RIGID RATIONAL HOMOTOPY THEORY AND MIXEDNESS

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1. Introduction

Suppose that k is a finite field, and X/k is a geometrically connected variety (= separated scheme of finite type).

Question. Is there a way to 'do algebraic topology on X'?

More specifically, we can ask if there are cohomology functors $H^i(X)$ which is some way behave like the singular cohomology $X \mapsto H^i(X(\mathbb{C}), \mathbb{Z})$ for $\mathbb{C}$-varieties X? Famously, the answer to this question is yes, if one is prepared to work with other coefficient rings than $\mathbb{Z}$.

Example. Let $K = W(k)[1/p]$. Then rigid cohomology $X \mapsto H^i_{\mathrm{rig}}(X/K)$ is such a cohomology theory, with coefficients in the field K.

Of course, the search for such cohomology theories was motivated by the Weil conjectures, the full proof of which comes from a detailed study of the action of Frobenius on these spaces.

Example. (Kedlaya, Chiarellotto) If X is smooth, then $H^i_{\mathrm{rig}}(X/K)$ admits a weight filtration for the action of Frobenius, with weights in the range $[i, 2i]$.

If we want to start doing homotopy theory, then we need a slightly more sophisticated cohomology theory. For example, categories of coefficients (local systems) determine linearised versions of the fundamental group.

Example. Let $\mathcal{N}\mathrm{Isoc}^\dagger(X/K)$ denote the category of unipotent overconvergent isocrystals on X/K. Then for every k-point $x \in X(k)$, there there is a unique pro-unipotent group $\pi_1^{\mathrm{rig}}(X/K, x)$ such that the pullback functor $x^* : \mathcal{N}\mathrm{Isoc}^\dagger(X/K) \to \mathrm{Vec}_K$ induces an equivalence of categories $\mathcal{N}\mathrm{Isoc}^\dagger(X/K) \xrightarrow{\sim} \mathrm{Rep}_K(\pi_1^{\mathrm{rig}}(X/K, x))$. That is, $\mathcal{N}\mathrm{Isoc}^\dagger(X/K)$ is a Tannakian category and x^* is a fibre functor.

There is also a natural action of Frobenius on $\pi_1^{\mathrm{rig}}(X/K, x)$, and Chiarellotto proved the existence of a 'weight filtration' in the smooth case.

Questions. Is there 'something common' lying behind both $\pi_1^{\mathrm{rig}}(X/K, x)$ and $H^i_{\mathrm{rig}}(X/K)$, from which we can deduce the weight filtration on both $\pi_1^{\mathrm{rig}}(X/K, x)$ and $H^i_{\mathrm{rig}}(X/K)$? Can we show that weight filtrations exist in the non-smooth case? What about higher homotopy groups?

2. Rational homotopy theory

The clue to what this object should be comes from traditional rational homotopy theory.

Definition. (1) A nilpotent space is a topological space X such that for every $x \in X$, $\pi_1(X,x)$ is a nilpotent group, and the action of $\pi_1(X,x)$ on $\pi_n(X,x)$ is nilpotent for $n \geqslant 2$.

(2) A rational space is a nilpotent topological space X such that $H^i(X,\mathbb{Z})$ is a finite dimensional $\mathbb{Q}$-vector space for all i.

Then every nilpotent space X can be 'localised' to given a rational space $X_{\mathbb{Q}}$, such that $H^i(X_{\mathbb{Q}},\mathbb{Z}) \cong H^i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $i \geqslant 0$ and $\pi_n(X_{\mathbb{Q}},x) \cong \pi_n(X,x) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \geqslant 1$ and $x \in X$. (Of course, we have to be slightly careful what we mean by $\pi_1(X,x) \otimes_{\mathbb{Z}} \mathbb{Q}$.)

Theorem. *(Sullivan, '77) The space $X_{\mathbb{Q}}$ can be recovered up to homotopy from the complex $\mathbf{R}\Gamma(X,\mathbb{Q})$ together with it's natural multiplicative structure.*

What do we mean by the 'natural multiplicative structure' on $\mathbf{R}\Gamma(X,\mathbb{Q})$? Well, since $\mathbb{Q}_X$ is a sheaf of $\mathbb{Q}$-algebras, the Godement resolution $G_\bullet(\mathbb{Q}_X)$ has a natural multiplication making it into a sheaf of $\mathbb{Q}$-cdga's, hence $\mathbf{R}\Gamma(X,\mathbb{Q}) = \Gamma(X, G_\bullet(\mathbb{Q}_X))$ is naturally a cdga over $\mathbb{Q}$. Then Sullivan's theorem tells us that this cdga completely determines $X_{\mathbb{Q}}$ up to homotopy.

