# The homotopy exact sequence for overconvergent isocrystals

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1 Introduction

2 Pro-algebraic fundamental groups

3 Overconvergent isocrystals

4 Proof of $p$-adic HES

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More sophisticated answer: a morphism $X \to S$ admitting a proper hypercover $X_\bullet \to X$ and a compactification $X_\bullet \to \overline{X}_\bullet$ such that $\overline{X}_\bullet \to S$ is smooth and proper and $\overline{X}_\bullet \setminus X_\bullet$ is a relative NCD.

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$f : X \to S$ is a smooth and projective morphism of varieties over a field $k$, with geometrically connected base and fibres. Fix $x \in X(k)$ and set $s = f(x)$.

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Thus we expect to see a right exact sequence

$$\pi_1(X_s, x) \to \pi_1(X, x) \to \pi_1(S, s) \to 1$$

of fundamental groups (whatever they are!).

# The étale fundamental group

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$$\text{FÉt}(Y) \cong \pi_1^{\text{ét}}(Y, \bar{y})\text{-FSet}$$

between finite étale covers of $Y$ and finite (discrete) $\pi_1^{\text{ét}}(Y, \bar{y})$-sets, such that the forgetful functor

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**Theorem (Grothendieck)**

*Assume the Basic Setup, and let $\bar{x} \to x$ be a geometric point over $x$, with corresponding geometric point $\bar{s}$ over $s$. Then the sequence*

$$\pi_1^{\text{ét}}(X_{\bar{s}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to \pi_1^{\text{ét}}(S, \bar{s}) \to 1$$

*of pro-finite fundamental groups is exact.*

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# Tannakian duality

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Such a functor is called a *fibre functor*. If we can choose $F' = F$ then we say that $\mathcal{T}$ is *neutral* Tannakian.

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**Theorem (Saavedra)**

*Let $\mathcal{T}$ be a neutral Tannakian category over $F$, with fibre functor*

$$\omega : \mathcal{T} \to \mathrm{Vec}_F.$$

*Then there exists a unique pro-algebraic group $G = G(\mathcal{T}, \omega)$ over $F$, and an equivalence $\mathrm{Rep}(G) \cong \mathcal{T}$ which identifies $\omega : \mathcal{T} \to \mathrm{Vec}_F$ with the forgetful functor $\mathrm{Rep}(G) \to \mathrm{Vec}_F$.*

## Examples

1. Let $Y$ be a normal, connected, Noetherian scheme, and $\mathrm{Loc}^{\text{ét}}_{\mathbb{Q}_\ell}(Y)$ the category of lisse $\mathbb{Q}_\ell$-sheaves on $Y_{\text{ét}}$. Then $\mathrm{Loc}^{\text{ét}}_{\mathbb{Q}_\ell}(Y)$ is neutral Tannakian over $\mathbb{Q}_\ell$, and any geometric point $\bar{y} \to Y$ provides a fibre functor

$$\begin{aligned} \mathrm{Loc}^{\text{ét}}_{\mathbb{Q}_\ell}(Y) &\to \mathrm{Vec}_{\mathbb{Q}_\ell} \\ \mathcal{F} &\mapsto \mathcal{F}_{\bar{y}}. \end{aligned}$$

The corresponding fundamental group $\pi_1^{\text{ét}}(Y, \bar{y})_{\mathbb{Q}_\ell}$ is the $\mathbb{Q}_\ell$-pro-algebraic completion of $\pi_1^{\text{ét}}(Y, \bar{y})$. This is 'well-behaved' only when $\ell$ is invertible on $Y$.

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2. $Y/k$ a smooth, geometrically connected variety over a field $k$ of characteristic 0. Then the category $\mathrm{MIC}(Y/k)$ of vector bundles with integrable connection on $Y$ is Tannakian over $k$. If there exists a rational point $y \in Y(k)$ it is moreover neutral Tannakian, and

$$y^* : \mathrm{MIC}(Y/k) \to \mathrm{Vec}_k$$

is a fibre functor. This gives rise to the de Rham fundamental group $\pi_1^{\mathrm{dR}}(Y, y)$.

# Examples (contd.)

3. More generally, if $Y/k$ is a smooth, geometrically connected variety over any field $k$, then the category $\mathrm{Strat}(X/k)$ of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is Tannakian over $k$. If there exists a rational point $y \in Y(k)$, then it is moreover neutral Tannakian, and

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is a fibre functor. This gives rise to the stratified fundamental group $\pi_1^{\mathrm{strat}}(Y, y)$. If $\mathrm{char}(k) = 0$ then $\pi_1^{\mathrm{dR}}(Y, y) = \pi_1^{\mathrm{strat}}(Y, y)$.

