rational homotopy type to be $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})$. This has some advantages over the previous definition - it is obvious that it only depends on $f:X\to S$ and not on any choice of compactification or framing system, and subject to certain base change results, it will give a Gauss–Manin connection on the relative homotopy type. However, it is not particularly computable, and in order to do any calculations, first definition is needed.

2.4.1 Another comparison theorem

In this section, I will prove a comparison theorem between the two approached to relative rigid rational homotopy types. Notation will be exactly as above. The realisation functor

$$\mathrm{Mod}(\mathcal{O}^\dagger_{S/K}) \to \mathrm{Mod}(i_S^{-1}\mathcal{O}_{]\overline{S}[_{\mathscr{S}}}) \tag{2.76}$$

$$E \mapsto E_{(S,]\overline{S}[_{\mathscr{S}})} \tag{2.77}$$

is exact, hence extends to a functor

$$\mathrm{Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{S/K})) \to \mathrm{Ho}(\mathrm{dga}(]S[_{\mathscr{S}}, i_S^{-1}\mathcal{O}_{]\overline{S}[_{\mathscr{S}}})) \tag{2.78}$$

$$\mathscr{A}^* \mapsto \mathscr{A}^*_{(S,]\overline{S}[_{\mathscr{S}})}. \tag{2.79}$$

Recall the morphism of topoi $\pi_{]\overline{S}[_{\mathscr{S}}} :]\overline{S}[_{\mathscr{S}0} \to]\overline{S}[_{\mathscr{S}}$, and that

$$\mathbf{R}\pi_{]\overline{S}[_{\mathscr{S}}*}(j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}) = \pi_{]\overline{S}[_{\mathscr{S}}*}(j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}) = i_{S*}i_S^{-1}\mathcal{O}_{]\overline{S}[_{\mathscr{S}}}. \tag{2.80}$$

Lemma 2.4.6. *The induced morphism* $\pi^{-1}_{]\overline{S}[_{\mathscr{S}}}(i_{S*}i_S^{-1}\mathcal{O}_{]\overline{S}[_{\mathscr{S}}}) \to j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}}$ *is flat.*

Proof. After replacing V_0 by the G-topology on V, what I must show is that for V a good analytic variety, $W \subset V$ a closed sub-variety, which is open for the G-topology, and $\pi_V : V_G \to V$ the natural map, the induced morphism $\pi_V^{-1}\pi_{V*}(j^\dagger_{W_G}\mathcal{O}_{V_G}) \to j^\dagger_{W_G}\mathcal{O}_{V_G}$ is flat. But this just follows because for any two G-open $U' \subset U$ subsets of V, the map $\Gamma(\mathcal{O}_{V_G}, U) \to \Gamma(\mathcal{O}_{V_G}, U')$ is flat. □

Hence there is an induced functor

$$i_S^{-1} \circ \mathbf{R}\pi_{]\overline{S}[_{\mathscr{S}}*} : \mathrm{Ho}(\mathrm{dga}(]\overline{S}[_{\mathscr{S}0}, j^\dagger\mathcal{O}_{]\overline{S}[_{\mathscr{S}0}})) \to \mathrm{Ho}(\mathrm{dga}(]S[_{\mathscr{S}}, i_S^{-1}\mathcal{O}_{]\overline{S}[_{\mathscr{S}}})). \tag{2.81}$$

Theorem 2.4.7. *There is a natural isomorphism*

$$\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,]\overline{S}[_{\mathscr{S}})} \to i_S^{-1} \circ \mathbf{R}\pi_{]\overline{S}[_{\mathscr{S}}*}(\mathrm{Th}(\mathbf{R}\mathfrak{f}_{K0*}(j^\dagger\Omega^*_{]\overline{Y}_\bullet[_{\mathscr{Y}_\bullet 0}/]\overline{S}[_{\mathscr{S}0}}))) \tag{2.82}$$

in $\mathrm{Ho}(\mathrm{dga}(]S[_{\mathscr{S}}, i_S^{-1}\mathcal{O}_{]\overline{S}[_{\mathscr{S}}}))$.

Proof. The proof is almost word for word the same as in the absolute case, taking into account the corresponding statement for cohomology, which is Theorem 2.3.4, and its proof, which is essentially contained in Chapter 3 of [49]. □

Remark 2.4.8. The comparison theorem can be easily extended to take Frobenius structures into account.

2.4.2 Crystalline complexes and the Gauss–Manin connection

One of the advantages of a 'crystalline' definition of the relative rational homotopy type is in the interpretation of the Gauss–Manin connection. Deriving the notion of a crystal gives a sensible definition of what it means for a complex, or dga, to be crystalline, and the existence of the Gauss–Manin connection is essentially equivalent to $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})$ being crystalline. Unfortunately, at the moment, I cannot prove that this is the case, I can only show that it would follow from a certain 'generic coherence' result, for which some evidence is given.

Definition 2.4.9. Suppose that $\mathcal{E}$ is a complex of $\mathcal{O}^\dagger_{X/K}$-modules.

1. Say that $\mathcal{E}$ is quasi-bounded above if each realisation $\mathcal{E}_{(Y,V)}$ is bounded above.
2. Say that $\mathcal{E}$ is crystalline if it is quasi-bounded above, and for each morphism $u : (Z,W) \to (Y,V)$ of overconvergent varieties over (X/K), the induced map $\mathbf{L}u^\dagger\mathcal{E}_{(Y,V)} \to \mathcal{E}_{(Z,W)}$ is an isomorphism in $D^-(i_Z^{-1}\mathcal{O}_W)$.

An $\mathcal{O}^\dagger_{X/K}$-dga $\mathscr{A}^*$ is said to be crystalline if the underlying complex is crystalline.

As note above, the reason that I am interested in crystalline dga's is that they give a good interpretation of the Gauss–Manin connection, as I now explain.

Suppose that there is a morphism of k-varieties $f : X \to S$ as above, and a smooth and proper triple $(S, \overline{S}, \mathscr{S})$, and that it can be shown that $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})$ is crystalline. Let $p_i :]S[_{\mathscr{S}^2} \to]S[_{\mathscr{S}}$ denote the two natural projections. Then the crystalline nature of $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K}) \in \mathrm{Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{S/K}))$ together with flatness of the p_i means that there are natural quasi-isomorphisms of dga's

$$p_1^\dagger \mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,\mathscr{S}_K)} \to \mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,\mathscr{S}^2_K)} \tag{2.83}$$

$$p_2^\dagger \mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,\mathscr{S}_K)} \to \mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,\mathscr{S}^2_K)} \tag{2.84}$$

and hence there is an isomorphism

$$p_1^\dagger \mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,\mathscr{S}_K)} \to p_2^\dagger \mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,\mathscr{S}_K)} \tag{2.85}$$

in $\mathrm{Ho}(\mathrm{dga}(]S[_{\mathscr{S}}, i_S^{-1}\mathcal{O}_{\mathscr{S}^2_K}))$. In other words, there is a Gauss–Manin connection on the realisation $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(S,\mathscr{S}_K)}$, which can be transported over into the rigid world using the comparison theorem between rigid and overconvergent relative rational homotopy types.

Proposition 2.4.10. *Assume that there is some $U \subset Y$ open such that every $\mathbf{R}^q f_*(\mathcal{O}^\dagger_{X/K})|_U$ is a finitely presented crystal. Then $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})$ is a crystalline dga.*

Of course, this is really a statement about complexes, rather than dga's. I first show that $\mathbf{R}f_*(\mathcal{O}^{\dagger}_{X/K})$ is quasi-bounded above.

Lemma 2.4.11. *Let (C,O) be an overconvergent variety, and $p : X \to C$ a k-variety over C. Then the complex $\mathbf{R}p_{X/O*}(\mathcal{O}^{\dagger}_{X/O}) \in D^+(i_C^{-1}\mathcal{O}_O)$ is bounded above.*

Proof. By the spectral sequence associated to a finite open covering (Corollary 3.6.4 of [49]), I may assume that X is affine, and hence p has a geometric realisation $(X,V) \to (C,O)$. In fact, I may choose a realisation of the following form. Let $(C \hookrightarrow \mathscr{S} \leftarrow O)$ be a triple representing (C,O), and choose an embedding $X \hookrightarrow \mathscr{P}$ of X into a smooth and proper formal $\mathcal{V}$-scheme. Then a geometric realisation of $X \to C$ is given by

$$\begin{array}{ccccc} X & \longrightarrow & \mathscr{P} \times_{\mathcal{V}} \mathscr{S} & \longleftarrow & V = \mathscr{P}_K \times_K O \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & \mathscr{S} & \longleftarrow & O. \end{array} \tag{2.86}$$

By Theorem 3.5.3 of *loc. cit.*, I must show that $\mathbf{R}p_{]X[_V*}(i_X^{-1}\Omega^*_{V/O})$ is bounded above. Since each term is a coherent $i_X^{-1}\mathcal{O}_V$-module, by the usual spectral sequence relating the cohomology of the complex to the cohomology of each term, it will suffice to show that $\mathbf{R}p_{]X[_V*}$ sends coherent $i_X^{-1}\mathcal{O}_V$-modules to complexes which are bounded above. In fact I claim that $\mathbf{R}^i p_{]X[_V*}\mathscr{F} = 0$ for any coherent $i_X^{-1}\mathcal{O}_V$-module $\mathscr{F}$ and any $i \geq 2$.

The question is local on O, which I may therefore assume to be affinoid (recall that all the analytic varieties are assumed good). Now, exactly as in ??? there exist affinoid subspaces $V_{n,m} \subset V$ such that $]X[_V \subset \bigcup_n V_{n,m}$ for all m, and for each n, $V_{n,m}$ is a cofinal system of neighbourhoods of $]X[_V \cap (\cup_m V_{n,m})$ in $\cup_m V_{n,m}$. Since coherent $\mathcal{O}$-modules are acyclic on affinoids, and filtered colimits of sheaves are exact, it follows that

$$\mathbf{R}p_{]X[_V*}\mathscr{F} \cong \mathbf{R}\varprojlim_n(\varinjlim_m p_{]X[_V*}\mathscr{F}). \tag{2.87}$$

The claim follows from the fact that $\mathbf{R}^i \varprojlim_n$ vanishes for $i \geq 2$. □

Corollary 2.4.12. *Let $f : X \to Y$ be a morphism of k-varieties. Then $\mathbf{R}f_*(\mathcal{O}^{\dagger}_{X/K})$ is quasi-bounded above.*

Proof. Combine the above proposition with Proposition 3.5.2 of [49]. □

Definition 2.4.13 ([49], Definition 3.6.1)**.** A complex of $\mathcal{O}^{\dagger}_{Y/K}$-modules E is said to be of Zariski type if for any overconvergent variety (C,O) over (Y/K), and any open $U \subset O$, with corresponding closed immersion $i :]U[_O \to]C[_O$, the natural map $i^{-1}E_{(C,O)} \simeq E_{(U,O)}$ is a quasi-isomorphism.

Remark 2.4.14. Note that the corresponding statement is always true for a closed subscheme $Z \subset C$, since then the tube $]Z[_O \subset]C[_O$ is open.