Of course, the cdga $\mathbf{R}\Gamma(X,\mathbb{Q})$ makes sense for any topological space X, in general we will be able to recover $H^i(X,\mathbb{Q})$ from this cdga (this is obvious!) but also we can recover the pro-unipotent completion of the fundamental group $\pi_1(X,x)_{\mathbb{Q}}$ (this is less obvious!). There is also a certain shadow of the higher homotopy groups $\pi_n(X,x)$ that can be recovered from $\mathbf{R}\Gamma(X,\mathbb{Q})$.

Slogan: "cohomology + multiplicative structure = rational homotopy theory".

3. Rigid homotopy theory

So wherever we have a good cohomology theory, there should be a corresponding rational homotopy theory, obtained by 'remembering the multiplicative structure'. For rigid cohomology this works as follows.

First suppose we can embed X in smooth smooth and proper formal $\mathcal{V}$-scheme $\mathfrak{P}$, and let Y be the closure. So we have the tubes $]X[_{\mathfrak{P}},]Y[_{\mathfrak{P}} \subset \mathfrak{P}_K$ inside the generic fibre of $\mathfrak{P}$. The rigid cohomology of X is then defined to be

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K) := \mathbf{R}\Gamma(]Y[_{\mathfrak{P}}, j^\dagger\Omega^*_{]Y[_{\mathfrak{P}}}) \tag{1}$$

the hyper-cohomology of the overconvergent de Rham complex - this can be show to be independent of the embedding $X \hookrightarrow \mathfrak{P}$, up to quasi-isomorphism.

By the same sort of general nonsense as above, since the wedge product of differential forms gives $j^\dagger\Omega^*_{]Y[_{\mathfrak{P}}}$ the structure of a sheaf of cdga's, $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K)$ has a natural structure as a cdga, rather than just a complex. Since the same is true of the underlying complex, this cdga which is well-defined (i.e. independent of the chosen embedding) and functorial in the homotopy category $\mathrm{Ho}(\mathrm{cdga}_K)$ (where quasi-isomorphisms are inverted). This will then represent the rigid rational homotopy type of X.

In general, we won't be able to embed X into some $\mathfrak{P}$ as above, and we will have to use descent to construct the rational homotopy type. That is, choose a Zariski/étale/smooth/proper (anything for which cohomological descent holds) hyper-cover $X_\bullet \to X$, so we get a cosimplicial object $\mathbf{R}\Gamma_{\mathrm{rig}}(X_\bullet/K)$ in $\mathrm{Ho}(\mathrm{cdga}_K)$. In fact, by being a little bit more careful about how we choose our hypercover, we can ensure that we can lift this to a cosimplicial object $\mathbf{R}\Gamma_{\mathrm{rig}}(X_\bullet/K)$ in the category of cdga's.

Definition. The rigid rational homotopy type of X is

$$\mathbf{R}\Gamma_{\mathrm{rig}}(X/K) := \mathrm{holim}_\Delta \mathbf{R}\Gamma_{\mathrm{rig}}(X_\bullet/K). \tag{2}$$

Remark.

(1) If you don't like hyper-covers, then just think of a usual Zariski cover $\{U_i \to X\}$. The associated hyper-cover $U_\bullet \to X$ has $U_0 = \coprod_i U_i$ and then $U_n = U_0 \times_X \ldots \times_X U_0 = \coprod_{i_0,\ldots,i_n} U_{i_0} \cap \ldots \cap U_{i_n}$.

(2) This homotopy limit can be made quite explicit, and (up to an explicit quasi-isomorphism) the underlying complex is very easy to understand - it is nothing more than the total complex associated to a cosimplicial cochain complex.

Proposition. *We can recover both $H^i_{\mathrm{rig}}(X/K)$ and $\pi_1^{\mathrm{rig}}(X,x)$ from $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K)$.*

Proof. That we can recover $H^i_{\mathrm{rig}}(X/K)$ follows from proper descent for rigid cohomology, and the above remark about the underlying complex of $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K)$. The recovery of $\pi_1^{\mathrm{rig}}(X,x)$ is far more involved, and I won't go into the details here. □

As usual, we get a Frobenius action $\phi \circlearrowleft \mathbf{R}\Gamma_{\mathrm{rig}}(X/K)$ on the rigid rational homotopy type, and we can ask in what sense this Frobenius action is mixed.