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4. If $K$ is a complete, valued field of characteristic 0, and $Y/K$ is a smooth, geometrically connected analytic variety, then the category $\mathrm{MIC}(Y/K)$ of analytic vector bundles with integrable connection on $Y$ is Tannakian over $K$. If $y \in Y(K)$ is a rational point, then it is neutral Tannakian, and
$$y^* : \mathrm{MIC}(Y/K) \to \mathrm{Vec}_K$$
is a fibre functor. The corresponding fundamental group is denoted $\pi_1^{\mathrm{dR}}(Y, y)$.

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   can be deduced over $\mathbb{C}$ (more or less) using the Riemann–Hilbert correspondence. In general, it was proved using transcendental methods by Zhang.

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4. We'll come back to this one!

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Now suppose $k$ is perfect and $\mathrm{char}(k) = p > 0$. A $p$-adic analogue of $\mathrm{MIC}(Y/k)$ or $\mathrm{Strat}(Y/k)$ is given by the category of overconvergent isocrystals.

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To describe this category, let $\mathcal{V}$ be a complete DVR with residue field $k$ and fraction field $K$ of characteristic 0. Assume that $Y$ is smooth, and that there exists a projective formal scheme $\mathfrak{Y}$ over $\mathcal{V}$ and an open embedding $Y \hookrightarrow \mathfrak{Y}_k$ such that:

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Let us call such an embedding $Y \hookrightarrow \mathfrak{Y}$ a 'good embedding'.

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is defined by $\{|t| \geq 1\}$. We can therefore consider the 'strict neighbourhoods' $]Y[ \subset V_\lambda \subset \mathfrak{Y}_K$ defined locally by $\{|t| \geq \lambda\}$ for $\lambda \to 1^-$.

# Overconvergent fundamental groups

If $\lambda$ is close enough to 1, then the $V_\lambda$ are smooth over $K$, and by definition

$$\mathrm{Isoc}^\dagger(Y/K) \subset 2\text{-colim}_\lambda \mathrm{MIC}(V_\lambda)$$

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**Theorem (Crew)**

*If $Y/k$ is geometrically connected, $\mathrm{Isoc}^\dagger(Y/K)$ is Tannakian over $K$. If $y \in Y(k)$ is a rational point, then it is neutral Tannakian, and*

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We define the overconvergent fundamental group $\pi_1^\dagger(Y, y)$ to be the associated pro-algebraic group over $K$.

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## Applications

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### Corollary

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## Proof.

Put $Y$ into a Lefschetz pencil $\widetilde{X} \to \mathbb{P}_k^1$ with a section $\mathbb{P}_k^1 \to \widetilde{X}$, where $\widetilde{X} \to X$ is a blowup. Now apply the HES over the smooth locus of $\widetilde{X} \to \mathbb{P}_k^1$. □

## Applications (contd.)

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$$\pi_1^\dagger(X, x) \to \pi_1^{\text{ét}}(X, x)$$

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*This induces an isomorphism* $\pi_0(\pi_1^\dagger(X, x)) \cong \pi_1^{\text{ét}}(X, x)$.

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## Corollary

*This induces an isomorphism* $\pi_0(\pi_1^\dagger(X,x))\cong\pi_1^{\text{ét}}(X,x)$.

## Proof.

We want to show that any $E\in\mathrm{Isoc}^\dagger(X/K)$ with finite monodromy group is trivialised by a finite étale cover of $X$. By a result of Crew, it suffices to show that $E$ admits a Frobenius structure. Using the Lefschetz theorem, this can be reduced to the case of curves, where in fact it suffices to show that $E$ can be trivialised by a finite *separable* map. We can now argue by lifting to characteristic 0. □

1 Introduction

2 Pro-algebraic fundamental groups

3 Overconvergent isocrystals

4 Proof of $p$-adic HES

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2. Show how to reduce the algebraic result for isocrystals over $k$ to the analytic result for vector bundles with integrable connection.

The first can be achieved by transporting dos Santos' methods from algebraic geometry to analytic geometry. I will focus on explaining the second.

# Tannakian criteria

Since $G$ can be recovered from $\mathrm{Rep}(G)$, it is natural to ask if we can phrase exactness of a sequence

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In practise, (1) and (2) are rather straightforward to check, but (3) almost impossible.