Lemma 2.4.15. *Let E be a quasi-bounded above complex of $\mathcal{O}^{\dagger}_{Y/K}$-modules of Zariski type. Let $j : U \to Y$ be an open immersion, with closed complement $i : Z \to Y$. Then E is crystalline iff j^*E and i^*E are both crystalline.*

Proof. Let $g : (C', O') \to (C, O)$ be a morphism of overconvergent varieties over (Y/K), then letting e.g. C_U denote $C \times_Y U$, there is a diagram

$$\begin{array}{ccccc} (C'_U, O') & \xrightarrow{j'} & (C', O') & \xleftarrow{i'} & (C'_Z, O') \\ \downarrow{\scriptstyle g_U} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_Z} \\ (C_U, O) & \xrightarrow{j} & (C, O) & \xleftarrow{i} & (C_Z, O) \\ \downarrow & & \downarrow & & \downarrow \\ (U/K) & \xrightarrow{j} & (Y/K) & \xleftarrow{i} & (Z/K) \end{array} \tag{2.88}$$

and since $]C'[_{O'}$ is covered by $]C'_U[_{O'}$ and $]C'_Z[_{O'}$, to prove that the morphism

$$\mathbf{L}g^{\dagger}E_{(C,O)} \to E_{(C',O')} \tag{2.89}$$

is a quasi-isomorphism, it suffices to prove that the two morphisms

$$i'^{-1}\mathbf{L}g^{\dagger}E_{(C,O)} \to i'^{-1}E_{(C',O')} \tag{2.90}$$

$$j'^{-1}\mathbf{L}g^{\dagger}E_{(C,O)} \to j'^{-1}E_{(C',O')} \tag{2.91}$$

are quasi-isomorphisms. But now using the hypothesis that E is of Zariski type and that j^*E and i^*E are crystalline, together with 2.3.2 of [49], it follows that

$$i'^{-1}\mathbf{L}g^{\dagger}E_{(C,O)} = \mathbf{L}i'^{\dagger}\mathbf{L}g^{\dagger}E_{(C,O)} = \mathbf{L}g_Z^{\dagger}\mathbf{L}i^{\dagger}E_{(C,O)} \tag{2.92}$$

$$= \mathbf{L}g_Z^{\dagger}i^{-1}E_{(C,O)} = \mathbf{L}g_Z^{\dagger}E_{(Z,O)} \tag{2.93}$$

$$\simeq E_{(Z',O')} = i'^{-1}E_{(C',O')} \tag{2.94}$$

and

$$j'^{-1}\mathbf{L}g^{\dagger}E_{(C,O)} = \mathbf{L}j'^{\dagger}\mathbf{L}g^{\dagger}E_{(C,O)} = \mathbf{L}g_U^{\dagger}\mathbf{L}j^{\dagger}E_{(C,O)} \tag{2.95}$$

$$= \mathbf{L}g_U^{\dagger}j^{-1}E_{(C,O)} \simeq \mathbf{L}g_U^{\dagger}E_{(U,O)} \tag{2.96}$$

$$\simeq E_{(U',O')} \simeq j'^{-1}E_{(C',O')}. \tag{2.97}$$

□

To apply this to $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})$, I will need the following result.

Lemma 2.4.16. *Let $f : X \to Y$ be a morphism of k-varieties. Then $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})$ is of Zariski type.*

Proof. Choose an overconvergent variety (C, O) over Y/K, and let $U \subset C$ be an open subset with corresponding inclusion $i :]U[_O \to]C[_O$ of tubes. Note that the question is local on both C and U so I may assume that $U \cong D(f)$ for some $f \in \Gamma(C, \mathcal{O}_C)$. By using cohomological descent first for proper hyper-covers and then for Zariski covers of X (Theorem 1.1 of [55]) I may in fact assume that X_C is smooth and affine, and admits a geometric realisation (X_C, V') where $V' =]\overline{X}_C[_{V'}$ is quasi-compact

Thus $X_C \to C$ is has a geometric realisation of the form $\mathbf{g} : (X_C, V) \to (C, O)$ where $V = V' \times_K O$. By quasi-compactness and the corresponding statement an affinoid analytic spaces over K, coherent $i_X^{-1}\mathcal{O}_V$-modules are $\mathbf{R}\mathbf{g}_{K*}$-acyclic, thus

$$\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(C,O)} = \mathbf{R}\mathbf{g}_{K*}(i_{X_C}^{-1}\Omega^*_{V/O}) = \mathbf{g}_{K*}(i_{X_C}^{-1}\Omega^*_{V/O}) \tag{2.98}$$

$$\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})_{(U,O)} = \mathbf{R}\mathbf{g}_{K*}(i_{X_U}^{-1}\Omega^*_{V/O}) = \mathbf{g}_{K*}(i_{X_U}^{-1}\Omega^*_{V/O}). \tag{2.99}$$

Write $\mathscr{F}^* = i_{X_C}^{-1}\Omega^*_{V/O}$, and let $i' :]X_U[_V \to]X_C[_V$ and $\mathbf{g}'_K :]X_U[_V \to]U[_O$ denote the induced maps. So there is a Cartesian square

$$\begin{array}{ccc}]X_U[_V & \xrightarrow{i'} &]X_C[_V \\ {\scriptstyle \mathbf{g}'_K}\downarrow & & \downarrow{\scriptstyle \mathbf{g}_K} \\ ]U[_O & \xrightarrow{i} &]C[_O \end{array} \tag{2.100}$$

and I need to show that the base change map

$$i^{-1}\mathbf{g}_{K*}\mathscr{F}^* \to \mathbf{g}'_{K*}i'^{-1}\mathscr{F}^* \tag{2.101}$$

is a quasi-isomorphism. Note that $]U[_O$ is given by $\{x \in]C[_O \mid |f(x)| \geq 1\}$, and $]X_U[_V$ by $\{y \in]X_C[_V \mid |f(\mathbf{g}_K(x))| \geq 1\}$. Hence for any open set W of $]C[_O$, a cofinal system of open neighbourhoods of $W \cap]U[_O$ in W is given by $T_\eta := W \cap \{x \in]C[_O \mid |f(x)| > \eta\}$ for $\eta < 1$, and a cofinal system of neighbourhoods of $\mathbf{g}_K^{-1}(W) \cap]X_U[_V$ in $\mathbf{g}_K^{-1}(W)$ is given by $\mathbf{g}_K^{-1}(W) \cap \{y \in]X_C[_V \mid |f(\mathbf{g}_K(x))| > \eta\} = \mathbf{g}_K^{-1}(T_\eta)$ for $\eta < 1$. Hence it follows straight from the definition that $i^{-1}\mathbf{g}_{K*} = \mathbf{g}'_{K*}i'^{-1}$ as required. □

Now to complete the reduction to proving a 'generic' crystalline result, I need a base change theorem for cohomology of the overconvergent site.

Lemma 2.4.17. *Suppose*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{2.102}$$

is a Cartesian diagram of k-varieties. Then for any sheaf $E \in (X/K)_{\mathrm{An}^\dagger}$ the base change homomorphism

$$g^*\mathbf{R}f_*E \to \mathbf{R}f'_*g'^*E \tag{2.103}$$

is an isomorphism.

Proof. Given the definitions, this is actually pretty formal - since $(X/K)_{\mathrm{An}^\dagger}$, $(Y'/K)_{\mathrm{An}^\dagger}$ and $(X'/K)_{\mathrm{An}^\dagger}$ can all be viewed as open subtopoi of $(Y/K)_{\mathrm{An}^\dagger}$. However, it can also be shown directly using realisations (and §3.5 of [49]) as follows. Let (C, O) be an overconvergent variety over (Y'/K). Then

$$(g^*\mathbf{R}f_*E)_{(C,O)} = (\mathbf{R}f_*E)_{(C,O)} \tag{2.104}$$
$$= \mathbf{R}p_{X\times_Y C/O*}E|_{X\times_Y C/O} \tag{2.105}$$
$$= \mathbf{R}p_{X'\times_{Y'} C/O*}E|_{X'\times_{Y'} C/O} \tag{2.106}$$
$$= \mathbf{R}p_{X'\times_{Y'} C/O*}(g^*E)|_{X'\times_{Y'} C/O} \tag{2.107}$$
$$= (\mathbf{R}f'_*g'^*E)_{(C,O)} \tag{2.108}$$

as required. □

Hence using Noetherian induction on Y, to prove that $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})$ is crystalline, it suffices to prove that it is generically crystalline, i.e. that there exists an open subset $U \subset Y$ such that $\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K})|_U$ is crystalline.

Lemma 2.4.18. *Suppose that $E \in D^+(\mathcal{O}^\dagger_{Y/K})$ is a quasi-bounded above complex of $\mathcal{O}^\dagger_{Y/K}$-modules. If $\mathcal{H}^q(E)$ is a finitely presented crystal for all q, then E is crystalline.*

Proof. The key point is to show that the realisations of a finitely presented $\mathcal{O}^\dagger_{Y/K}$-module are flat. Indeed, this implies that for any morphism $\mathbf{g} : (C', O') \to (C, O)$ of overconvergent varieties over (Y/K),

$$\mathcal{H}^q(\mathbf{Lg}^\dagger_K E_{(C,O)}) \cong \mathbf{g}^\dagger_K \mathcal{H}^q(E_{(C,O)}) \cong \mathcal{H}^q(E_{(C',O')}) \tag{2.109}$$

and hence $\mathbf{Lg}^\dagger_K E_{(C,O)} \to E_{(C',O')}$ is a quasi-isomorphism.

Since crystals are of Zariski type, the question is local on Y, which I may therefore assume to be affine, and hence have a geometric realisation (Y, V). I first claim that for a finitely presented crystal F, $F_{(Y,V)}$ is a flat $i_Y^{-1}\mathcal{O}_V$-module. Let F_0 be the corresponding $j^\dagger\mathcal{O}_{]\overline{Y}[_{V_0}}$ module with overconvergent connection - this is locally free and is mapped to $i_{Y*}F_{(Y,V)}$ under the tensor equivalence of categories

$$\pi_{V*} : \mathrm{Coh}(j^\dagger\mathcal{O}_{]\overline{Y}[_{V_0}}) \cong \mathrm{Coh}(i_{Y*}i_Y^{-1}\mathcal{O}_V) \tag{2.110}$$

which implies that the latter is flat. In general, just note that locally any overconvergent variety (C, O) over Y/K admits a morphism to (Y, V) and hence the result follows from the fact that the pull-back of a flat module is flat. □

Proof of Proposition 2.4.10. Just combine the previous lemmata. □

A certain amount of evidence for the 'generic overconvergence' hypothesis of the proposition is given by the following translation of the main result of [48] into the language of the overconvergent site.

Proposition 2.4.19. *Let $f : X \to Y$ be a morphism of k-varieties, which extends to a morphism of pairs $(X, \overline{X}) \to (Y, \overline{Y})$ with $\overline{X}$ and $\overline{Y}$ proper. Then there exists an open subset $U \subset Y$ and a full subcategory $\mathcal{C}$ of triples over $(U, \overline{Y})$ satisfying the following condition.*

For any $q \geq 0$ there exists a finitely presented crystal E^q on U such that for any $(Z, \overline{Z}, \mathscr{X}) \in \mathcal{C}$ there is an isomorphism

$$\mathbf{R}^q f_*(\mathcal{O}^\dagger_{X/K})_{(Z,\mathscr{X}_K)} \cong E^q_{(Z,\mathscr{X}_K)} \tag{2.111}$$

of $i_Z^{-1}\mathcal{O}_{\mathscr{X}_K}$-modules, which functorial in $(Z, \overline{Z}, \mathscr{X})$.

Proof. Let $U, \mathcal{C}, \mathcal{F}^q$ be as in Theorem 0.3 of [48]. Let E^q be the finitely presented $\mathcal{O}^\dagger_{U/K}$-module corresponding to the overconvergent isocrystal $\mathcal{F}^q$. Let $\pi :]\overline{Z}[_{\mathscr{X}_0} \to]\overline{Z}[_{\mathscr{X}}$ denote the natural map. Since

$$\mathbf{R}^q f_{(X\times_Y Z, \overline{X}\times_{\overline{Z}}\overline{Y})/\mathscr{X},\mathrm{rig}*}(\mathcal{O}^\dagger_{X/K}) \cong \mathcal{F}^q_{(Z,\overline{Z},\mathscr{X})} \tag{2.112}$$

is $j^\dagger\mathcal{O}_{]\overline{Z}[_{\mathscr{X}_0}}$-coherent, Theorem 2.3.4 together with the fact that π_* is exact for coherent $j^\dagger\mathcal{O}_{]\overline{Z}[_{\mathscr{X}_0}}$-modules implies that

$$\mathbf{R}^q f_*(\mathcal{O}^\dagger_{X/K})_{(Z,\mathscr{X}_K)} \cong i_Z^{-1}\pi_*(\mathbf{R}^q f_{(X\times_Y Z, \overline{X}\times_{\overline{Z}}\overline{Y})/\mathscr{X},\mathrm{rig}*}(\mathcal{O}^\dagger_{X/K})) \tag{2.113}$$

where I am abusing notation slightly and writing $i_Z :]Z[_{\mathscr{X}} \to]\overline{Z}[_{\mathscr{X}}$. Hence it suffices simply to note that $i_Z^{-1}\pi_*(\mathcal{F}^q_{(Z,\overline{Z},\mathscr{X})}) \cong E^q_{(Z,\mathscr{X}_K)}$. □

Remark 2.4.20. Of course, I have not said what the category $\mathcal{C}$ is, so the proposition as stated is not particularly useful. A full description of $\mathcal{C}$ comes from a precise statement of

Shiho's result, which is Theorem 5.1 of [48]. Another way to look at the proposition is that it is saying $\mathbf{R}^q f_*(\mathcal{O}^\dagger_{X/K})$ is generically a finitely presented crystal on some full subcategory of $(Y/K)_{\mathrm{An}^\dagger}$.

