ℓ-independence over local function fields

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k = field, k^s = separable closure, $G_k = \mathrm{Gal}(k^s/k)$
X/k variety (separated scheme of finite type)
$\ell \neq \mathrm{char}(k)$ prime $\rightsquigarrow$

$$H^i_\ell(X) := H^i_{\text{ét}}(X_{k^s}, \mathbb{Q}_\ell)$$

$$\rho_\ell : G_k \to \mathrm{GL}(H^i_\ell(X))$$

Question

How does ρ_ℓ depend on ℓ? It it 'independent of ℓ' in some sense?

Example (Deligne)

Suppose that $k = \mathbb{F}_q$ is finite, and that X/k is smooth and proper. Then for all $n \in \mathbb{Z}$

$$\mathrm{Tr}(\mathrm{Frob}_k^n | H^i_\ell(X))$$

is in $\mathbb{Q}$ and is independent of $\ell \neq p$.

Can also phrase this as follows: let $W_k \subset G_k$ consist of integral powers of Frob_k. Then $\forall\ \ell, \ell' \neq p$, and any alg. closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$,

$$(\rho_\ell|_{W_k})^{\mathrm{ss}} \otimes \Omega \cong (\rho_{\ell'}|_{W_k})^{\mathrm{ss}} \otimes \Omega$$

Remark

Conjecturally $(\rho_\ell|_{W_k})^{\mathrm{ss}} = (\rho_\ell|_{W_k})$.

In general, should exist an abelian category $\mathcal{MM}_{k,\mathbb{Q}}$ of (rational) mixed motives over k, cohomology groups $H^i_{\mathrm{mot}}(X) \in \mathcal{MM}_{k,\mathbb{Q}}$ and realisation functors

$$- \otimes \mathbb{Q}_\ell : \mathcal{MM}_{k,\mathbb{Q}} \to \mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)$$

for all $\ell \neq \mathrm{char}(k)$ such that

$$H^i_{\mathrm{mot}}(X) \otimes \mathbb{Q}_\ell \cong H^i_\ell(X)$$

Example

Can construct a category of 1-motives $\mathcal{MM}^{\leq 1}_{k,\mathbb{Q}}$ 'by hand' $\rightsquigarrow$ independence results for curves and abelian varieties.

Now take F a local field with finite residue field k, $\ell \neq \mathrm{char}(k)$.

Theorem (Grothendieck)

Every ℓ-adic representation of G_F is quasi-unipotent.

Can use this to construct

$$\mathrm{WD} : \mathrm{Rep}_{\mathbb{Q}_\ell}(G_F) \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\mathrm{WD}_F)$$

with target the category of Weil–Deligne representations. These are continuous representations

$$\rho : W_F \to \mathrm{GL}(V)$$

of the Weil group (for the discrete topology on V) together with a nilpotent map $N : V \to V(1)$.

Conjecture (Fontaine $C_{\mathrm{WD}}(X,i)$)

X/F variety, $i \geq 0$. Then for any $\ell, \ell' \neq p$ and any alg. closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$ we have

$$\mathrm{WD}(H^i_\ell(X)) \otimes \Omega \cong \mathrm{WD}(H^i_{\ell'}(X)) \otimes \Omega$$

as object of $\mathrm{WD}_\Omega(W_F)$.

Conjecture (Fontaine $C_{\mathrm{WD}}(X,i)_{\mathrm{faible}}$)

Same, but replacing $\mathrm{WD}(H^i_\ell(X))$ with $\mathrm{WD}(H^i_\ell(X))^{F\text{-ss}}$.

For any family of Weil–Deligne representations $\{E_\ell\}_{\ell\in\mathbf{P}}$ we will say that they are (weakly) independence of ℓ if they satisfy the above conjecture.

If V is a Weil–Deligne representation, $\exists!$ increasing filtration $M_\bullet V$ such that

$$N^k : \mathrm{Gr}^M_k V \xrightarrow{\sim} \mathrm{Gr}^M_{-k} V(k)$$

Lemma (Deligne)

A family $\{E_\ell\}_{\ell \in \mathbf{P}}$ of Weil–Deligne representations is weakly independent of ℓ iff $\forall k \in \mathbb{Z}$

$$\mathrm{Tr}(- \mid Gr^M_k E_\ell) : W_F \to \mathbb{Q}_\ell$$

takes values in $\mathbb{Q}$ and is independent of ℓ.

Today: concentrate on the case when $F \cong k((t))$ is a local field of equicharacteristic.

1. How to extend these conjectures to include $\ell = p$?
2. Prove them when X/F is smooth and proper.

$k =$ finite field, characteristic p, $F \cong k((t))$, $K = W(k)[1/p]$, $\sigma =$ Frobenius. For $\ell \neq p$ the functor

$$\mathrm{WD} : \mathrm{Rep}_{\mathbb{Q}_\ell}(G_F) \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\mathrm{WD}_F)$$

arises from the ℓ-adic local monodromy theorem $\rightsquigarrow$ want to replace this with the p-adic monodromy theorem.

Definition

The Robba ring $\mathcal{R}$ over K is the ring of analytic functions over K convergent on some half-open annulus $\{\eta \leq |t| < 1\}$.