# Weak exactness

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**Definition**

We say that a sequence

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## Proposition

*Let*

$$K \xrightarrow{a} G \xrightarrow{b} H \to 1$$

*be a sequence of pro-algebraic groups, such that $b \circ a$ is trivial, and $b$ is surjective. Then the sequence is weakly exact iff the following two conditions hold.*

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2. *If $V \in \mathrm{Rep}(G)$, and $U_0 \subset a^*(V)$ is the largest trivial sub-object, then there exists some $V_0 \subset V$ such that $a^*(V_0) = U_0$.*

# Geometric push-forwards

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**Proposition**

*Assume the Basic Setup, with $k$ perfect of characteristic $p > 0$, and $S$ smooth. Then there exists a push-forward functor*

$$f_* : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(S/K)$$

*right adjoint to $f^*$, such that*

$$s^* f_* E \cong H^0_{\mathrm{rig}}(X_s/K, E|_{X_s})$$

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**Corollary**

*The sequence*

$$\pi_1^\dagger(X_s, x) \to \pi_1^\dagger(X, x) \to \pi_1^\dagger(S, s) \to 1$$

*is weakly exact.*

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$$\widetilde{X} := \left\{ (x, H) \in X \times_S \check{\mathbb{P}}^n_S \middle| \, x \in H \right\} \subset X \times_S \widetilde{S}.$$

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\widetilde{X}_{\tilde{s}} & \longrightarrow & \widetilde{X}_U & \longrightarrow & U \\
\| & & \downarrow & & \downarrow \\
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\downarrow & & \downarrow & & \downarrow \\
X_s & \longrightarrow & X & \longrightarrow & S
\end{array}$$

where $(\widetilde{X}_U, \tilde{x}) \to (U, \tilde{s})$ is as in the Basic Setup, but with relative dimension $d-1$.

We therefore have the diagram

$$\begin{array}{ccccccc}
\pi_1^\dagger(\widetilde{X}_{\tilde{s}},\tilde{x}) & \longrightarrow & \pi_1^\dagger(\widetilde{X}_U,\tilde{x}) & \longrightarrow & \pi_1^\dagger(U,\tilde{s}) & \longrightarrow & 1 \\
\Vert & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
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*The normal closure of the image of $\pi_1^\dagger(\widetilde{X}_{\tilde{s}},\tilde{x}) \to \pi_1^\dagger(X_s,x)$ is the whole of $\pi_1^\dagger(X_s,x)$.*

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By some diagram chasing we can therefore deduce that if the homotopy sequence for $\widetilde{X}_U \to U$ is exact, then so is the homotopy sequence for $X \to S$. By induction we may therefore assume that $d = 1$.

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In particular, we can lift $S$ to a smooth affine $\mathcal{V}$-scheme $\mathrm{Spec}\,(A)$, and the family $X \to S$ to a smooth projective family of curves over $\mathrm{Spec}\,(A)$.

Thus there exist good embeddings $S \hookrightarrow \mathfrak{S}$ and $X \hookrightarrow \mathfrak{X}$ and a commutative, *Cartesian* diagram

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If we now let $W_\lambda$ the associated 'strict neighbourhoods' of $]X[$ and $V_\lambda$ those of $]S[$, then for $\lambda$ closed enough to 1 there are induced *smooth and projective* maps $W_\lambda \to V_\lambda$.

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$$\pi_1^{\mathrm{dR}}(\mathfrak{X}_{K,\tilde{s}}, \tilde{x}) \to \pi_1^{\mathrm{dR}}(W_\lambda, \tilde{x}) \to \pi_1^{\mathrm{dR}}(V_\lambda, \tilde{s}) \to 1$$

of pro-algebraic groups over $K$, for all $\lambda$ close enough to 1.

Since we definition we have

$$\begin{aligned} \mathrm{Isoc}^\dagger(X/K) &\subset 2\text{-colim}_\lambda \mathrm{MIC}(W_\lambda/K) \\ \mathrm{Isoc}^\dagger(S/K) &\subset 2\text{-colim}_\lambda \mathrm{MIC}(V_\lambda/K) \\ \mathrm{Isoc}^\dagger(X_s/K) &\subset \mathrm{MIC}(\mathfrak{X}_{K,\tilde{s}}/K) \end{aligned}$$

stable by sub-quotients, we get a commutative diagram

$$\begin{array}{ccccccc} \pi_1^{\mathrm{dR}}(\mathfrak{X}_{K,\tilde{s}},\tilde{x}) & \longrightarrow & \varprojlim_\lambda \pi_1^{\mathrm{dR}}(W_\lambda,\tilde{x}) & \longrightarrow & \varprojlim_\lambda \pi_1^{\mathrm{dR}}(V_\lambda,\tilde{s}) & \longrightarrow & 1 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\ \pi_1^\dagger(X_s,x) & \longrightarrow & \pi_1^\dagger(X,x) & \longrightarrow & \pi_1^\dagger(S,s) & \longrightarrow & 1 \end{array}$$

with exact top row. Again, some diagram chasing together with weak exactness lets us deduce exactness of the bottom row.

Thank-you!