2.5 Rigid fundamental groups and homotopy obstructions

In the previous sections I have defined absolute and relative rigid rational homotopy types. These are dga's, and the bar construction can be applied to obtain algebraic models of path spaces. Thus pro-unipotent groups can be extracted which in some sense deserve to be called unipotent fundamental groups. However, there are already definitions of these - in the absolute case as the Tannaka dual of the category of unipotent isocrystals, and in the relative (smooth and proper) case, there is a definition of the unipotent fundamental group given in the previous chapter. One would like to compare these constructions and show that they give the same answer, and in this section I do so in the absolute case.

Here, Olsson's proof for convergent homotopy types of smooth and proper varieties carries over almost verbatim. Recall that there are functors

$$D : \mathrm{Ho}(\mathrm{dga}_K) \to \mathrm{Ho}(\mathrm{Alg}_K^\Delta) \tag{2.114}$$

$$\mathbf{R}\mathrm{Spec} : \mathrm{Ho}(\mathrm{Alg}_K^\Delta)^\circ \to \mathrm{Ho}(\mathrm{SPr}(K)) \tag{2.115}$$

and Olsson has shown in his preprint [41] that the bar construction π_1 of a dga A coincides with the topological π_1 of the simplicial presheaf $\mathbf{R}\mathrm{Spec}(D(A))$. Hence it suffices to prove the comparison between this topological π_1 of the rational homotopy type

$$(X/K)_{\mathrm{rig}} := \mathbf{R}\mathrm{Spec}(D(\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K})))) \tag{2.116}$$

and the Tannakian π_1 of X/K.

In the smooth and proper case, working with the convergent site, this is proved by Olsson in §2 of [42], and his proof adapts fairly easily to the rigid case. Rather than writing out the whole proof in this slightly different situation, I will just make a few comments that I hope will convince the reader that the necessary changes are easily made.

Thanks to the comparison results both of §2.3 above and of Le Stum's paper [49], everywhere in the construction of $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ rigid spaces can be replaced by Berkovich spaces. It is also easy to construct the 'cohomology complexes' of ind-coherent crystals of $\mathcal{O}^\dagger_{X/K}$-modules on the overconvergent site, exactly as in §2.24 of [42] by taking framing systems and realisations on these framing systems. This allows me to define the pointed stack $(\widetilde{X}/K)_{\mathrm{rig}}$ analogously to §2.29 of *loc. cit.*, but instead taking $\widetilde{G}$ to be the pro-unipotent Tannakian fundamental group rather than the whole pro-algebraic fundamental group. (Note that in this case, because I am only working with unipotent isocrystals, $G = 1$).

The proof of Proposition 2.35 and Lemma 2.36 needs to be slightly modified as follows. Let π denote the functor of $\widetilde{G}$-invariants (of sheaves or modules), and let $C^\bullet(-)$ denote the cohomology complex of an ind-coherent crystal of $\mathcal{O}^\dagger_{X/K}$-modules. Let $\mathbf{L}(\mathcal{O}_{\widetilde{G}})$ be the overconvergent version of Olsson' object of the same name. Then as in Proposition 2.35 I need to compare $\mathbf{R}\Gamma_{\mathrm{rig}}(\mathcal{V})$ and $\mathbf{R}\pi(\mathbf{R}\Gamma_{\mathrm{an}}(C^\bullet(\mathcal{V}\otimes\mathbf{L}(\mathcal{O}_{\widetilde{G}}))))$ for a unipotent overconvergent isocrystal $\mathcal{V}$, which is equivalent to comparing $\mathbf{R}\Gamma_{\mathrm{rig}}(\mathcal{V})$ and $\mathbf{R}\pi(\mathbf{R}\Gamma_{\mathrm{rig}}(\mathcal{V}\otimes\mathbf{L}(\mathcal{O}_{\widetilde{G}})))$. Since π and Γ_{rig} commute, as in the proof of Lemma 2.36 it suffices to show that $\mathcal{V} \cong \mathbf{R}\pi(\mathcal{V} \otimes \mathbf{L}(\mathcal{O}_{\widetilde{G}}))$, and the proof of this follows exactly as in *loc. cit.*, using the overconvergent rather than the convergent site. Hence the following theorem has been proved.

Theorem 2.5.1. *The Tannakian unipotent fundamental group of a k-variety X at a point $x \in X(k)$ coincides with the unipotent fundamental group obtained from the augmented dga $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X/K}))$ via the bar construction. In particular, if k is a finite field, then the linear Frobenius structure on the (co-ordinate ring of the) former is mixed.*

Remark 2.5.2. Mixed structures on the unipotent Tannakian fundamental group have already been studied by Chiarellotto in [17], where he defines a weight filtration on the completed universal enveloping algebra of the Lie algebra of the unipotent Tannakian fundamental group.

Recall from the previous chapter the function field analogue of Kim's non-abelian period map

$$X(S) \to H^1_{F,\mathrm{rig}}(S, \pi_1^{\mathrm{rig}}(X/S, p)) \tag{2.117}$$

which takes sections of a smooth and proper scheme $f : X \to S$ over a curve over k to a certain set classifying F-torsors under the relative unipotent fundamental group, at some base point $p \in X(S)$. Basic functoriality of relative rational homotopy types in this situation gives a map

$$X(S) \to [\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K}), \mathcal{O}^\dagger_{S/K}]_{F\text{-Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{S/K}))} \tag{2.118}$$

where the RHS is maps in the homotopy category. I would ideally like to compare these two maps, and to do so, I would certainly need to compare the Tannakian construction of the relative fundamental group with the relative rational homotopy type.

2.5.1 A rather silly example

Recall that when discussing homotopy obstructions in the absolute case, I noted that the non-existence of a section of the map $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$, $x \mapsto x^2$ could not be detected on the level of rational homotopy, because the rational homotopy type of $\mathbb{A}^1_k$ is trivial. However, this non-existence can be detected on the level of relative rational homotopy types. Indeed, it clearly suffices to show that there is no section of the $x \mapsto x^2$ map on $\mathbb{A}^1_k \setminus \{0\}$, and here I can explicitly describe the (isomorphism class of the) push-forward of the constant isocrystal $f_*(\mathcal{O}^\dagger_{\mathbb{A}^1_k\setminus\{0\}/K})$. It is a free rank 2 module over $K\langle t, t^{-1}\rangle^\dagger$, with connection and algebra structures defined by

$$\nabla \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} df \\ dg - g\frac{dt}{2t} \end{pmatrix}, \quad \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1f_2 + g_1g_2 \\ f_1g_2 + f_2g_1 \end{pmatrix}. \tag{2.119}$$

It is simple to verify that there cannot be a morphism $f_*(\mathcal{O}^\dagger_{\mathbb{A}^1_k\setminus\{0\}/K}) \to \mathcal{O}^\dagger_{\mathbb{A}^1_k\setminus\{0\}/K}$ compatible with both the algebra structures and the connection, and hence that there can be no section of f on $\mathbb{A}^1_k \setminus \{0\}$.

Of course this example is rather stupid - one does not need the huge machinery of homotopy theory and the overconvergent site to show that there is no square root of t in $k[t]$! However, this example is instructive for two reasons:

- it shows that the relative rational homotopy type contains strictly more information that just looking at the map between the absolute rational homotopy types;
- the algebra structure was crucial in showing the non-existence of a section of homotopy types - there certainly is a section of the cohomology, but it is not multiplicative.

3 Étale rational homotopy types and crystalline homotopy sections

One of the motivations for my study of rational homotopy theory in Diophantine geometry comes from the potential to study rational points over function fields, that is, if $f : X \to S$ is a family of varieties, parametrised by some curve S over a finite field, functoriality induces a map

$$X(S) \to [\mathbf{R}f_*(\mathcal{O}^\dagger_{X/K}), \mathcal{O}^\dagger_{S/K}]_{F\text{-Ho}(\mathrm{dga}(\mathcal{O}^\dagger_{S/K}))}. \tag{3.1}$$

In this section, I ask the question of what the analogous picture is when S is replaced by a local, mixed characteristic base, for example a finite (possibly ramified) extension of $\mathbb{Z}_p$. If $\mathfrak{X}/\mathbb{Z}_p$ is a smooth and proper scheme, say, then the intuition that it's (p-adic) cohomology should form some sort of 'local system' over the base is made precise by p-adic Hodge theory - the Galois representation $H^i_{\text{ét}}(\mathfrak{X}_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is crystalline, and the associated filtered ϕ-module is the crystalline cohomology of $\mathfrak{X}_{\mathbb{F}_p}$.

The extension of this to rational homotopy theory has in a large part been accomplished by Martin Olsson in [43]. There, he constructs an étale homotopy type $G\mathcal{C}(X, \mathbb{Q}_p)$ associated to any variety X over any field K of characteristic $\neq p$ - this is a dga over $\mathbb{Q}_p$, together with an action of $G_K = \mathrm{Gal}(\overline{K}/K)$ on $G\mathcal{C}(X, \mathbb{Q}_p)$ in the homotopy category $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$. When X is the generic fibre of a (log) smooth and proper scheme over a complete mixed characteristic DVR, he also proves a non-abelian version of the p-adic Hodge theory comparison theorem, relating the étale homotopy type to the crystalline rational homotopy type of the special fibre.

This construction, however, is slightly inadequate, given my interest in sections. The point is that if X is geometrically connected, then there is a unique G_K-invariant morphism $G\mathcal{C}(X, \mathbb{Q}_p) \to \mathbb{Q}_p$ in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$, and hence is of no use in studying rational points. The main idea to rectify this is that the G_K-action should be lifted from $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_\ell})$ to $\mathrm{dga}_{\mathbb{Q}_\ell}$. More specifically, I would like to define an appropriate homotopical category $G_K\text{-dga}_{\mathbb{Q}_\ell}$ of G_K-equivariant $\mathbb{Q}_\ell$-dga's, and for any variety X/K an object $G\mathcal{C}(X, \mathbb{Q}_\ell)$ which maps to Olsson's object under the natural map

$$\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_\ell}) \to G_K\text{-Ho}(\mathrm{dga}_{\mathbb{Q}_\ell}). \tag{3.2}$$

The idea behind the construction is simply to reformulate Olsson's definition in such a way that there is a natural action on the complex before passing to the homotopy category. Of course, if it were just a case of constructing such an object, then this could be done in a rather straightforward way using Godement resolutions. However, I also want to prove a suitably lifted version of the p-adic Hodge theory comparison theorem, modelled on Olsson's original proof. Hence I have chosen a slightly more complicated construction, however, one that is very closely related to Olsson's original construction, and thus one that will allow me to easily adapt his proof of the comparison theorem.

I thus obtain a reasonable notion for what should constitute the 'local system' of p-adic rational homotopy types associated to a smooth and proper scheme $\mathfrak{X}/\mathbb{Z}_p$, and the next question I need to answer is what a 'global section' of this rational homotopy type is. Put another way, what constitutes good reduction for a section $G\mathcal{C}(\mathfrak{X}_{\mathbb{Q}_p}, \mathbb{Q}_p) \to \mathbb{Q}_p$ in $\mathrm{Ho}(G_{\mathbb{Q}_p}\text{-dga}_{\mathbb{Q}_\ell})$? The clue comes from non-abelian cohomology of the fundamental group - the appropriate set of torsors to look at in this context is those that trivialise upon base change to B_{cr}. Of course, exactly what this 'trivialisation' means in the context of a a morphism $G\mathcal{C}(\mathfrak{X}_{\mathbb{Q}_p}, \mathbb{Q}_p) \to \mathbb{Q}_p$ is not entirely straightforward but the answer is provided by the p-adic Hodge theory comparison theorem, since this gives a *distinguished* morphism $G\mathcal{C}(\mathfrak{X}_{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \tilde{B}_{\mathrm{cr}} \to \tilde{B}_{\mathrm{cr}}$ (here $\tilde{B}_{\mathrm{cr}}$ is a suitable localisation of B_{cr}). I can then use this to get a 'crystalline' refinement of the étale homotopy obstruction studied by Barnea-Schlank, Harpaz-Schlank and Pál in [2, 31, 44].