Definition. We say that a Frobenius cdga A^* over K is mixed if there exists an increasing, Frobenius invariant, multiplicative filtration $W_\bullet A^*$ of A^* such that $H^{q-p}\mathrm{Gr}^W_p(A^*)$ is pure of weight q.

Remark. Actually, ensuring that we get a genuine Frobenius structure on the cdga, rather than just in the homotopy category, is somewhat technical, and I won't explain it precisely here.

Remark. If we want to get a weight filtration on the homotopy groups of X, we really need a filtration on some complex representing it's cohomology, it is not enough to have weight filtrations on each cohomology group of this complex, which is the usual definition of mixedness for complexes.

Theorem. *There exists a Frobenius-invariant quasi-isomrphism $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K) \simeq A^*_X$ where A^*_X is a mixed Frobenius cdga over K.*

Proof. The proof consists of three main steps:

(1) Showing that the result is true when X is smooth, and the complement of a strict normal crossings divisor in some smooth and proper k-scheme $\overline{X}$

(2) Use de Jong's alterations and cohomological descent to show that for any X/k we can choose some simplicial k-scheme $X_\bullet \to X$, with each X_n of the above form, such that $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K) \simeq \mathrm{holim}_\Delta \mathbf{R}\Gamma_{\mathrm{rig}}(X_\bullet/K)$ as Frobenius cdga's.

(3) Show that if $A^{*,\bullet}$ is a cosimplicial cdga, each with Frobenius structure, and each $A^{*,n}$ admits a weight filtration, then (an explicit model for) $\mathrm{holim}_\Delta(A^{*,\bullet})$ admits a weight filtration.

The first step was essentially done by Kim and Hain - in this case we can compute the rational homotopy type using log-crystalline cohomology rather than rigid cohomology. The details here are rather complicated but the basic idea is that you locally lift to characteristic zero, where exactly as in the Hodge case the weight filtration is defined by order of log poles around the complementary divisor.

The existence of such a 'resolution' $X_\bullet \to X$ in the second step is a straightforward and very standard application of de Jong's theorem - the isomorphism $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K) \simeq \mathrm{holim}_\Delta \mathbf{R}\Gamma_{\mathrm{rig}}(X_\bullet/K)$ then follows more or less immediately from cohomological descent for rigid cohomology.

To explain the third step, let suppose instead that we were working with complexes rather than cdga's. So we have a cosimplicial Frobenius complex $A^{*,\bullet}$, together with a weight filtration on each $A^{*,n}$. We this get a filtered double complex $N(A^{*,\bullet})$ by using the Dold-Kan correspondence. Hence we get two filtrations on the total complex $\mathrm{Tot}(N(A^{*,\bullet}))$ - one coming from the weight filtration on $A^{*,\bullet}$, and one from the filtration by cosimplicial degree. Taking their convolution gives a filtration on $\mathrm{Tot}(N(A^{*,\bullet}))$ which an easy check shows to be a weight filtration, i.e. the cohomology groups of the graded pieces are pure. To extend this to include multiplicative structures, we simply need to note that the explicit quasi-isomorphism $\mathrm{Tot}(N(A^{*,\bullet})) \cong \mathrm{holim}_\Delta A^{*,\bullet}$ (for some given explicit model of the homotopy limit) can be extend to an explicit filtered quasi-isomorphism if $A^{*,\bullet}$ comes with a natural filtration. Again, the exact details here are rather messy. □

In what sense is the weight filtration independent of the chosen resolution $X_\bullet \to X$ (and the chosen compactification of each X_n)? To answer this question, suppose that we have a morphism $f : A^* \to B^*$ of filtered Frobenius cdga's, which is a quasi-isomorphism of the underlying (non-filtered) complexes. If the filtrations are weight filtrations, i.e. each $H^{q-p}\mathrm{Gr}_p^W(-)$ is pure of weight q, then the spectral sequences associated to the filtrations degenerate at the E_2 page. Since the weight filtrations on cohomology are unique, it follows that f must induce an isomorphism between the E_2-pages of the associated spectral sequences. (This is slightly weaker than being a filtered quasi-isomorphism, i.e. inducing an isomorphism on there E_1-page.)

Corollary. *The weight filtration on $\mathbf{R}\Gamma_{\mathrm{rig}}(X/K)$ is well-defined and functorial up to E_1-quasi-isomorphism.*