Rigid cohomology over Laurent series fields

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F= global function field, $\mathrm{char}(F) = p$, $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$, $k =$ field of constants of F.

$$\{\text{algebraic varieties } X \text{ over } F\} \to \{\ell\text{-adic Galois representations } \ell \neq p\}$$
$$X \mapsto H^*_{\text{ét}}(X_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell) \curvearrowleft G_F$$

Question

- Can we describe the Galois representation $H^*_{\text{ét}}(X_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell)$?
- What arithmetic properties of X are reflected in $H^*_{\text{ét}}(X_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell)$?

If v is a place of F, then we can also consider the restriction of these Galois representations to a decomposition group $G_{F_v} \subset G_F$ at v.

Question

- Can we describe $H^*_{\text{ét}}(X_{F^{\text{sep}}}, \mathbb{Q}_\ell)$ as a representation of G_{F_v}?
- What about varieties X/F_v?

Recall that we say a Galois representation is unramified/unipotent if the inertia group $I_{F_v} \subset G_{F_v}$ acts trivially/unipotently.

Example

1. Weight monodromy conjecture describe the eigenvalues of Frobenius elements acting on $H^*_{\text{ét}}(X_{F^{\text{sep}}}, \mathbb{Q}_\ell)$.
2. If X (smooth and proper over F_v) has good/semistable reduction at v then $H^*_{\text{ét}}(X_{F^{\text{sep}}}, \mathbb{Q}_\ell)$ is unramified/unipotent. There exist partial converse results.

Key fact about ℓ-adic Galois representations is the following.

Theorem (Grothendieck's ℓ-adic local monodromy theorem)

Let $\rho : G_{F_v} \to \mathrm{GL}(V)$ be an ℓ-adic Galois representation. Then there exists a finite separable extension F' of F_v such that $\rho|_{G_{F'}}$ is unipotent, i.e. $I_{F'}$ acts unipotently.

- This is a 'cohomological' version of semistable reduction.
- It also allows us to compare $H^*_{\text{ét}}(X_{F_v^{\text{sep}}}, \mathbb{Q}_\ell)$ for different ℓ (as G_{F_v}-representations) by attaching Weil–Deligne representations to them.

What is the p-adic analogue of this story?
$C =$ smooth projective curve whose function field is F.

$\{$geometric Galois representations$\} \leftrightarrow \{\mathbb{Q}_\ell$ sheaves on some $U \subset C\}$

The correct p-adic analogue of the objects appearing on the RHS is well-known, these are *overconvergent F-isocrystals* on U.

$K = W(k)[1/p]$, $\mathfrak{C} =$ lift of C to a rigid analytic curve over K.
Have $\mathrm{sp} : \mathfrak{C} \to C$, each fibre $]x[_{\mathfrak{C}}$ is an open unit disc over K. An overconvergent F-isocrystal on U is then a vector bundle with integrable connection on

$$\mathfrak{C} \setminus \bigcup_{x \in C \setminus U}]x[_{\mathfrak{C}}$$

which extends slightly into the missing discs, together with a 'Frobenius structure'.

To any variety X/F, take a flat model $f : \mathcal{X} \to C$ and lift to a morphism $\mathfrak{X} \to \mathfrak{C}$ of analytic varieties. Then $\mathbf{R}^q_{dR} f_*(\mathcal{O}_{\mathfrak{X}})$ becomes an overconvergent isocrystal on some $U \subset C$.

$$\{p\text{-adic cohomology of } Y\} \leftrightarrow \{\text{de Rham cohomology of a lift } \mathfrak{Y}\}$$

Local analogues will be modules with connection over an appropriate lift of F_v, they should also arise by taking the fibre of an overconvergent F-isocrystal near a 'missing' residue disc $]x[_{\mathfrak{C}}$.

Assume the residue field of v is k, and fix an isomorphism $F_v \cong k(\!(t)\!)$.

Definition

The Robba ring $\mathcal{R}_K$ over K is the ring of analytic functions over K convergent on some half-open annulus $\{\eta \leq |z| < 1\}$.

So it is the ring of series $\sum_{i=-\infty}^{\infty} a_i z^i$ with $a_i \in K$ such that $|a_i| \rho^i \to 0$ as $i \to +\infty$ for all $\rho < 1$, and $|a_i| \eta^i \to 0$ as $i \to -\infty$ for some $\eta < 1$. Let $\mathcal{R}_K^{\mathrm{int}} \subset \mathcal{R}_K$ denote the series with $a_i \in W(k)$ for all i.