However, this notion of a section of the étale rational homotopy type, is still somewhat inadequate. The main reason for this is that continuity is not taken into account - the analogue would be calculating extension group $\mathrm{Ext}^1_{G_K}(H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p), \mathbb{Q}_p)$ in the category of representations of G_K as an abstract group, rather than as a pro-finite group. In the final part of this chapter, I tentatively propose an approach to the question of how to tell whether a section $G\mathcal{C}(\mathfrak{X}_{\mathbb{Q}_p}, \mathbb{Q}_p) \to \mathbb{Q}_p$ in the homotopy category $\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_p})$ is continuous. A more detailed introduction is given at the beginning of §3.3.

3.1 Construction of étale rational homotopy types

To define dga's representing étale homotopy types, Olsson first defines them for $K(\pi, 1)$'s, using group cohomology of the geometric fundamental group, and then for general X, he takes a hyper-covering $U_\bullet \to X$ by $K(\pi, 1)$'s and uses the functor of Thom-Sullivan cochains. In order to do so, he first picks a geometric generic point $\mathrm{Spec}(\Omega) \to X$ which will act as a base-point for the group cohomology calculations, and the Galois action comes from the fact that there is an action of $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X)$, which, after passing to the homotopy category, factors through

$$\mathrm{Aut}(\mathrm{Spec}(\Omega)/X) \to G_K. \tag{3.3}$$

To explain how to get a Galois action before passing to the homotopy category, consider first the case when the given K-variety X is a $K(\pi, 1)$, and admits a rational point $x \in X(K)$. There is an action of $\mathrm{Gal}(\overline{K}/K)$ on the geometric fundamental group $\pi_1^{\text{ét}}(X_{\overline{K}}, \bar{x})$, and hence on group cohomology with coefficients in an equivariant sheaf.

In general, however, such a rational point will not exist, and even if it did, it would not be liftable to a system of compatible rational points on a hyper-covering $U_\bullet \to X$ when X is not a $K(\pi, 1)$. Instead, I will work with the whole geometric fundamental groupoid, which will have an action of the Galois group in such a way that the dga's representing rational homotopy types (computed using groupoid cohomology) will have a Galois action before passing to the homotopy category. So let X be a variety over K, and fix an algebraic closure $\mathrm{Spec}(\overline{K}) \xrightarrow{\alpha} \mathrm{Spec}(K)$.

Definition 3.1.1. Let S be a scheme over $\mathrm{Spec}(K)$. The étale fundamental groupoid of S is the groupoid with objects geometric points $\bar{x} \to S$ together with a given morphism $\bar{x} \to \mathrm{Spec}(\overline{K})$ fitting into a commuting diagram

$$\begin{array}{ccc} \bar{x} & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{K}) & \xrightarrow{\alpha} & \mathrm{Spec}(K). \end{array} \tag{3.4}$$

Morphisms are isomorphisms of the corresponding fibre functors

$$\bar{x}^* : S_{\text{fét}} \to (\mathrm{Sets}) \tag{3.5}$$

from the category of schemes finite étale over S to sets. Denote this groupoid by $\pi_f^{\text{ét}}(S)$, and the set of morphisms between any two objects $\bar{x}, \bar{y}$ will be denoted $\pi_f^{\text{ét}}(S)(\bar{x}, \bar{y})$.

Remark 3.1.2. From now on, I will use the term 'geometric point' to mean an object of $\pi_f^{\text{ét}}$, that is a geometric point as in the above definition.

Remark 3.1.3. The fundamental groupoid can be endowed with a topology, by which I mean a topology on each $\pi_f^{\text{ét}}(S)(\bar{x}, \bar{y})$, as follows. If $\bar{x} = \bar{y}$, then give $\pi_f^{\text{ét}}(S)(\bar{x}, \bar{x}) = \pi_1^{\text{ét}}(S, \bar{x})$ the usual pro-finite topology, in general give $\pi_f^{\text{ét}}(S)(\bar{x}, \bar{y})$ the topology induced by any isomorphism (of sets) $\pi_f^{\text{ét}}(S)(\bar{x}, \bar{y}) \cong \pi_1^{\text{ét}}(S, \bar{x})$ or $\pi_f^{\text{ét}}(S)(\bar{x}, \bar{y}) \cong \pi_1^{\text{ét}}(S, \bar{y})$. For any morphism of schemes over $\mathrm{Spec}(K)$, the induced morphism of fundamental groupoids is continuous.

Definition 3.1.4. Let S be a scheme over $\mathrm{Spec}(K)$. A Galois module on S (with coefficients in $\mathbb{Q}_\ell$) is a continuous $\mathbb{Q}_\ell$-representation of $\pi_f^{\text{ét}}(S)$. Specifically, this is a collection of topological $\mathbb{Q}_\ell$-vector spaces $V_{\bar{x}}$, one for each $\bar{x} \in \mathrm{Ob}(\pi_f^{\text{ét}}(S))$, and topological isomorphisms $\rho_g : V_{\bar{x}} \to V_{\bar{y}}$ for every $g : \bar{x}^* \Rightarrow \bar{y}^*$ in $\pi_f^{\text{ét}}(S)(\bar{x}, \bar{y})$, which are compatible with composition, taking inverses, and the identity. Also required is that the map

$$\pi_f^{\text{ét}}(S)(\bar{x}, \bar{y}) \times V_{\bar{x}} \to V_{\bar{y}} \tag{3.6}$$

is continuous for all $\bar{x}, \bar{y}$.

The two main cases of interest for me will be $S = X$ and $S = X_{\overline{K}}$ for a variety X/K, and it is an easy check that in these cases the category of Galois modules just defined is equivalent to the category defined by Olsson in §5.2 of [43].

The reason for making these new definitions is that there is an action of G_K on $\pi_f^{\text{ét}}(X_{\overline{K}})$ as I now describe. Given $\sigma \in G_K$ and an object $\bar{x} \in \pi_f^{\text{ét}}(X_{\overline{K}})$ corresponding to a diagram

$$\begin{array}{ccc} \bar{x} & \longrightarrow & X_{\overline{K}} \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(\overline{K}) \end{array} \tag{3.7}$$

take $\sigma(\bar{x})$ to be the 'outer triangle' of the diagram

$$\begin{array}{ccccc} \bar{x} & \longrightarrow & X_{\overline{K}} & \xrightarrow{\sigma^* \otimes \mathrm{id}} & X_{\overline{K}} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(\overline{K}) & \xrightarrow{\sigma^*} & \mathrm{Spec}(\overline{K}). \end{array} \tag{3.8}$$

To get the action on morphisms, suppose that $g : \bar{x}^* \Rightarrow \bar{y}^*$ is a natural isomorphism of fibre functors. Then $\sigma(g) : \sigma(\bar{x})^* \Rightarrow \sigma(\bar{y})^*$ is the natural isomorphism whose evaluation on $\mathcal{F} \in X_{\overline{K},\text{ét}}$ is

$$g_{(\sigma^* \otimes \mathrm{id})^* \mathcal{F}} : \bar{x}^*(\sigma^* \otimes \mathrm{id})^* \mathcal{F} = \sigma(\bar{x})^* \mathcal{F} \to \bar{y}^*(\sigma^* \otimes \mathrm{id})^* \mathcal{F} = \sigma(\bar{y})^* \mathcal{F}. \tag{3.9}$$

Remark 3.1.5. This is the reason for taking base points compatible with the given algebraic closure of K in the definition of the fundamental groupoid.

Definition 3.1.6. A G_K-equivariant (continuous, $\mathbb{Q}_\ell$-valued) representation of $\pi_f^{\text{ét}}(X_{\overline{K}})$ is a continuous $\mathbb{Q}_\ell$-representation (V, ρ) of $\pi_f^{\text{ét}}(X_{\overline{K}})$, together with topological isomorphisms $c_\sigma : V_{\bar{x}} \to V_{\sigma(\bar{x})}$ for all $\bar{x} \in \mathrm{Ob}(\pi_f^{\text{ét}}(X_{\overline{K}}))$, $\sigma \in G_K$, such that the equality

$$c_\sigma(\rho_g(\lambda)) = \rho_{\sigma(g)}(c_\sigma(\lambda)) \tag{3.10}$$

holds for all $g \in \pi_f^{\text{ét}}(X_{\overline{K}})(\bar{x}, \bar{y})$, $\sigma \in G_K$ and $\lambda \in V_{\bar{x}}$.

Remark 3.1.7. This definition is not completely satisfactory, as there is no requirement for continuity of the G_K-action. This will not be a problem, however.

Lemma 3.1.8. *Any representation of $\pi_f^{\text{ét}}(X)$ naturally gives rise to a G_K-equivariant representation of $\pi_f^{\text{ét}}(X_{\overline{K}})$ via pull-back along $\pi_f^{\text{ét}}(X_{\overline{K}}) \to \pi_f^{\text{ét}}(X)$.*

Proof. Let $\sigma^* \otimes \text{id} : X_{\overline{K}} \to X_{\overline{K}}$ and $a : X_{\overline{K}} \to X$ be the natural maps. Then by definition $\sigma(\bar{x}) = (\sigma^* \otimes \text{id}) \circ \bar{x}^*$, and since $a = a \circ (\sigma^* \otimes \text{id})$, for a $\pi_f^{\text{ét}}(X)$ representation V, there is a natural identification

$$(a^*V)_{\bar{x}} = \bar{x}^* a^* V = \bar{x}^* (\sigma^* \otimes \text{id})^* a^* V = (a^*V)_{\sigma(\bar{x})}. \tag{3.11}$$

Define the map $c_\sigma : (a^*V)_{\bar{x}} \to (a^*V)_{\sigma(\bar{x})}$ to be this canonical identification. □

Now let X/K be a variety, and L a differential graded algebra in the ind-category of Galois modules on X - thanks to the above lemma L can be viewed as a dga in the ind-category of G_K-equivariant Galois modules on $X_{\overline{K}}$; the groupoid cohomology of such objects is defined as follows. For any $n \geq 0$, let $\pi_f^{\text{ét}}(X_{\overline{K}})^n$ be the set of all possible n-tuples of morphisms in $\pi_f^{\text{ét}}(X_{\overline{K}})$, all of whose targets are the same. The set $\pi_f^{\text{ét}}(X_{\overline{K}})$ inherits a topology via the identification

$$\pi_f^{\text{ét}}(X_{\overline{K}})^n = \coprod_{\bar{x} \in \text{Ob}(\pi_f^{\text{ét}}(X_{\overline{K}}))} \left(\bigcup_{I \in \text{Ob}(\pi_f^{\text{ét}}(X_{\overline{K}}))^n} \prod_{\bar{y} \in I} \pi_f^{\text{ét}}(X_{\overline{K}})(\bar{y}, \bar{x}) \right) \tag{3.12}$$

and has a 'left action' by $\pi_f^{\text{ét}}(X_{\overline{K}})$, whereby a morphism g whose source is the common target of an object $(g_1, \ldots, g_n) \in \pi_f^{\text{ét}}(X_{\overline{K}})^n$ acts on this tuple via pre-composition to give $(gg_1, \ldots, gg_n) \in \pi_f^{\text{ét}}(X_{\overline{K}})^n$. Define the cosimplicial dga of 'homogeneous cochains' associated to the dga L by taking $\Gamma^n(\pi_f^{\text{ét}}(X_{\overline{K}}), L)_r$ to be the set of all continuous morphisms

$$c : \pi_f^{\text{ét}}(X_{\overline{K}})^{n+1} \to \bigoplus_{\bar{x} \in \text{Ob}(\pi_f^{\text{ét}}(X_{\overline{K}}))} L_{r,\bar{x}} \tag{3.13}$$

such that $c(g_1, \ldots, g_n) \in L_{r,\bar{x}}$ (where $\bar{x}$ is the common target of the g_i), and $c(g(g_1, \ldots, g_n)) = g \cdot c(g_1, \ldots, g_n)$, at least wherever this makes sense. This inherits a vector space structure, a multiplication and a differential in the 'r' direction from L. The cosimplicial structure in 'n' comes from the fact that every morphism $[n] \to [m]$ in the simplicial category induces a morphism $\pi_f^{\text{ét}}(X_{\overline{K}})^{m+1} \to \pi_f^{\text{ét}}(X_{\overline{K}})^{n+1}$ and hence a morphism $\Gamma^n(\pi_f, L) \to \Gamma^m(\pi_f, L)$. Thus $\Gamma^\bullet(\pi_f, L)_*$ becomes a cosimplicial $\mathbb{Q}_\ell$-dga. Define the 'groupoid cohomology' dga of

L to be

$$\mathbf{R}\Gamma(\pi_f^{\text{ét}}(X_{\overline{K}}), L) := \mathrm{Th}(\Gamma^\bullet(\pi_f, L)). \tag{3.14}$$

Since L is a G_K-equivariant representation of π_f, there is an action of G_K on this 'groupoid cohomology' dga. Thus $\mathbf{R}\Gamma(\pi_f^{\text{ét}}(X_{\overline{K}}), L)$ can be viewed as an object in G_K-dga$_{\mathbb{Q}_\ell}$.

