A HOMOTOPY EXACT SEQUENCE AND UNIPOTENT FUNDAMENTAL GROUPS OVER FUNCTION FIELDS

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Suppose that X/K is a smooth and proper variety over a number field, and $x \in X(K)$ is a rational point. Then we can consider the geometric étale fundamental group $\pi_1^{\text{ét}}(X_{\overline{K}}, x)$ of X, which is a pro-finite group together with a continuous action of the Galois group G_K. For each prime ℓ we can consider the pro-unipotent completion $\pi_1^{\text{ét}}(X_{\overline{K}}, x)_{\mathbb{Q}_\ell}$ of this group, which is a pro-unipotent group over $\mathbb{Q}_\ell$ with a Galois action, continuous in the sense that its Hopd algebra $\mathcal{A}_\ell(x)$ is a direct limits of finite dimensional, continuous ℓ-adic representations of G_K. Also note that this group can be interpreted as classifying unipotent ℓ-adic sheaves on $X_{\overline{K}}$.

These unipotent groups for varying ℓ in some sense form a 'compatible' system, for example the pieces of the filtration coming from the lower central series form compatible collections of ℓ-adic Galois representations. We expect that they should all come from 'realising' an appropriate motivic fundamental group $\pi_1^{\text{mot}}(X, x)$, the definition of which (in this generality) is currently not quite within reach.

If we now take a variety X/F over a global function function field of characteristic p instead, then we can play the same game and consider the family $\{\pi_1^{\text{ét}}(X_{\overline{F}})_{\mathbb{Q}_\ell}\}$ of pro-unipotent groups with Galois action, however, there is an anomaly in this family at $\ell = p$. Of course, this phenomenon can be seen already at the level of cohomology: the ℓ-adic cohomology only behaves well for $\ell \neq p$. The question I want to ask in this talk (at least for X smooth and proper) is what should be the correct p-adic unipotent fundamental group of a variety over a global function field.

To answer this question, let us examine the ℓ-adic case once more. If v is a place of good reduction for X, not dividing ℓ, then this Galois action is unramified at v, so if $\mathcal{X} \to C$ is a smooth and proper model for X, with C some smooth (possibly open) curve over a finite field k, then $\mathcal{A}_\ell(x)$ can be considered as a direct limit of ℓ-adic sheaves on C. In other words, we can consider $\pi_1^{\text{ét}}(X_{\overline{F}}, x)_{\mathbb{Q}_\ell}$ as an ℓ-adic sheaf of pro-unipotent groups on C. We know that the stalk of this sheaf at a geometric generic point is exactly the geometric fundamental group of the corresponding fibre, but this also holds for closed points too. Note as well that the rational point which we used as a base point extends uniquely to a section of $\mathcal{X} \to C$ which is the 'base-point' for this ℓ-adic sheaf of unipotent fundamental groups.

It is this reformulation that we can find an analogue of in the p-adic case - it is saying that we should try to construct a p-adic local system of unipotent groups on C whose stalks are exactly the p-adic unipotent groups of the fibres of f. The point is that we now know exactly what should constitute a p-adic local system - namely an overconvergent F-isocrystal, and what the p-adic analogue of the unipotent fundamental group of a variety over a finite field is - namely the rigid fundamental group.

To explain what this latter object is recall that we could also view the ℓ-adic fundamental group as classifying unipotent lisse $\mathbb{Q}_\ell$-sheaves on our variety. This is a general phenomena - whenever we have a fundamental group, it's representations will correspond to certain local systems. In the p-adic world, we turn this on its head and the theorem becomes a definition. If Y/k is a variety over a finite field, the category of unipotent overconvergent isocrystals on X is Tannakian, and hence, given a point $y \in Y$ which acts as a fibre functor for this Tannakian category, is equivalent to the category of representations of some affine group scheme, which we call the rigid fundamental group of Y, $\pi_1^{\mathrm{rig}}(Y, y)$. We can also look at other categories of local systems, for example, by taking the full category of overconvergent isocrystals, which gives rise to the full pro-algebraic fundamental group $\pi_1^{\mathrm{alg}}(Y, y)$. Note that the rigid fundamental group arises as the maximal pro-unipotent quotient of $\pi_1^{\mathrm{alg}}(Y, y)$.