Lemma

The map $z \mapsto t$ induces an isomorphism $\mathcal{R}_K^{\mathrm{int}}/p \cong k(\!(t)\!)$.

Let $\partial_z : \mathcal{R}_K \to \mathcal{R}_K$ denote differentiation with respect to z, and $\sigma : \mathcal{R}_K \to \mathcal{R}_K$ the map $\sum_i a_i z^i \mapsto \sum_i \sigma(a_i) z^{ip}$.

Definition

A (φ, ∇)-module over $\mathcal{R}_K$ is a finite free $\mathcal{R}_K$-module M together with:

- a K-linear map $\nabla : M \to M$ such that

$$\nabla(fm) = \partial_z(f)m + f\nabla(m)$$

  for all $f \in \mathcal{R}_K, m \in M$;
- a horizontal isomorphism $\varphi : M \otimes_{\mathcal{R}_K,\sigma} \mathcal{R}_K \to M$.

These are p-adic analogues of ℓ-adic representations of G_{F_v}.

One reason that these are good candidates for p-adic analogues of Galois representations is the following.

Theorem (Andre–Mebkhout, Kedlaya)

Let M be a (φ, ∇)-module over $\mathcal{R}_K$. Then after making a finite separable extension of $k((t))$, the connection ∇ acts via a unipotent matrix.

Just as in the ℓ-adic case, we can use this result to attach p-adic Weil–Deligne representations to (φ, ∇)-modules over $\mathcal{R}_K$ (this was originally done by Marmora).

ℓ-adic picture:

$$\begin{array}{ccc} \{\text{varieties over } F\} & \longrightarrow & \{\text{varieties over } F_v\} \\ \downarrow & & \downarrow \\ \{\ell\text{-adic representations of } G_F\} & \longrightarrow & \{\ell\text{-adic representations of } G_{F_v}\} \end{array}$$

p-adic picture:

$$\begin{array}{ccc} \{\text{varieties over } F\} & \longrightarrow & \{\text{varieties over } F_v\} \\ \downarrow & & \downarrow ? \\ \{\text{overconvergent } F\text{-isocrystals }\} & \longrightarrow & \{(\varphi, \nabla)\text{-modules over } \mathcal{R}_K\} \end{array}$$

Question

- How can we associate (φ, ∇)-modules over $\mathcal{R}_K$ to varieties X over $k((t))$?
- What can we say about the relationship between these (φ, ∇)-modules and the arithmetic of X?
- Can we compare these (φ, ∇)-modules to the ℓ-adic representations $H^*_{\text{ét}}(X_{F_v^{\text{sep}}}, \mathbb{Q}_\ell)$?

Rigid cohomology is a p-adic cohomology theory for varieties over fields of characteristic p, constructed via de Rham cohomology in characteristic 0. We'll work over k for simplicity, and write $W = W(k)$.

Definition

A frame over W is a triple $(X, Y, \mathfrak{P})$ with $X \hookrightarrow Y$ an open immersion of k-varieties, and $Y \hookrightarrow \mathfrak{P}$ a closed immersion of formal W-schemes. We say that the frame is proper if Y is proper over k and smooth if $\mathfrak{P}$ is smooth over W.

$\mathfrak{P}_K$ = generic fibre of $\mathfrak{P}$, considered as an adic space over K.

$$\mathrm{sp} : \mathfrak{P}_K \to \mathfrak{P}$$

$$]Y[_{\mathfrak{P}} = \mathrm{sp}^{-1}(Y)^{\circ}, \quad]X[_{\mathfrak{P}} := \overline{\mathrm{sp}^{-1}(X)} \subset]Y[_{\mathfrak{P}}$$

$$j :]X[_{\mathfrak{P}} \to]Y[_{\mathfrak{P}}$$

Example

1. $(\mathbb{A}^1_k, \mathbb{P}^1_k, \widehat{\mathbb{P}}^1_W)$. Then $]Y[_{\mathfrak{P}} = \mathfrak{P}_K = \mathbb{P}^{1,\mathrm{an}}_K$ and $]X[_{\mathfrak{P}} = \overline{\mathbb{D}_K(0, 1^+)}$.
2. $(\mathrm{Spec}(k), \mathrm{Spec}(k), \widehat{\mathbb{A}}^1_W)$. Then $]X[_{\mathfrak{P}} =]Y[_{\mathfrak{P}} = \mathbb{D}_K(0, 1^-)$.