Remark 3.1.9. Of course, the general definition of groupoid cohomology works for any topological groupoid.

Proposition 3.1.10. *Suppose that G is a connected topological groupoid, and that $x \in$ $\mathrm{Ob}(G)$. Then for any continuous G-representation V there is a natural quasi-isomorphism of complexes*

$$\mathbf{R}\Gamma(G, V) \to \mathbf{R}\Gamma(G(x), V_x) \tag{3.15}$$

where $G(x)$ denotes the automorphism group of x.

Proof. Entirely similar to the discrete case. □

Corollary 3.1.11. *Let L be dga in the ind-category of Galois modules on X, and $\mathrm{Spec}(\Omega) \to$ X a geometric generic point on X. Then there is a natural quasi-isomorphism of dga's (no G_K-action)*

$$\mathbf{R}\Gamma(\pi_f^{\text{ét}}(X_{\overline{K}}), L) \to \mathrm{Th}(\mathcal{C}^\bullet(X, \mathrm{Spec}(\Omega), L)) \tag{3.16}$$

where the $\mathcal{C}^\bullet(X, \mathrm{Spec}(\Omega), L)$ appearing on the RHS is the object defined by Olsson in §5.5 of [43].

Definition 3.1.12. Recall that a connected variety Y over an algebraically closed field is called a $K(\pi, 1)$ if the natural map from group cohomology of the fundamental group to étale cohomology is an isomorphism. A variety U over K is called a $K(\pi, 1)$ if every connected component of $U_{\overline{K}}$ is.

Now let X/K be a smooth variety, and choose an étale hyper-covering $U_\bullet \to X$ such that each U_n is a $K(\pi, 1)$. For L a dga in the ind-category of Galois modules on X, define

$$\mathbf{R}\Gamma_{\text{ét}/K}(X, L)_{U_\bullet} := \mathrm{Th}(\mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{\bullet,\overline{K}}), L|_{U_\bullet})) \tag{3.17}$$

which is a $\mathbb{Q}_\ell$-dga with G_K-action.

Let the category of such $\mathbb{Q}_\ell$-dga's with Galois action be denoted G_K-dga$_{\mathbb{Q}_\ell}$, and the category obtained by inverting quasi-isomorphisms $\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_\ell})$. Note that by Proposition A.2.8.2 of [36] there are at least two model category structures on G_K-dga$_{\mathbb{Q}_\ell}$ for which the weak equivalences are exactly the quasi-isomorphisms, so $\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_\ell})$ is a locally small category.

Lemma 3.1.13. *The object* $\mathbf{R}\Gamma_{\text{ét}/K}(X,L)_{U_\bullet} \in \mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_\ell})$ *is independent of the hyper-covering* $U_\bullet$*, up to canonical isomorphism.*

Proof. Exactly the same as in 5.22 of *loc. cit.* □

Hence the object $\mathbf{R}\Gamma_{\text{ét}/K}(X,L) := \mathbf{R}\Gamma_{\text{ét}/K}(X,L)_{U_\bullet} \in \mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_\ell})$ is independent of the chosen hyper-covering $U_\bullet \to X$.

Proposition 3.1.14. *Assume that* X/K *is geometrically connected, and choose a geometric generic point* $\mathrm{Spec}(\Omega) \to X$*. Then after applying the forgetful functor*

$$\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_\ell}) \to G_K\text{-}\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_\ell}) \tag{3.18}$$

there is a natural quasi-isomorphism

$$\mathbf{R}\Gamma_{\text{ét}/K}(X,L) \to G\mathcal{C}(L, \mathrm{Spec}(\Omega)) \tag{3.19}$$

where the latter is the object defined in §5.22-5.26 of loc. cit.

Proof. Choose a geometric generic point $E = \mathrm{Spec}(\Omega) \to X_{\overline{K}}$, such that Ω is a separable closure of $\overline{K}(X)$. For any étale $U \to X$, write $E_U := \coprod_{\mathrm{Hom}_X(E,U)} E$ - this is a disjoint union of geometric generic points in $U_{\overline{K}}$, and there is a natural map $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X) \to \mathrm{Aut}(E_U/U)$.

Choose an étale hyper-covering $U_\bullet \to X$ by $K(\pi,1)$'s. The above construction induces an action of $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X)$ on each $G\mathcal{C}(U_{n,\overline{K}}, E_{U_n}, L)$ (in the notation of §5.21 of *loc. cit.*), which thus induces an action of $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X)$ on $G\mathcal{C}(L,\mathrm{Spec}(\Omega)) := \mathrm{Th}(G\mathcal{C}(U_{\bullet,\overline{K}}, E_{U_\bullet}, L))$. After passing to the homotopy category, this factors through $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X) \to G_K$, which defines the Galois action on $G\mathcal{C}(L,\mathrm{Spec}(\Omega))$.

For each n, there is a natural morphism of complexes

$$\mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) \to G\mathcal{C}(U_{n,\overline{K}}, E_{U_n}, L) \tag{3.20}$$

which is defined as follows. Choosing an isomorphism $\iota : E_{U_n} \to \coprod_{i=0}^m \mathrm{Spec}(\Omega)$ and letting $\bar\eta : \mathrm{Spec}(\Omega) \to U_{n,\overline{K}}$ denote some fixed geometric generic point gives an ordering of the components of E_{U_n}. Then for each $p \geq 0$, there is a map

$$\begin{array}{ccc} \Gamma^p(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) & \longrightarrow & \mathcal{C}^p(U_{n,\overline{K}}, E_{U_n}, L) \\ & & \| \\ & & \prod_{\mathrm{Fun}([p],[m])} \mathrm{Hom}^{\mathrm{cts}}_{\pi_1^{\text{ét}}(U_{n,\overline{K}},\bar\eta)}(\pi_1^{\text{ét}}(U_{n,\overline{K}},\bar\eta)^{p+1}, L) \end{array} \tag{3.21}$$

which takes $c \in \Gamma^p(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L)$ to the map whose component at any function $\alpha : [p] \to [m]$ takes $(g_0, \ldots, g_p) \in \pi_1^{\text{ét}}(U_{n,\overline{K}}, \bar{\eta})^{p+1}$ to $c(g_0\iota_{\alpha(0)}, \ldots, g_p\iota_{\alpha(p)})$, where ι_i denotes the induced isomorphism from the ith component of E_U (with the given ordering) to $\mathrm{Spec}(\Omega)$.

Thanks to Corollary 3.1.11 and §5.9 of *loc. cit.* (which taken together imply that this map is a quasi-isomorphism) to prove the proposition, it suffices to prove that after passing to the homotopy category, this map is equivariant for the action of G_K, since the proposition then follows by applying Th to both sides. But now just note that for $\sigma \in G_K$ there is a commutative diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) & \longrightarrow & G\mathcal{C}(U_{n,\overline{K}}, E_{U_n}, L) \\ \downarrow \sigma & & \downarrow \sigma \\ \mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) & \longrightarrow & G\mathcal{C}(U_{n,\overline{K}}, \sigma(E)_{U_n}, L) \end{array} \tag{3.22}$$

and choosing an automorphism $c_\sigma \in \mathrm{Aut}(\mathrm{Spec}(\Omega)/X)$ over $\sigma \in G_K$ induces an isomorphism $\sigma(E) \cong E$ over $X_{\overline{K}}$, and hence a commutative diagram

$$\begin{array}{ccccc} \mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) & \longrightarrow & G\mathcal{C}(U_{n,\overline{K}}, E_{U_n}, L) & & \\ \downarrow \sigma & & \downarrow \sigma & \searrow^{c_\sigma} & \\ \mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) & \longrightarrow & G\mathcal{C}(U_{n,\overline{K}}, \sigma(E)_{U_n}, L) & \longrightarrow & G\mathcal{C}(U_{n,\overline{K}}, E_{U_n}, L). \end{array} \tag{3.23}$$

The outer square of this descends to a commutative diagram of isomorphisms

$$\begin{array}{ccc} \mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) & \longrightarrow & G\mathcal{C}(U_{n,\overline{K}}, E_{U_n}, L) \\ \downarrow \sigma & & \downarrow \sigma \\ \mathbf{R}\Gamma(\pi_f^{\text{ét}}(U_{n,\overline{K}}), L) & \longrightarrow & G\mathcal{C}(U_{n,\overline{K}}, E_{U_n}, L) \end{array} \tag{3.24}$$

in the homotopy category, implying the result. □

Now let $\mathcal{V}$ be the ring of integers of K, $B_{\text{cris}}(\mathcal{V})$ Fontaine's ring of crystalline periods for $\mathcal{V}$ and $\widetilde{B}_{\text{cris}}$ its localisation as defined in §6.8 of *loc. cit* . Let X/V be a smooth, proper scheme with a relative normal crossings divisor $D \subset X$, and write $X_K^\circ = X_K \setminus D_K$, $X_k^\circ = X_k \setminus D_k$. Let $G_K\text{-dga}_{\widetilde{B}_{\text{cris}}}$ be the category of G_K-semi-linear $\widetilde{B}_{\text{cris}}$-dga's, and $\mathrm{Ho}(G_K\text{-dga}_{\widetilde{B}_{\text{cris}}})$ the localisation of this category at the class of quasi-isomorphisms. The functors

$$- \otimes_{\mathbb{Q}_p} \widetilde{B}_{\text{cris}} : G_K\text{-dga}_{\mathbb{Q}_p} \to G_K\text{-dga}_{\widetilde{B}_{\text{cris}}} \tag{3.25}$$

$$- \otimes_K \widetilde{B}_{\text{cris}} : \mathrm{dga}_K \to G_K\text{-dga}_{\widetilde{B}_{\text{cris}}} \tag{3.26}$$

preserve quasi-isomorphisms, and hence descend to the homotopy categories.

Theorem 3.1.15. *There is a natural quasi-isomorphism*

$$\mathbf{R}\Gamma_{\text{ét}/K}(X_K^\circ, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\text{cris}} \to \mathbf{R}\Gamma_{\text{Th}}(\Omega^*(\mathcal{O}^\dagger_{X_k^\circ/K})) \otimes_K \widetilde{B}_{\text{cris}} \tag{3.27}$$

in $\text{Ho}(G_K\text{-dga}_{\widetilde{B}_{\text{cris}}})$.

Proof. I first claim that I can replace the rigid rational homotopy type in the statement $\mathbf{R}\Gamma_{\text{Th}}(\Omega^*(\mathcal{O}^\dagger_{X_k^\circ/K}))$ by the log-convergent rational homotopy type $\mathbf{R}\Gamma_{\text{cris}}\left(\mathcal{O}_{((X_k,D_k)/K)_{\text{conv}}}\right)$ considered in [43]. The proof is almost identical to the comparison between the rigid and the convergent homotopy type in the smooth and proper case in §2.2.3.

I next claim that I can replace the LHS by the groupoid cohomology dga computed using the full sub-groupoid $\pi_f^{\text{ét,gen}}(X_{\overline{K}})$ on geometric generic points, which is stable under Galois, this follows from Proposition 3.1.10. Of course, this can be done for any dga L in the ind-category of Galois modules on X_K°, and similarly for any other scheme over $\text{Spec}(K)$.

If instead it were a question of proving a quasi-isomorphism in $G_K\text{-Ho}(\text{dga}_{\widetilde{B}_{\text{cris}}})$, then this is done by Olsson in §6.17 of *loc. cit.* However, the proof carries over almost word for word - the point is that all the objects and morphisms that appear in diagrams 6.17.1 and 6.17.5 can be replaced by Galois-equivariant counterparts.