So to summarise - given a smooth and proper morphism $f : \mathcal{X} \to C$ of varieties over a finite field, with a section p, we would like to show that there is a natural action of $\pi_1^{\mathrm{alg}}(C, c)$ on $\pi_1^{\mathrm{rig}}(\mathcal{X}_c, p(c))$, which will encode the fact that $\pi_1^{\mathrm{rig}}(\mathcal{X}_c, p(c))$ arises as the stalk of a p-adic local system on C.

How should such an object be constructed? The most straightforward analogy is probably the case of a topological fibration, although the same phenomenon can be seen for étale fundamental groups. So suppose we are given a Serre fibration $X \to B$ of topological spaces, and let F be the homotopy fibre. Assume that both B and F are connected. Then there is an exact sequence of homotopy groups

$$\pi_2(B) \to \pi_1(F) \to \pi_1(X) \to \pi_1(B) \to 1$$

and a section $B \to X$ splits this sequence, giving an action of $\pi_1(B)$ on $\pi_1(F)$. This gives rise to a locally constant sheaf of groups $\pi_1(X/B)$ on B, whose stalk at any point $b \in B$ is just the fundamental group of the fibre F_b over b.

Thus for a smooth and proper morphism $\mathcal{X} \to C$ of varieties over a finite field, with fibre $\mathcal{X}_c$ over some $c \in C$, we should look for an appropriate homotopy exact sequence of p-adic fundamental groups

$$\pi_1^{\mathrm{rig}}(\mathcal{X}_c, p(c)) \to \pi_1^{?}(\mathcal{X}, p(c)) \to \pi_1^{\mathrm{alg}}(C, c) \to 1$$

which, given a section, is both left exact and split. The key question is then which fundamental group of the total space $\mathcal{X}$ will fit into such an exact sequence, since neither the unipotent nor full pro-algebraic will do. The correct fundamental group to look at on the total space $\mathcal{X}$ will be that classifying overconvergent isocrystals which are *relatively* unipotent, that is those which are iterated extensions of isocrystals pulled back from C. For a base-point $x \in \mathcal{X}$, let $\pi_1^{\mathrm{rel}}(\mathcal{X}, x)$ denote the Tannaka dual of this category.

Theorem. *Let $f : \mathcal{X} \to C$ be a smooth and proper morphism of varieties over a finite field k, and let $x \in \mathcal{X}(k)$ be a rational point, mapping to $c \in C(k)$. Assume that both C and $\mathcal{X}_c$ are geometrically connected. Then there is an exact sequence of fundamental groups*

$$\pi_1^{\mathrm{rig}}(\mathcal{X}_c, x) \to \pi_1^{\mathrm{rel}}(\mathcal{X}, x) \to \pi_1^{\mathrm{alg}}(C, c) \to 1.$$

If $p : C \to \mathcal{X}$ is a section, and $x = p(c)$ then this induces a split *exact sequence*

$$1 \to \pi_1^{\mathrm{rig}}(\mathcal{X}_c, x) \to \pi_1^{\mathrm{rel}}(\mathcal{X}, x) \leftrightarrows \pi_1^{\mathrm{alg}}(C, c) \to 1.$$

Proof. I'll stick to the second claim. The existence of a section immediately implies surjectivity of the right hand map, and it is clear that the composite of the two maps is trivial. The remaining claims of exactness can be phrased in Tannakian terms as follows:

(1) If E is a relatively unipotent isocrystal on $\mathcal{X}$, whose restriction to $\mathcal{X}_c$ is constant, then E is the pullback of an isocrystal on C.

(2) If E is relatively unipotent, the there exists a sub-isocrystal $E_0 \subset E$ such that for every $c \in C$, $E_0|_{\mathcal{X}_c} \subset E_{\mathcal{X}_c}$ is the largest constant sub-isocrystal.

(3) Every unipotent isocrystal on $\mathcal{X}_c$ is a sub-quotient of one which extends to $\mathcal{X}$.

(1) and (2) are essentially proved by defining a push-forward functor f_* on isocrystals, which is right adjoint to the pull-back f^*, and satisfies a base change formula. This uses Caro's theory of overholonomic $\mathcal{D}$-modules. The point is that then $f^*f_*E \subset E$ is the largest sub-isocrystal which is constant on fibres.