Definition

$$j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}} := j_* j^{-1} \mathcal{O}_{]Y[_{\mathfrak{P}}}$$

$$H^i_{\mathrm{rig}}(X/K) := H^i(]Y[_{\mathfrak{P}}, j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}} \otimes \Omega^*_{]Y[_{\mathfrak{P}}/K})$$

Then $j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}}$ consists of 'overconvergent functions', that is functions which converge on some open neighbourhood V of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$. If $(X, Y, \mathfrak{P})$ is smooth and proper then the rigid cohomology $H^i_{\mathrm{rig}}(X/K)$ doesn't depend on any of the choices.

Example

1. $(\mathbb{A}^1_k, \mathbb{P}^1_k, \widehat{\mathbb{P}}^1_W)$. Then $j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}}$ is the ring of functions $f = \sum_{i \geq 0} a_i z^i$ which converge on some disc $|z| \leq \rho$ with $\rho > 1$ (depending of f). This has trivial de Rham cohomology. If we just took a lift of X without worrying about overconvergence, then we would get the ring of functions which converge on $|z| \leq 1$, which has infinite dimensional de Rham cohomology.
2. $(\mathrm{Spec}(k), \mathrm{Spec}(k), \widehat{\mathbb{A}}^1_W)$. Then $j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}} = \mathcal{O}_{]Y[_{\mathfrak{P}}} = \mathcal{O}_{\mathbb{D}_K(0,1^-)}$. Again, this has trivial de Rham cohomology.

In Berthelot's theory, we can replace k with *any* field L of characteristic p, as long as we replace K with a complete, discretely valued field of characteristic 0 whose residue field L.

So what about $k((t))$? Unfortunately the Robba ring $\mathcal{R}_K$ is *not* a complete discretely valued p-adic field (it's not even a field). What we can use instead is the Amice ring.

Definition

The Amice ring $\mathcal{E}_K$ over K consists of series $\sum_i a_i z^i$ with $a_i \in K$ such that $\sup_i |a_i| < \infty$ and $a_i \to 0$ as $i \to -\infty$.

This *is* a complete, discretely valued field with residue field $k((t))$ (via $z \mapsto t$), and rigid cohomology gives (φ, ∇)-modules $H^i_{\mathrm{rig}}(X/\mathcal{E}_K)$ associated to $X/k((t))$.

We have $\mathcal{E}_K, \mathcal{R}_K \subset K[\![t, t^{-1}]\!]$ but $\mathcal{R}_K \not\subset \mathcal{E}_K$ and $\mathcal{E}_K \not\subset \mathcal{R}_K$.

Definition

We define the bounded Robba ring $\mathcal{E}_K^\dagger := \mathcal{E}_K \cap \mathcal{R}_K$, this is also equal to $\mathcal{R}_K^{\mathrm{int}}[1/p]$.

$$\mathcal{E}_K \leftarrow \mathcal{E}_K^\dagger \rightarrow \mathcal{R}_K$$

$\mathcal{E}_K^\dagger$ is a henselian d.v.f. whose residue field is $k(\!(t)\!)$, but is no longer complete.

Aim

Define a version of rigid cohomology taking values in vector spaces over $\mathcal{E}_K^\dagger$.

Main idea: interpret $\mathcal{E}_K^\dagger$ as something of the form $j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}}$, then our theory will just be a relative version of rigid cohomology.

- The Robba ring $\mathcal{R}_K$ consists of functions which converge 'on some neighbourhood of the boundary' of the open unit disc $\{|z| < 1\}$.
- The bounded Robba ring $\mathcal{E}_K^\dagger$ is exactly the functions in $\mathcal{R}_K$ which are bounded on some annulus $\eta \leq |z| < 1$.

So we want to consider a space which looks like the open unit disc $\mathbb{D}_K(0, 1^-)$ over K, but which has 'boundary points' and whose functions look like functions on $\mathbb{D}_K(0, 1^-)$ which are actually bounded.

Define the formal scheme $\mathbb{D}^b_W := \mathrm{Spf}(W[\![t]\!])$, using the p-adic topology. We let $\mathbb{D}^b_K = \mathrm{Spec}(S_K, W[\![t]\!])$ denote its generic fibre, where $S_K = W[\![t]\!] \otimes_W K$ is the ring of bounded analytic functions on $\mathbb{D}_K(0, 1^-)$.

