p-adic cohomology over local fields of characteristic p

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1 Motivation

Suppose that F is a 'p-adic' local field (i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$), and that ℓ is a prime different from p.

Theorem (Grothendieck). *Let (ρ, V) be a continuous ℓ-adic representation of $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Then (ρ, V) is potentially semi-stable, that is there exists a finite extension F'/F such that the inertia group $I_{F'} \subset G_F$ acts unipotently.*

This allows one to attach Weil–Deligne representations to ℓ-adic Galois representations, and these Weil–Deligne representations determine the Galois representation uniquely.

If $\ell = p$, then this theorem is no longer true, for example, when F has characteristic zero, then the inertia action on $\mathbb{Q}_p(1)$ is never quasi-unipotent. This also tells us that even if we restrict to representations 'coming from geometry' then we don't get potentially semistable representations in the naive sense - we need a different notion.

This is provided by p-adic Hodge theory - we have a certain 'ring of semistable periods' $\mathbf{B}_{\mathrm{st}}$, this is a $\mathbb{Q}_p$-algebra together with a G_K-action, and we say that V is semistable if

$$\dim_{F_0} \left(\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\right)^{G_F} = \dim_{\mathbb{Q}_p} V$$

where F_0 is the maximal absolutely unramified subfield of F. A representation is then potentially semistable if there exists some finite extension F'/F such that it is semistable as a $G_{F'}$-representation.

Theorem (Faltings, Berger, Kedlaya, André, Mebkhout). *All p-adic representations coming from geometry are potentially semistable.*

As before, if a representation is potentially semistable, then one can attach a Weil–Delinge representation to it, although now the Galois representation is no longer uniquely determined by its associated Weil-Deligne representation.

This still leaves open the case of what happens in the p-adic case when F is of characteristic p, i.e. $F \cong k((t))$ for some finite field k. But what sort of objects should we even be looking at? What sort of objects arise as the p-adic cohomology of varieties over $k((t))$?

Consider the Amice ring

$$\mathscr{E}_K := \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \,\middle|\, a_i \in K,\ \sup_i |a_i| < \infty,\ |a_i| \to 0 \text{ as } i \to -\infty \right\}$$

where K is complete discrete valuation field with residue field k. This is a complete discrete valuation field of mixed characteristic, with residue field $k((t))$, and Berthelot's rigid cohomology gives, for any $k((t))$-variety X, a graded φ-module

$$H^*_{\mathrm{rig}}(X/\mathscr{E}_K)$$

over $\mathscr{E}_K$, that is a finite dimensional graded vector space together with a 'Frobenius structure'. Actually, we get more - $\mathscr{E}_K$ has a natural differential structure (differentiation with respect to t) and one can show that rigid cohomology groups naturally come with a connection, that is they are (φ,∇)-modules over $\mathscr{E}_K$.

Definition. A (φ,∇)-module over $\mathscr{E}_K$ is a finite dimensional vector space M together with a Frobenius structure φ and a connection ∇ such that φ is horizontal with respect to the connection.

These objects also arise in a closely related context - if G is a p-divisible group over $k((t))$, then crystalline Diedonné theory constructs an associated (φ,∇)-module over $\mathscr{E}_K$, $D(G)$, which completely determines G up to isogeny.

So we have the following natural questions:

- What is the right notion of 'potential semistability' for (φ,∇)-modules over $\mathscr{E}$?
- Can we prove that $H^*_{\text{rig}}(X/\mathscr{E}_K)$ satisfies this condition for all $k((t))$-varieties X?

2 Potential Semistability

To answer the first question, we introduce a certain subring of $\mathscr{E}_K$, the bounded Robba ring

$$\mathscr{E}_K^\dagger = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \in \mathscr{E}_K \,\middle|\, \exists \eta < 1 \text{ s.t. } |a_i|\eta^i \to 0 \text{ as } i \to -\infty \right\},$$

this is a Henselian discrete valuation field, again with residue field $k((t))$. We can similarly define the notion of a (φ,∇)-module over $\mathscr{E}_K^\dagger$, and there is an obvious base extension functor from (φ,∇)-modules over $\mathscr{E}_K^\dagger$ to those over $\mathscr{E}_K$. Actually, thanks to a theorem of Kedlaya, this base extension functor is fully faithful, so it makes sense to speak of a (φ,∇)-module over $\mathscr{E}_K$ being overconvergent, that is coming from one over $\mathscr{E}_K^\dagger$. We have the following results.