First note that the Galois module $B_{\text{cris}}(U^\wedge)$ on U defined in §6 of *loc. cit.* can be considered as a Galois module in the sense of this chapter, after replacing $\pi_f^{\text{ét}}(U)$ by $\pi_f^{\text{ét,gen}}(U)$. For the 'group-cohomology' dga's, simply take groupoid cohomology instead, endow the crystalline cohomology dga's with the trivial Galois action, and endow anything tensored with B_{cris} with the diagonal Galois action.

The only place where it is not clear that Olsson's proof carries over is in the analogue of the map

$$\mathbf{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\text{cris}}) \to G\mathcal{C}(U^{\wedge\circ}_{\overline{K}}, \widehat{E}_U, \mathbf{R}^\bullet(B_{\text{cris}}(U^\wedge))) \tag{3.28}$$

which acts as the bridge between étale cohomology and crystalline cohomology. Working with groupoid cohomology instead, there is a collection of vector spaces $\mathbf{R}^\bullet(B_{\text{cris}}(U^\wedge))_{\bar{\eta}}$, one for each geometric generic point $\bar{\eta}$, and there are natural maps

$$\mathbf{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\text{cris}}) \to \mathbf{R}^\bullet(B_{\text{cris}}(U^\wedge))_{\bar{\eta}} \tag{3.29}$$

which induce a map

$$\mathbf{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\text{cris}}) \to \mathbf{R}\Gamma(\pi_f^{\text{ét,gen}}(U^{\wedge\circ}_{\overline{K}}), \mathbf{R}^\bullet(B_{\text{cris}}(U^\wedge))). \tag{3.30}$$

Hence using the B_{cris}-module structure of the RHS this extends to a map

$$\mathbf{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\mathrm{cris}}) \otimes_K B_{\mathrm{cris}} \to \mathbf{R}\Gamma(\pi_f^{\text{ét,gen}}(U^{\wedge\circ}_{\overline{K}}), \mathbf{R}^\bullet(B_{\mathrm{cris}}(U^\wedge))) \tag{3.31}$$

which I claim is compatible with the G_K-actions. Since the RHS is a semi-linear B_{cris}-module, it suffices to show that the image of

$$\mathbf{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\mathrm{cris}}) \to \mathbf{R}\Gamma(\pi_f^{\text{ét,gen}}(U^{\wedge\circ}_{\overline{K}}), \mathbf{R}^\bullet(B_{\mathrm{cris}}(U^\wedge))) \tag{3.32}$$

lands in the subspace of G_K-invariants. But chasing through the definitions, this follows from the fact that there is a commutative diagram of formal schemes

$$\begin{array}{ccc} B_{\mathrm{cris}}(U^\wedge)_{\bar\eta} & \xrightarrow{\quad\sigma\quad} & B_{\mathrm{cris}}(U^\wedge)_{\sigma(\bar\eta)} \\ & \searrow \quad \swarrow & \\ & U^\wedge & \end{array} \tag{3.33}$$

since the map $\mathbf{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\mathrm{cris}}) \to \mathbf{R}^\bullet(B_{\mathrm{cris}}(U^\wedge))_{\bar\eta}$ (resp. $\sigma(\bar\eta)$) arises via pull-back along the left (resp. right) diagonal morphism in the diagram. □

3.2 Crystalline sections

Now that the correct comparison isomorphism is in place, I can start to explain how to use it to obtain a notion of 'crystalline sections' for the p-adic rational homotopy type. I will stick to the situation of the previous section, but change notation slightly for convenience. Let X/K be a smooth variety satisfying the following condition: there exists a smooth, proper scheme $\overline{\mathfrak{X}} \to \mathrm{Spec}(\mathcal{V})$ and a relative normal crossings divisor $\mathfrak{D} \subset \overline{\mathfrak{X}}$ such that $X = \overline{\mathfrak{X}}_K \setminus \mathfrak{D}_K$, and $X_k := \overline{\mathfrak{X}}_k \setminus \mathfrak{D}_k$ is geometrically connected. Write $\mathfrak{X} = \overline{\mathfrak{X}} \setminus \mathfrak{D}$. In this situation, there is a comparison quasi-isomorphism

$$\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}} \to \mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X_k/K})) \otimes_K \widetilde{B}_{\mathrm{cris}} \tag{3.34}$$

in $\mathrm{Ho}(G_K\text{-}\mathrm{dga}_{\widetilde{B}_{\mathrm{cris}}})$. Now, since X_k is geometrically connected, there is a unique morphism $\mathbf{R}\Gamma_{\mathrm{Th}}(\Omega^*(\mathcal{O}^\dagger_{X_k/K})) \to K$ in $\mathrm{Ho}(\mathrm{dga}_K)$, and hence, via the comparison theorem a distinguished morphism

$$\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}} \to \widetilde{B}_{\mathrm{cris}} \tag{3.35}$$

in $\mathrm{Ho}(G_K\text{-dga}_{\widetilde{B}_{\mathrm{cris}}})$. Thus there is a distinguished class of sections

$$\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p) \to \mathbb{Q}_p \tag{3.36}$$

in $\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_p})$, namely those which are equal to this distinguished section after base changing to $\widetilde{B}_{\mathrm{cris}}$.

Remark 3.2.1. This explains why I need an object (and comparison theorem for such an object) in $\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_p})$ rather than $G_K\text{-Ho}(\mathrm{dga}_{\mathbb{Q}_p})$ - if X is geometrically connected then there is a unique morphism $\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p) \to \mathbb{Q}_p$ in the latter category, and hence there would not be an 'interesting' subset of the set of such morphisms!

Letting $[\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p), \mathbb{Q}_p]$ denote $\mathrm{Hom}_{\mathrm{Ho}(G_K\text{-dga}_{\mathbb{Q}_p})}(\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p), \mathbb{Q}_p)$, define the subset

$$[\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p), \mathbb{Q}_p]^{\mathrm{cris}} \subset [\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p), \mathbb{Q}_p] \tag{3.37}$$

of crystalline sections to be those sections whose base change to $\widetilde{B}_{\mathrm{cris}}$ is the distinguished section. By functoriality of all the above constructions, there is an induced commutative diagram

$$\begin{array}{ccc} X(K) & \longrightarrow & [\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p), \mathbb{Q}_p] \\ \uparrow & & \uparrow \\ \mathfrak{X}(\mathcal{V}) & \longrightarrow & [\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_p), \mathbb{Q}_p]^{\mathrm{cris}}. \end{array} \tag{3.38}$$

In their preprint [2], Barnea and Schlank define a model structure on $\mathrm{Pro}(G_K\text{-sSet})$, the category of pro-discrete simplicial G_K-sets, and for any K-variety X, define a relative étale homotopy type $|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}} \in \mathrm{Pro}(G_K\text{-sSet})$. This construction is functorial, and hence gives rise to a map

$$X(K) \to |X_{\text{ét}}|^{hG_K}_{\mathrm{Spec}(K)_{\text{ét}}} \tag{3.39}$$

where the superscript refers to the functor of homotopy fixed points. If I can show that the construction of the ℓ-adic rational homotopy type factors through theirs, then this will give rise to a crystalline obstruction set

$$|X_{\text{ét}}|^{hG_K,\mathrm{cris}}_{\mathrm{Spec}(K)_{\text{ét}}} \subset |X_{\text{ét}}|^{hG_K}_{\mathrm{Spec}(K)_{\text{ét}}} \tag{3.40}$$

in which the image of integral points must land. The essential point in show this factorisation is that the equivalence class of the fundamental groupoid $\pi_f^{\text{ét}}(X_{\overline{K}})$, together with its Galois action, can be recovered from $|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}}$. This is done as follows.

For any $\mathcal{S} \in \mathrm{Pro}(\mathrm{sSet})$, define the fundamental groupoid to have objects the set of morphisms $\bullet \to \mathcal{S}$ in $\mathrm{Pro}(\mathrm{sSet})$, and with morphisms the pro-set of homotopy classes of paths in geometric realisations. More precisely, if $\mathcal{S} = \{\mathcal{S}_\alpha\}$, and $\bullet \to \{\mathcal{S}_\alpha\}$ is a point, then there is an induced point $\bullet \in |\mathcal{S}_\alpha|$ in each geometric realisation. Thus for any two such points, the set of homotopy classes of paths between the two induced points in the geometric realisations $|\mathcal{S}_\alpha|$ form a pro-set. This gives rise to a groupoid with a pro-discrete topology on each hom-set. Moreover, if each realisation $|\mathcal{S}_\alpha|$ is finite, in the sense that all homotopy groups are finite, then this topology will be pro-finite.

By functoriality, if $\mathcal{S}$ arises from an object of $\mathrm{Pro}(G_K\text{-sSet})$, via the forgetful functor, then there is an induced action of G_K on $\pi_f(\mathcal{S})$. Thus $\pi_f(|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}})$ will be a groupoid with a G_K-action.

Proposition 3.2.2. *The fundamental groupoid $\pi_f(|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}})$ is pro-finite, and there is a continuous, G_K-equivariant, functorial equivalence*

$$\pi_f^{\text{ét}}(X_{\overline{K}}) \cong \pi_f(|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}}). \tag{3.41}$$

Proof. Let $|X_{\overline{K},\text{ét}}| \in \mathrm{Pro}(\mathrm{sSet})$ denote the absolute étale topological type of $X_{\overline{K}}$ in the sense of *loc. cit.* Then there is a G_K-action on $|X_{\overline{K},\text{ét}}|$ induced by the usual G_K-action on $X_{\overline{K}}$ - thus $|X_{\overline{K},\text{ét}}|$ can be considered as an object in $G_K\text{-Pro(sSet)}$. By applying the natural functor

$$\mathrm{Pro}(G_K\text{-sSet}) \to G_K\text{-Pro(sSet)} \tag{3.42}$$

the pro-simplicial set $|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}}$ can also be considered as an object in the latter category. I claim that there is an isomorphism

$$|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}} \cong |X_{\overline{K},\text{ét}}| \tag{3.43}$$

in $G_K\text{-Pro(sSet)}$. Indeed, applying the fibrant resolution functor to $\bullet \in \mathrm{Pro}(X_{\text{ét}}^{\Delta^\circ})$, to obtain a pro-simplicial sheaf $\mathcal{T}$ on $X_{\text{ét}}$, the pull-back $\mathcal{T}_{\overline{K}}$ to $X_{\overline{K}}$ is a fibrant replacement for $\bullet \in \mathrm{Pro}(X_{\overline{K},\text{ét}}^{\Delta^\circ})$ by Proposition 10.1 of *loc. cit.*. Given the definition of the relative and absolute étale homotopy types in *loc. cit.*, namely

$$|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}} := \mathrm{Pro}(f_!)(\mathcal{T}) \tag{3.44}$$

$$|X_{\overline{K},\text{ét}}| := \mathrm{Pro}(f_{\overline{K},!})(\mathcal{T}_{\overline{K}}) \tag{3.45}$$

the claim follows from the fact for a sheaf of sets $\mathcal{F}$ on $X_{\text{ét}}$, then via the identification $\mathrm{Spec}(K)_{\text{ét}} \cong G_K\text{-Set}$, the sheaf $f_!(\mathcal{F})$ on $\mathrm{Spec}(K)_{\text{ét}}$ (where $f : X \to \mathrm{Spec}(K)$ is the structure morphism) can be identified with the set $f_{\overline{K},!}(\mathcal{F}_{\overline{K}})$ together with its G_K-action arising via functoriality.

By the definition of the Galois action on $\pi_f^{\text{ét}}(X_{\overline{K}})$, to prove the proposition, it thus suffices to prove that there is an equivalence of groupoids $\pi_f^{\text{ét}}(X_{\overline{K}}) \cong \pi_f(|X_{\overline{K},\text{ét}}|)$ which is natural in $X_{\overline{K}} \to \mathrm{Spec}(\overline{K})$ (recall that geometric points are required to be compatible with the given algebraic closure of K). But this now follows because the Artin-Mazur homotopy type can be recovered from $|X_{\overline{K},\text{ét}}|$, and hence the category of finite local systems on $X_{\overline{K},\text{ét}}$, and hence the étale fundamental groupoid, all in a functorial manner.

□

Remark 3.2.3. This result will only be helpful if equivalent groupoids have quasi-isomorphic cohomology. As already mentioned in justifying replacing $\pi_f^{\text{ét}}(X_{\overline{K}})$ by $\pi_f^{\text{ét,gen}}(X_{\overline{K}})$ in the proof of Theorem 3.1.15, this follows from Proposition 3.1.10.