Then $\mathbb{D}^b_K = \mathbb{D}_K(0, 1^-) \cup \{\xi_-, \xi\}$ where ξ is an open point such that $\overline{\{\xi\}} = \{\xi_-, \xi\}$. Moreover, for any open subset $U \supset \{\xi_-, \xi\}$,

$$\Gamma(U, \mathcal{O}_{\mathbb{D}^b_K}) = \{f \in \Gamma(U \cap \mathbb{D}_K(0, 1^-), \mathcal{O}_{\mathbb{D}_K(0,1^-)}) \mid f \text{ bounded}\}$$

The frame we will consider is therefore the triple

$$\left(\mathrm{Spec}(k((t))), \mathrm{Spec}(k[\![t]\!]), \mathbb{D}^b_W\right)$$

and we have a specialisation map

$$\mathrm{sp} : \mathbb{D}^b_K \to \mathbb{D}^b_W$$

such that

$$]\mathrm{Spec}(k((t)))[_{\mathbb{D}^b_K} = \overline{\mathrm{sp}^{-1}(\mathrm{Spec}(k((t))))} = \{\xi_-, \xi\}.$$

Lemma

$\Gamma(\mathbb{D}^b_K, j^\dagger_{\mathrm{Spec}(k((t)))}\mathcal{O}_{\mathbb{D}^b_K}) = \mathcal{E}^\dagger_K$, and $\Gamma(\mathbb{D}^b_K, -)$ induces an equivalence of categories between coherent $j^\dagger_{\mathrm{Spec}(k((t)))}\mathcal{O}_{\mathbb{D}^b_K}$-modules and finite dimensional $\mathcal{E}^\dagger_K$-vector spaces.

Therefore the key definition is the following.

Definition

A frame over $W[\![t]\!]$ is a triple $(X, Y, \mathfrak{P})$ where $X \hookrightarrow Y$ is an open immersion of a $k(\!(t)\!)$-variety into a $k[\![t]\!]$-scheme of finite type, and $Y \hookrightarrow \mathfrak{P}$ is a closed immersion of Y into a p-adic formal scheme, topologically of finite type over $W[\![t]\!]$. We say that the frame is proper if Y is so over $k[\![t]\!]$, and smooth if $\mathfrak{P}$ is so over $W[\![t]\!]$.

Then everything proceeds as in the classical case. We have a closed immersion

$$j :]X[_{\mathfrak{P}} \to]Y[_{\mathfrak{P}}$$

of tubes, and we define

$$H^i_{\mathrm{rig}}(X/\mathcal{E}^\dagger_K) = H^i(]Y[_{\mathfrak{P}}, j_*j^{-1}\mathcal{O}_{]Y[_{\mathfrak{P}}} \otimes \Omega^*_{]Y[_{\mathfrak{P}}/S_K}).$$

whenever $(X, Y, \mathfrak{P})$ is smooth and proper.

Theorem (L., Pál)

The groups $H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger)$ are well-defined and finite dimensional over $\mathcal{E}_K^\dagger$. Moreover, they are naturally equipped with the structure of (φ, ∇)-modules over $\mathcal{E}_K^\dagger$ and there is a natural base change isomorphism

$$H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger) \otimes_{\mathcal{E}_K^\dagger} \mathcal{E}_K \to H^i_{\mathrm{rig}}(X/\mathcal{E}_K).$$

The following definition thus makes sense.

Definition

We define $H^i_{\mathrm{rig}}(X/\mathcal{R}_K) := H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger) \otimes_{\mathcal{E}_K^\dagger} \mathcal{R}_K$. These are (φ, ∇)-modules over $\mathcal{R}_K$.

$$\begin{array}{ccc} \{\text{varieties over } F\} & \longrightarrow & \{\text{varieties over } F_v\} \\ \downarrow & & \downarrow \\ \{\text{overconvergent } F\text{-isocrystals }\} & \longrightarrow & \{(\varphi, \nabla)\text{-modules over } \mathcal{R}_K\} \end{array}$$

We can now attach p-adic Weil–Deligne representations to varieties over $k((t))$, using the local monodromy theorem.

Theorem (L., Pál)

Let $X/k((t))$ be a curve. Then the p-adic Weil–Deligne representation attached to $H^i_{\mathrm{rig}}(X/\mathcal{R}_K)$ satisfies the weight-monodromy conjecture.

Theorem (L., Pál)

Let $X/k((t))$ be a curve. Then the p-adic Weil–Deligne representation attached to $H^i_{\mathrm{rig}}(X/\mathcal{R}_K)$ is 'compatible' with the family of ℓ-adic ones attached to $H^i_{\acute{e}t}(X_{k((t))^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ for $\ell \neq p$.