Theorem (Trihan)**.** *Let G be a p-divisible group over $k((t))$ with semistable reduction. Then the Dieudonné module $D(G)$ is overconvergent.*

Theorem (Pal, unpublished)**.** *Let G be a semistable p-divisible group over $k((t))$, and $H \subset G$ a sub-p-divisible group. Then H is semistable if and only if $D(H)$ is overconvergent.*

In addition, it is not hard to see that if $F/k((t))$ is a finite separable extension, with corresponding extension $\mathscr{E}_K^F/\mathscr{E}_K$, then a (φ,∇)-module M over $\mathscr{E}_K$ is overconvergent if and only if $M \otimes_{\mathscr{E}_K} \mathscr{E}_K^F$ is. We are therefore tempted to make the following definition.

Definition. Say a (φ,∇)-module over $\mathscr{E}_K$ is potentially semistable if it is overconvergent, i.e. admits an $\mathscr{E}_K^\dagger$-lattice.

Another clue that this is the correct notion is that for such overconvergent modules we can form associated Weil–Deligne representations. Let $\mathscr{R}_K$ denote the Robba ring

$$\mathscr{R}_K = \left\{ \sum_i a_i t^i \in K[\![t,t^{-1}]\!] \,\middle|\, \begin{array}{c} \exists \eta < 1 \text{ s.t. } |a_i|\eta^i \to 0 \text{ as } i \to -\infty \\ \forall \rho < 1,\ |a_i|\rho^i \to 0 \text{ as } i \to \infty \end{array} \right\}$$

then we have $\mathscr{E}_K^\dagger = \mathscr{R}_K \cap \mathscr{E}_K$, and so we can base change (φ,∇)-modules over $\mathscr{E}_K^\dagger$ to those over $\mathscr{R}_K$.

The p-adic local monodromy theorem then tells us that every such module over the Robba ring is quasi-unipotent, just as in the $\ell \neq p$ case, and Marmora used this to show that actually the category of such modules is actually equivalent to the category of Weil–Deligne representations of $G_{k((t))}$. Thus to an overconvergent (φ,∇)-module one can associate a Weil–Deligne representation, although again this does not determine the module, it only determines its base change to $\mathscr{R}_K$.

3 (φ,∇)-modules coming from geometry

Our approach to showing that the (φ,∇)-modules $H^*_{\mathrm{rig}}(X/\mathscr{E}_K)$ are overconvergent is to build a new cohomology theory $H^*_{\mathrm{rig}}(X/\mathscr{E}_K^\dagger)$ taking values in $\mathscr{E}_K^\dagger$ rather than $\mathscr{E}_K$. If we can show that this gives an $\mathscr{E}_K^\dagger$ lattice inside $H^*_{\mathrm{rig}}(X/\mathscr{E}_K)$ (ignoring the (φ,∇)-structures) then it will actually follow almost entirely straightforwardly from the construction that it will form a lattice as a (φ,∇)-module, so we forget about this extra structure for now.

Let us first explain how to construct 'classical' rigid cohomology. One takes a scheme X over $k((t))$, one embeds it into a proper scheme Y over $k((t))$, and then embeds Y into a smooth formal scheme $\mathfrak{P}$ over $\mathscr{O}_{\mathscr{E}_K}$. There is a specialisation map

$$\mathrm{sp} : \mathfrak{P}_{\mathscr{E}_K} \to \mathfrak{P}_0$$

from the generic fibre of $\mathfrak{P}$ to the special fibre, and we consider the tubes

$$]X[_{\mathfrak{P}} = \mathrm{sp}^{-1}(X),\ ]Y[_{\mathfrak{P}} = \mathrm{sp}^{-1}(Y),$$

as well as the natural inclusion

$$j :]X[_{\mathfrak{P}} \to]Y[_{\mathfrak{P}}.$$

One looks at the subsheaf $j^\dagger\mathscr{O}_{]Y[_{\mathfrak{P}}}$ of $j_*\mathscr{O}_{]X[_{\mathfrak{P}}}$ consisting of functions which converge on some strict neighbourhood of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$, the rigid cohomology of X is then the 'overconvergent' de Rham cohomology

$$H^*(]Y[_{\mathfrak{P}}, j^\dagger\mathscr{O}_{]Y[_{\mathfrak{P}}} \otimes \Omega^*_{]Y[_{\mathfrak{P}}}),$$

this only depends on X and not on the choice of Y or $\mathfrak{P}$. Our observation as to how to construct an $\mathscr{E}_K^\dagger$-valued theory is that $\mathscr{E}_K^\dagger$ itself can be viewed as an overconvergent algebra of the form $j^\dagger\mathscr{O}_{]Y[_{\mathfrak{P}}}$, if we are prepared to work with slightly more general formal schemes and rigid varieties.