(3) is proved by constructing 'universal' unipotent isocrystals $\{U_n\}$ on the fibre $\mathcal{X}_c$ and then extending these universal isocrystals to the total space $\mathcal{X}$. These universal isocrystals are constructed inductively, each U_{n+1} is the extension of U_n by $\mathcal{O}^\dagger_{\mathcal{X}_c} \otimes_K H^1_{\mathrm{rig}}(\mathcal{X}_c, U_n^\vee)^\vee$ given by choosing the object of

$$\mathrm{Ext}^1(U_n, \mathcal{O}^\dagger_{\mathcal{X}_c} \otimes_K H^1_{\mathrm{rig}}(\mathcal{X}_c, U_n^\vee)^\vee) \cong \mathrm{End}(H^1_{\mathrm{rig}}(\mathcal{X}_c, U_n^\vee))$$

corresponding to the identity. Then the extensions of these are constructed by letting W_{n+1} be an appropriate extension of W_n by $(f^*\mathbf{R}^1 f_* W_n^\vee)^\vee$. The Leray spectral sequence gives

$$\begin{array}{ccccc}
H^1_{\mathrm{rig}}(\mathcal{X}, W_n^\vee \otimes (f^*\mathbf{R}^1 f_* W_n^\vee)^\vee) & \longrightarrow & H^0_{\mathrm{rig}}(C, \mathbf{R}^1 f_* W_n^\vee \otimes (\mathbf{R}^1 f_* W_n^\vee)^\vee) & \longrightarrow & H^2_{\mathrm{rig}}(C, (\mathbf{R}^1 f_* W_n^\vee)^\vee) \\
\Vert & & \Vert & & \downarrow \\
\mathrm{Ext}(W_n, (f^*\mathbf{R}^1 f_* W_n^\vee)^\vee) & \longrightarrow & \mathrm{End}(\mathbf{R}^1 f_* W_n^\vee) & & H^2_{\mathrm{rig}}(\mathcal{X}, W_n^\vee \otimes (\mathbf{R}^1 f_* W_n^\vee)^\vee)
\end{array}$$

and W_{n+1} will extend U_{n+1} if the extension class maps to the identity in $\mathrm{End}(\mathbf{R}^1 f_* W_n^\vee)$. We can now use the section to split the map

$$H^2_{\mathrm{rig}}(C, (\mathbf{R}^1 f_* W_n^\vee)^\vee) \to H^2_{\mathrm{rig}}(\mathcal{X}, W_n^\vee \otimes (\mathbf{R}^1 f_* W_n^\vee)^\vee)$$

and hence show that the map

$$\mathrm{Ext}(W_n, (f^*\mathbf{R}^1 f_* W_n^\vee)^\vee) \to \mathrm{End}(\mathbf{R}^1 f_* W_n^\vee)$$

is surjective. Hence we can choose an appropriate W_{n+1}. ∎

Corollary. *Let $f : \mathcal{X} \to C$ be as above, and p a section. Then there exists an affine group scheme $\pi_1^{\mathrm{rig}}(\mathcal{X}/C, p)$ in the category of overconvergent isocrystals on C such that for every closed point $c \in C$, the stalk $\pi_1^{\mathrm{rig}}(\mathcal{X}/C, p)_c$ is naturally isomorphic to the rigid fundamental group $\pi_1^{\mathrm{rig}}(\mathcal{X}_c, p_c)$ of the fibre.*

This is our p-adic analogue of the unipotent fundamental group of a smooth and proper variety over function field F - note that the action of $\pi_1^{\text{alg}}(C,c)$ on $\pi_1^{\text{rig}}(\mathcal{X}_c, p(c))$ is exactly analogous to the action of the unramified Galois group $G_{F,S}$ on $\pi_1^{\text{ét}}(X_{\overline{F}}, x)_{\mathbb{Q}_\ell}$.

The fact that $\pi_1^{\text{rig}}(\mathcal{X}/C, p)$ is an overconvergent F-isocrystal, i.e. comes with a Frobenius structure follows simply from functoriality.

As an application of why I am interested in these sorts of ideas, note that we can use entirely similar ideas to construct path torsors - namely if we take another section $q \in \mathcal{X}(C)$, then we can construct $\pi_1^{\text{rig}}(\mathcal{X}/C; p, q)$ which is a torsor under the relative fundamental group $\pi_1^{\text{rig}}(\mathcal{X}/C, p)$. Thus we get a period map

$$\mathcal{X}(C) \to H^1_{F,\text{rig}}(C, \pi_1^{\text{rig}}(\mathcal{X}/C, p))$$

which should be viewed as a function field analogue of Kim's global period map.