Thus $\mathbf{R}\Gamma_{\text{ét}/K}(X,\mathbb{Q}_p)$ can be recovered from $|X_{\text{ét}}|_{\text{Spec}(K)_{\text{ét}}}$ using the same recipe of computing groupoid cohomology on a hyper-covering by $K(\pi,1)$'s. Of course, this is functorial in X, and hence there is a commutative diagram

$$\begin{array}{ccc} X(K) & \longrightarrow & |X_{\text{ét}}|^{hG_K}_{\text{Spec}(K)_{\text{ét}}} \\ & \searrow & \downarrow \\ & & [\mathbf{R}\Gamma_{\text{ét}/K}(X,\mathbb{Q}_p),\mathbb{Q}_p]; \end{array} \tag{3.46}$$

define $|X_{\text{ét}}|^{hG_K,\text{cris}}_{\text{Spec}(K)_{\text{ét}}} \subset |X_{\text{ét}}|^{hG_K}_{\text{Spec}(K)_{\text{ét}}}$ by the Cartesian diagram

$$\begin{array}{ccc} |X_{\text{ét}}|^{hG_K,\text{cris}}_{\text{Spec}(K)_{\text{ét}}} & \longrightarrow & |X_{\text{ét}}|^{hG_K}_{\text{Spec}(K)_{\text{ét}}} \\ \downarrow & & \downarrow \\ [\mathbf{R}\Gamma_{\text{ét}/K}(X,\mathbb{Q}_p),\mathbb{Q}_p]^{\text{cris}} & \longrightarrow & [\mathbf{R}\Gamma_{\text{ét}/K}(X,\mathbb{Q}_p),\mathbb{Q}_p]. \end{array} \tag{3.47}$$

By commutativity of the above diagram, the image of $\mathfrak{X}(\mathcal{V}) \subset X(K)$ must land in $|X_{\text{ét}}|^{hG_K,\text{cris}}_{\text{Spec}(K)_{\text{ét}}}$.

3.3 Continuity

There is a certain sense in which the set $[\mathbf{R}\Gamma_{\text{ét}/K}(X,\mathbb{Q}_\ell),\mathbb{Q}_\ell]$ considered in the previous section is not the 'correct' set of sections of the ℓ-adic rational homotopy type, and the reason for this is that continuity has not been taken into account. For example, if E/K is an elliptic curve, the completed Kummer map $\widehat{E(K)} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to H^1(G_K, V_\ell(E))$ has target the continuous group cohomology of the Tate module $V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, not abstract group cohomology. The set $[\mathbf{R}\Gamma_{\text{ét}/K}(X,\mathbb{Q}_\ell),\mathbb{Q}_\ell]$, however, really is an analogue of abstract group cohomology. Thus the following two questions naturally arise.

1. In what sense is the G_K-action on the ℓ-adic rational homotopy type $\mathbf{R}\Gamma_{\text{ét}/K}(X,\mathbb{Q}_\ell)$ continuous?

2. What is the correct notion of a 'continuous' section $\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$, or in other words, what is the appropriate homotopy category of $\mathbb{Q}_\ell$-dga's with continuous G_K-action?

Answers to both these questions are essential in order to get the 'correct' definition of the set of homotopy sections $[\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_\ell), \mathbb{Q}_\ell]$, as well as the set of crystalline homotopy sections when $\ell = p$. In this section I will propose answers to both these questions, and the two key ideas to the construction of the appropriate homotopy category are as follows:

- mimic the construction of the ℓ-adic derived category in [25], but keeping track of multiplicative structures;
- replace dga's by cosimplicial algebras to make things work in positive characteristic.

For a K-variety X, consider the ringed topos $(X_{\text{ét}}^{\mathbb{N}}, \{\mathbb{Z}/\ell^n\})$ of projective systems of $\{\mathbb{Z}/\ell^n\}$-modules on $X_{\text{ét}}$, and the associated category $\text{Alg}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta$ of cosimplicial algebras in this ringed topos. Consider the natural functors

$$\text{Alg}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta \to \text{Ch}^{\geq 0}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \hookrightarrow \text{Ch}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \tag{3.48}$$

the first arrow of which is the composite of the 'forget the multiplication' functor and the Dold-Kan equivalence. In [25], the triangulated category $D_c^b(X_{\text{ét}}, \mathbb{Z}_\ell)$ is constructed as a suitable full subcategory of a localisation of a full subcategory of the derived category $D(\text{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}))$ of 'essentially bounded' complexes. One then tensors the hom-modules with $\mathbb{Q}_\ell$ to form $D_c^b(X_{\text{ét}}, \mathbb{Q}_\ell)$.

In order to construct a good homotopy category of dga's, it is necessary to first localise with respect to $\mathbb{Z}_\ell \to \mathbb{Q}_\ell$, to properly keep track of multiplicative structures. I will first show that this can be made to work for complexes, and then use this to get a good theory of cosimplicial $\mathbb{Q}_\ell$-algebras. Throughout I will use the terminology of [25].

Call a projective system $M_\bullet$ of $\{\mathbb{Z}/\ell^n\}$-modules on $X_{\text{ét}}$ *essentially zero* if there exists an étale covering $U \to X$ such that for all $n \geq$ there exists some $m \geq n$ such that $M_m|_U \to M_n|_U$ is the zero map. Let $\text{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})$ denote the category of essentially bounded complexes of projective systems on $X_{\text{ét}}$, that is the full subcategory of complexes of projective systems of $\{\mathbb{Z}/\ell^n\}$-modules whose cohomology sheaves are essentially zero outside some bounded range. Let $D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})$ denote its derived category. Let $D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})$ denote the full subcategory of $\mathbb{Z}_\ell$-complexes, that is the full subcategory on systems of complexes $M_\bullet$ such that the mapping cone of the natural morphism $M_\bullet \to (\mathbf{R}\varprojlim_n M_n) \otimes^{\mathbf{L}} \mathbb{Z}/\ell^\bullet$ is essentially zero. Say that a morphism in $D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})$ is an essential quasi-isomorphism if it's mapping cone is essentially zero.

Consider the following diagram of categories, where $- \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is denoted by $(-)_{\mathbb{Q}_\ell}$.

$$\begin{array}{ccc} \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) & \longrightarrow & \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \\ \downarrow & & \downarrow \\ D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) & \longrightarrow & D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \\ \uparrow & & \uparrow \\ D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) & \longrightarrow & D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \\ \downarrow & & \downarrow \\ D^b(X_{\text{ét}}, \mathbb{Z}_\ell) & \longrightarrow & D^b(X_{\text{ét}}, \mathbb{Z}_\ell)_{\mathbb{Q}_\ell} \end{array} \tag{3.49}$$

where the horizontal arrows are all the canonical localisations, the downwards/upwards vertical arrows on the LHS are the localisations/embeddings described above, and the vertical arrows on the RHS are the induced functors on localisations.

Lemma 3.3.1. *The induced functors*

$$\mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \to D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \tag{3.50}$$
$$D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \to D^b(X_{\text{ét}}, \mathbb{Z}_\ell)_{\mathbb{Q}_\ell} \tag{3.51}$$

are categorical localisations, and the induced functor

$$D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \to D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \tag{3.52}$$

is fully faithful.

Remark 3.3.2. A functor $\mathcal{C} \to \mathcal{D}$ is called a categorical localisation if, denoting by W the class of morphisms in $\mathcal{C}$ which map to isomorphisms in $\mathcal{D}$, the natural map $\mathcal{C}[W^{-1}] \to \mathcal{D}$ is an equivalence.

Proof. The last claim concerning full faithfulness is straightforward, and the first follows from simply using the universal property of localisations to show that if $\mathcal{C} \to \mathcal{D}$ is a $\mathbb{Z}_\ell$-linear localisation, then $\mathcal{C}_{\mathbb{Q}_\ell} \to \mathcal{D}_{\mathbb{Q}_\ell}$ is also a localisation. □

Definition 3.3.3. Say that a morphism in $\mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a quasi-isomorphism if it becomes an isomorphism in $D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Say that a morphism in $D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell}$ is an essential quasi-isomorphism if it becomes an isomorphism in $D^b(X_{\text{ét}}, \mathbb{Z}_\ell)_{\mathbb{Q}_\ell}$.

Thus $D^b_c(X_{\text{ét}}, \mathbb{Q}_\ell)$ can be constructed from $\text{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell}$ as follows. First localise at the class of quasi-isomorphisms, then take the full subcategory on $\mathbb{Z}_\ell$-complexes ($\mathbb{Q}_\ell$-complexes might be a more appropriate term), then localise at essential quasi-isomorphisms, and then take a full subcategory. The point of this is that this construction can now be imitated with complexes replaced by cosimplicial algebras.

Let $\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta$ denote the category of cosimplicial algebras over $\{\mathbb{Z}/\ell^n\}$ on $X_{\text{ét}}$, with essentially bounded cohomology. Define the category

$$\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell} \tag{3.53}$$

to be the category whose objects are the same as $\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta$, but whose morphisms are pairs (α, k) where $\alpha : \{A_n\} \to \{B_n\}$ is a morphism in $\text{Mod}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta$ and $k \in \mathbb{Z}_{\geq 0}$ is such that $\ell^k \alpha(xy) = \alpha(x)\alpha(y)$ for all local sections x, y of $\{A_n\}$. The idea is that (α, k) represents the multiplicative morphism $\frac{\alpha}{\ell^k} : \{A_n\} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to \{B_n\} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Thus (α, k) and (β, p) are said to represent the same morphism if $\ell^p \alpha = \ell^k \beta$.

There are functors

$$\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta \to \text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell} \tag{3.54}$$

$$\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \to \text{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \tag{3.55}$$

which send $f : A \to B$ and $(f, n) : A \to B$ to $(f, 0) : A \to B$ and $f \otimes \frac{1}{\ell^n} : A \to B$ respectively.

Call a morphism in $\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell}$ a quasi-isomorphism if it is so on the underlying complexes, that is as a morphism in $\text{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell}$, and define the category

$$\text{Ho}(\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell}) \tag{3.56}$$

to be the localisation of $\text{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell}$ at the class of quasi-isomorphisms. Define $\text{Ho}(\text{Alg}^{eb}_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell})$ to be the full subcategory whose underlying complex is a $\mathbb{Z}_\ell$-complex (again, $\mathbb{Q}_\ell$-complex might be a more appropriate name).

Say that a morphism in $\text{Ho}(\text{Alg}^{eb}_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell})$ is an essential quasi-isomorphism if it is so in $D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell}$, and define the category $\text{Ho}(\text{Alg}^{eb}(X_{\text{ét}}, \mathbb{Q}_\ell)^\Delta)$ to be the localisation of $\text{Ho}(\text{Alg}^{eb}_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell})$ at the class of essential quasi-isomorphisms.

Definition 3.3.4. Define the category of bounded constructible cosimplicial $\mathbb{Q}_\ell$-algebras

$$\text{Ho}^b_c(\text{Alg}(X_{\text{ét}}, \mathbb{Q}_\ell)^\Delta) \subset \text{Ho}(\text{Alg}^{eb}(X_{\text{ét}}, \mathbb{Q}_\ell)^\Delta) \tag{3.57}$$

to be the full subcategory on objects with constructible cohomology.

Remark 3.3.5. Perhaps a word or two is needed to convince the reader that this category has enough morphisms. In the topological case, over $\mathbb{C}$, the functor from the homotopy category of sheaves of $\mathbb{Q}_\ell$-algebras to the derived category of $\mathbb{Q}_\ell$-modules is faithful, and hence when these cosimplicial $\mathbb{Q}_\ell$-algebras have bounded constructible cohomology, every morphism between them is in particular a morphism in the category $D^b_c(X(\mathbb{C}), \mathbb{Q}_\ell)$. Hence every morphism, after multiplying by some ℓ^n, will come from a linear morphism between sub-cosimplicial $\mathbb{Z}_\ell$-algebras. The given algebraic definition is exactly designed to capture all these morphisms.