The triple we want to take is $X = \mathrm{Spec}(k((t)))$, $Y = \mathrm{Spec}\left(k[[t]]\right)$, $\mathfrak{P} = \mathrm{Spf}\left(\mathscr{V}[[t]]\right)$, where $\mathscr{V}$ is the valuation ring of K, and $\mathscr{V}[[t]]$ is equipped with the p-adic, rather than maximal adic topology. Luckily, in Huber's world of adic space, or equivalently Fujuwara/Kato's world of 'rigid Zariski/Riemann spaces', we can make sense of things like the generic fibre of $\mathrm{Spf}\left(\mathscr{V}[[t]]\right)$, i.e. as $\mathrm{Spa}\left(S_K, \mathscr{V}[[t]]\right)$ where $S_K = \mathscr{V}[[t]] \otimes_{\mathscr{V}} K$. We can similarly form the construction $j^\dagger\mathscr{O}_{]Y[_{\mathfrak{P}}}$, of 'functions which converge on a strict neighbourhood of $]\mathrm{Spec}\, k((t))[_{\mathrm{Spf}(\mathscr{V}[[t]])}$', and we find the following:

- The global sections of $j^\dagger\mathscr{O}_{]Y[_{\mathfrak{P}}}$ (with $X, Y, \mathfrak{P}$ as above) is $\mathscr{E}_K^\dagger$.

- The global sections functor is an equivalence between coherent $j^\dagger\mathcal{O}_{]Y[_{\mathfrak{P}}}$-modules and finite dimensional $\mathcal{E}_K^\dagger$-vector spaces.

This suggests that what we should be looking for is a 'relative' version of rigid cohomology, where we work relative to the triple $(k((t)), k[\![t]\!], \mathcal{V}[\![t]\!])$. Thus we make the following definition

Definition. A smooth and proper frame over $\mathcal{V}[\![t]\!]$ is a triple $(X, Y, \mathfrak{P})$ where $X \hookrightarrow Y$ is an open immersion of a $k((t))$ variety into a proper $k[\![t]\!]$-scheme, and $Y \hookrightarrow \mathfrak{P}$ is a closed immersion of Y into a smooth, p-adic, formal $\mathcal{V}[\![t]\!]$-scheme.

Exactly as in the classical case, we can then define the 'overconvergent' de Rham cohomology

$$H^*(]Y[_{\mathfrak{P}}, j^\dagger\mathcal{O}_{]Y[_{\mathfrak{P}}} \otimes \Omega^*_{]Y[/S_K})$$

using Huber's theory of adic spaces to be able to systematically work on the generic fibres of p-adic formal schemes over $\mathcal{V}[\![t]\!]$.

Theorem (L., Pal)**.** *These groups only depend on X, and not on Y or $\mathfrak{P}$, we therefore get well-defined cohomology groups $H^*_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger)$ which are vector spaces over $\mathcal{E}_K^\dagger$.*

The proof is exactly the same as in classical rigid cohomology. Of course the main thing we would like to prove is that $H^*_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger)$ is a lattice inside $H^*_{\mathrm{rig}}(X/\mathcal{E}_K)$. At the moment, we can only do this in dimension 1.

Theorem (L., Pal)**.** *Let $X/k((t))$ be a smooth curve. Then the natural map*

$$H^*_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger) \otimes_{\mathcal{E}_K^\dagger} \mathcal{E}_K \to H^*_{\mathrm{rig}}(X/\mathcal{E}_K)$$

is an isomorphism.

The proof goes by locally using a pushforward construction via a finite étale map to $\mathbb{A}^1$, which reduces to the case of $\mathbb{A}^1$, but with coefficients. We then prove a version of the p-adic local monodromy theorem, which then implies base change via a direct computation.

4 Applications/Speculations

Let $X/k((t))$ be a smooth and proper curve, we can therefore attach a p-adic Weil–Deligne representation to its cohomology

$$H^i_{\mathrm{rig}}(X/\mathcal{R}_K) := H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger) \otimes_{\mathcal{E}_K^\dagger} \mathcal{R}_K.$$

Conjecture (Independence of ℓ)**.** *The family of Weil–Deligne representations attached to*

$$\left\{ \left\{ H^i_{\text{ét}}(X_{k((t))^{\mathrm{sep}}}, \mathbb{Q}_\ell) \right\}_{\ell \neq p}, H^i_{\mathrm{rig}}(X/\mathcal{R}_K) \right\}$$

is a compatible family.

We can also formulate a version of the weight-monodromy conjecture.

Conjecture. *The kth graded part of the monodromy filtration on $H^i_{\mathrm{rig}}(X/\mathcal{R}_K)$ is pure of weight $i+k$.*

These should both be theorems soon!