The following diagram might also be helpful.

$$\begin{array}{ccccc}
\mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) & \rightarrow & \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} & \longleftarrow & \mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell} \\
\downarrow & & \downarrow & & \downarrow \\
D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) & \longrightarrow & D^b(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} & \leftarrow & \mathrm{Ho}(\mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell}) \\
\uparrow & & \uparrow & & \uparrow \\
D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) & \rightarrow & D^b_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} & \leftarrow & \mathrm{Ho}(\mathrm{Alg}^{eb}_{\mathbb{Z}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell}) \\
\downarrow & & \downarrow & & \downarrow \\
D^b(X_{\text{ét}}, \mathbb{Z}_\ell) & \longrightarrow & D^b(X_{\text{ét}}, \mathbb{Z}_\ell)_{\mathbb{Q}_\ell} & \longleftarrow & \mathrm{Ho}(\mathrm{Alg}^{eb}(X_{\text{ét}}, \mathbb{Q}_\ell)^{\Delta}) \\
\uparrow & & \uparrow & & \uparrow \\
D^b_c(X_{\text{ét}}, \mathbb{Z}_\ell) & \longrightarrow & D^b_c(X_{\text{ét}}, \mathbb{Q}_\ell) & \longleftarrow & \mathrm{Ho}^b_c(\mathrm{Alg}(X_{\text{ét}}, \mathbb{Q}_\ell)^{\Delta})
\end{array} \tag{3.58}$$

To define derived functors, one might start with the following.

Conjecture 3.3.6. *There is a model category structure on* $\mathrm{Alg}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}$ *for which the weak equivalences are quasi-isomorphisms.*

In the usual way, the push-forward and pull-back functors for a morphism $f : X \to Y$ would then derive to functors

$$\mathbf{R}f_* : \mathrm{Ho}(\mathrm{Alg}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}) \to \mathrm{Ho}(\mathrm{Alg}(Y_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}) \tag{3.59}$$

$$f^* : \mathrm{Ho}(\mathrm{Alg}(Y_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}) \to \mathrm{Ho}(\mathrm{Alg}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}) \tag{3.60}$$

and one might try to prove that these induce functors on $\mathrm{Ho}^b_c(\mathrm{Alg}(-, \mathbb{Q}_\ell)^{\Delta})$. Instead, I will Godement resolutions to define derived functors. Denote the Godement resolution functor by $G^\bullet : \mathrm{Sh}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \to \mathrm{Sh}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}$, since algebraic structures are preserved by $G^\bullet$, it induces the following commutative squares, which are compatible with the forgetful functor from cosimplicial algebras to complexes.

$$\begin{array}{ccc} \mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta} & \longrightarrow & (\mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta})^{\Delta} \\ \downarrow & & \downarrow \\ \mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell} & \longrightarrow & (\mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta})^{\Delta}_{\mathbb{Q}_\ell} \end{array} \tag{3.61}$$

$$\begin{array}{ccc} \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) & \longrightarrow & \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta} \\ \downarrow & & \downarrow \\ \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} & \longrightarrow & \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell}. \end{array} \tag{3.62}$$

Now define

$$\mathbf{R}f_* := \mathrm{diag} \circ f_* \circ G^{\bullet} : \mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta} \to \mathrm{Alg}(Y_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta} \tag{3.63}$$

$$\mathbf{R}f_* := \mathrm{diag} \circ f_* \circ G^{\bullet} : \mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell} \to \mathrm{Alg}(Y_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell} \tag{3.64}$$

$$\mathbf{R}f_* := \mathrm{Tot}_N \circ f_* \circ G^{\bullet} : \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \to \mathrm{Ch}(Y_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \tag{3.65}$$

$$\mathbf{R}f_* := \mathrm{Tot}_N \circ f_* \circ G^{\bullet} : \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \to \mathrm{Ch}(Y_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} \tag{3.66}$$

where diag is the diagonal of a double cosimplicial object, induced by the diagonal functor $\Delta \to \Delta^2$, and Tot_N takes a cosimplicial complex to the total complex of the associated normalised double complex.

Remark 3.3.7. By the Eilenberg-Zilber theorem $\mathbf{R}f_*$ does not depend on whether an object is considered as a cosimplicial algebra or as a complex, up to natural quasi-isomorphism.

Proposition 3.3.8. $\mathbf{R}f_*$ *descends to a functor*

$$\mathbf{R}f_* : \mathrm{Ho}^b_c(\mathrm{Alg}(X_{\text{ét}}, \mathbb{Q}_\ell)^{\Delta}) \to \mathrm{Ho}^b_c(\mathrm{Alg}(Y_{\text{ét}}, \mathbb{Q}_\ell)^{\Delta}). \tag{3.67}$$

Proof. One easily reduces to the case of showing that

$$\mathbf{R}f_* : \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \to \mathrm{Ch}(Y_{\text{ét}}, \{\mathbb{Z}/\ell^n\}) \tag{3.68}$$

descends to

$$\mathbf{R}f_* : D^b_c(X_{\text{ét}}, \mathbb{Z}_\ell) \to D^b_c(Y_{\text{ét}}, \mathbb{Z}_\ell), \tag{3.69}$$

where it follows from the fact that $\mathbf{R}f_*$ as defined above is naturally quasi-isomorphic to the usual derived functor $\mathbf{R}f_*$. □

Definition 3.3.9. The étale rational homotopy type of a variety $f : X \to \mathrm{Spec}(K)$ is

$$\mathbf{R}f_*(\mathbb{Q}_\ell) \in \mathrm{Ho}_c^b(\mathrm{Alg}(\mathrm{Spec}(K)_{\text{ét}}, \mathbb{Q}_\ell)^\Delta). \tag{3.70}$$

As usual, functoriality induces a map

$$X(K) \to [\mathbf{R}f_*(\mathbb{Q}_\ell), \mathbb{Q}_\ell] \tag{3.71}$$

where the RHS is the set of morphisms in $\mathrm{Ho}_c^b(\mathrm{Alg}(\mathrm{Spec}(K)_{\text{ét}}, \mathbb{Q}_\ell)^\Delta)$.

To tie this in with the results of the previous section, I would next need to prove a comparison theorem between $\mathbf{R}f_*(\mathbb{Q}_\ell)$ and $\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_\ell)$ which would enable me to define crystalline sections, as well as a comparison with $|X_{\text{ét}}|_{\mathrm{Spec}(K)_{\text{ét}}}$, which would enable me to define the 'correct' set of crystalline sections in

$$|X_{\text{ét}}|^{hG_K}_{\mathrm{Spec}(K)_{\text{ét}}} \tag{3.72}$$

by taking continuity of the ℓ-adic rational homotopy type into account. I hope to be able to do this in a future work.

3.4 Relation with other work

3.4.1 Pridham's pro-algebraic homotopy types

Pro-algebraic ℓ-adic homotopy types have also been studied by Pridham in [45], where he defines, for any K-variety X, and any Zariski dense representation

$$\rho : \pi_f^{\text{ét}}(X) \to R(\mathbb{Q}_\ell) \tag{3.73}$$

with R a pro-reductive pro-algebraic groupoid over $\mathbb{Q}_\ell$, a simplicial pro-algebraic groupoid $X^{R,\mathrm{Mal}}_{\overline{K},\text{ét}}$ over $\mathbb{Q}_\ell$, together with a Galois action in an appropriate homotopy category, which he proves to be 'algebraic' in an appropriate sense. When ρ is the trivial representation, his object is closely related to the object $\mathbf{R}\Gamma_{\text{ét}/K}(X, \mathbb{Q}_\ell)$.

The natural question to ask is then whether or not this Galois action can be 'lifted' from the homotopy category, and in what sense this 'lifted' action is algebraic. One could then hope to prove a suitably lifted version of Pridham's p-adic Hodge theory for pro-algebraic homotopy types, and use this to study crystalline sections. This way one could access data about pro-algebraic homotopy types, without reference to base points, rather than just pro-unipotent homotopy types.

It may also be that Pridham's work provides a better setting for studying a suitably lifted version of the p-adic Hodge theory comparison map. This is because 'algebraic' Galois actions would make sense over $\widetilde{B}_{\mathrm{cris}}$, whereas there is no real candidate over $\widetilde{B}_{\mathrm{cris}}$ for the notion of 'continuity' developed in §6.2 for Galois-equivariant dga's over $\mathbb{Q}_p$.

3.4.2 The pro-étale topology of Bhatt/Scholze

Recent work of Bhatt and Scholze also suggests another approach to ℓ-adic rational homotopy theory. In [7], they define, for any scheme X, the pro-étale site $X_{\text{proét}}$ of X, and show how to associate, to any topological ring A, a constant sheaf $\underline{A}$ on $X_{\text{proét}}$. They then prove that when X is a variety over some field k, there is a natural equivalence of categories

$$D^b_c(X_{\text{proét}}, \underline{\mathbb{Q}_\ell}) \cong D^b_c(X_{\text{ét}}, \mathbb{Q}_\ell) \tag{3.74}$$

where the LHS is interpreted literally as the subcategory of the derived category of $\underline{\mathbb{Q}_\ell}$-modules on $X_{\text{proét}}$ with bounded, constructible cohomology, and the RHS is the usual ℓ-adic derived category. By the general theory, there is a model category structure on $\mathrm{dga}(X_{\text{proét}}, \underline{\mathbb{Q}_\ell})$ (again, interpreted literally), and the push-forward functor

$$f_* : \mathrm{dga}(X_{\text{proét}}, \underline{\mathbb{Q}_\ell}) \to \mathrm{dga}(\mathrm{Spec}(k)_{\text{proét}}, \underline{\mathbb{Q}_\ell}) \tag{3.75}$$

is right Quillen. Hence there is a derived functor $\mathbf{R}f_*$, and then the ℓ-adic rational homotopy type is simply

$$\mathbf{R}f_*(\underline{\mathbb{Q}_\ell}) \in \mathrm{Ho}(\mathrm{dga}(\mathrm{Spec}(k)_{\text{proét}}, \underline{\mathbb{Q}_\ell})). \tag{3.76}$$

The relation with the approach of the previous section is the following.

Theorem 3.4.1. *For all varieties X/k, with k a field of characteristic $\neq \ell$, there is a fully faithful functor* $\mathrm{Ho}^b_c(\mathrm{Alg}(X_{\text{ét}}, \mathbb{Q}_\ell)^\Delta) \to \mathrm{Ho}(\mathrm{dga}(X_{\text{proét}}, \underline{\mathbb{Q}_\ell}))$ *which commutes with the functors $f^*, \mathbf{R}f_*$.*

Sketch of proof. I show how to construct the functor. Since sheaves of cosimplicial $\mathbb{Z}/\ell^n$-algebras on $X_{\text{ét}}$ can be viewed as sheaves on $X_{\text{proét}}$, taking homotopy limits will induce a functor

$$\underset{\leftarrow}{\mathrm{holim}} : \mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta \to \mathrm{Alg}^\Delta(X_{\text{proét}}, \underline{\mathbb{Z}_\ell}) \tag{3.77}$$

which, noting the equivalence between $\underline{\mathbb{Q}_\ell}$-dga's and cosimplicial $\underline{\mathbb{Q}_\ell}$-algebras, will induce a functor

$$\mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^\Delta_{\mathbb{Q}_\ell} \to \mathrm{dga}(X_{\text{proét}}, \underline{\mathbb{Q}_\ell}) \tag{3.78}$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Alg}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell} & \longrightarrow & \mathrm{dga}(X_{\text{proét}}, \underline{\mathbb{Q}_\ell}) \\ \downarrow & & \downarrow \\ \mathrm{Ch}^{eb}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})_{\mathbb{Q}_\ell} & \longrightarrow & \mathrm{Ch}(X_{\text{proét}}, \underline{\mathbb{Q}_\ell}) \end{array} \tag{3.79}$$

commutes, where the lower horizontal arrow is induced by $\mathbf{R}\varprojlim$. Thus the upper horizontal arrow sends quasi-isomorphisms to quasi-isomorphisms, and hence descends to a functor

$$\mathrm{Ho}(\mathrm{Alg}^{eb}_{\mathbb{Q}_\ell}(X_{\text{ét}}, \{\mathbb{Z}/\ell^n\})^{\Delta}_{\mathbb{Q}_\ell}) \to \mathrm{Ho}(\mathrm{dga}(X_{\text{proét}}, \underline{\mathbb{Q}_\ell})). \tag{3.80}$$

which is easily seen to commute with f^*. It also commutes with $\mathbf{R}f_*$, since the same is true in $D^b_c(X_{\text{et}}, \mathbb{Q}_\ell)$. □

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Bibliography

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