Have a Frobenius $\sigma : \mathcal{R} \to \mathcal{R}$ and a derivation $\partial_t : \mathcal{R} \to \mathcal{R} \rightsquigarrow$ notion of a (φ, ∇)-module over $\mathcal{R}$. Denote the category $\underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{R}}$.

Theorem (André, Mebkhout, Kedlaya)

Every (φ, ∇)-module M over $\mathcal{R}$ is quasi-unipotent.

The theorem means that after making a finite separable extension of $F = k((t))$, and formally adjoining $\log t$, M admits a basis of horizontal sections.

Corollary

Let K^{un} denote the maximal unramified extension of K. Then there exists an exact, faithful functor

$$\underline{\mathbf{M}\Phi}_{\mathcal{R}}^{\nabla} \to \mathrm{Rep}_{K^{\mathrm{un}}}(\mathrm{WD}_F)$$

So what we want a theory of p-adic cohomology landing in the category $\underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{R}}$, modelled on rigid/crystalline cohomology.

Set

$$\mathcal{E} := \widehat{W[\![t]\!][t^{-1}]}[1/p]$$

this is a complete DVF with residue field $F = k(\!(t)\!)$.
$\Rightarrow$ rigid cohomology for varieties over F is a functor

$$H^*_{\mathrm{rig}}(-/\mathcal{E}) : \mathrm{Var}_F \to \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}}$$

to (φ, ∇)-modules over $\mathcal{E}$.

Note that we can write

$$\mathcal{R} = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \;\middle|\; \begin{array}{c} \forall \rho < 1, \quad |a_i|\,\rho^i \to 0 \text{ as } i \to \infty \\ \exists \lambda < 1 \text{ s.t. } \quad |a_i|\,\lambda^i \to 0 \text{ as } i \to -\infty \end{array} \right\}$$

$$\mathcal{E} = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \;\middle|\; \begin{array}{c} \sup_i |a_i| < \infty \\ |a_i| \to 0 \text{ as } i \to -\infty \end{array} \right\}$$

So that $\mathcal{E} \not\subseteq \mathcal{R}$ and $\mathcal{R} \not\subseteq \mathcal{E}$.

Definition

$$\mathcal{E}^\dagger := \mathcal{E} \cap \mathcal{R} = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \;\middle|\; \begin{array}{c} \sup_i |a_i| < \infty \\ \exists \lambda < 1 \text{ s.t. } \quad |a_i|\,\lambda^i \to 0 \text{ as } i \to -\infty \end{array} \right\}$$

$\mathcal{E}^\dagger$ is a Henselian DVF with residue field F, and we have

$$\begin{tikzcd} \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}} & & \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{R}} \\ & \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}^\dagger} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Theorem (Kedlaya)

The functor $\underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}^\dagger} \to \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}}$ is fully faithful, and if $X \in \mathrm{Var}_F$ is smooth and proper, $H^i_{\mathrm{rig}}(X/\mathcal{E})$ is in the essential image.

Should think of $\underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}^\dagger} \to \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}}$ as analogous to the inclusion $\mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(G_K) \subset \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

Theorem (L., Pál)

Rigid cohomology descends to the bounded Robba ring $\mathcal{E}^\dagger$, in other words $\exists$ functor

$$H^*_{\mathrm{rig}}(-/\mathcal{E}^\dagger) : \mathrm{Var}_F \to \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}^\dagger}$$

satisfying all the axioms of an 'extended' Weil cohomology theory, whose base change to $\mathcal{E}$ is isomorphic to $H^*_{\mathrm{rig}}(-/\mathcal{E}^\dagger)$.

There also are versions with compact support, as well as support in a closed subscheme, and categories of coefficients $(F\text{-})\mathrm{Isoc}^\dagger(X/\mathcal{E}^\dagger)$ and $(F\text{-})\mathrm{Isoc}^\dagger(X/K)$ for this theory.

Corollary

Let X/F be a variety, then we can define a p-adic Weil–Deligne representation $H^i_p(X)$ associated to X via

$$H^i_{\mathrm{rig}}(X/\mathcal{R}) := H^i_{\mathrm{rig}}(X/\mathcal{E}^\dagger) \otimes \mathcal{R}.$$

Hence we can extend Fontaine's conjectures $C_{\mathrm{WD}}(X, i)$ and $C_{\mathrm{WD}}(X, i)_{\mathrm{faible}}$ to include $\ell = p$.

Note that the p-adic Weil–Deligne representations are defined over $\mathbb{Q}'_p := K^{\mathrm{un}}$. Set $\mathbb{Q}'_\ell = \mathbb{Q}_\ell$ if $\ell \neq p$.

Theorem (Chiarellotto, L.)

Let X/F be smooth and proper, and $i \geq 0$. Then $C_{\mathrm{WD}}(X,i)_{\mathrm{faible}}$ holds.

The first key step consists of reducing to the following case.

Definition

Let X/F be smooth and proper, with semistable reduction. We say that F is globally defined if there exists a smooth curve C over k, a rational point $c \in C(k)$ with $\widehat{k(C)}_c \cong F$ and a proper, flat scheme $\mathcal{X} \to C$, smooth away from c and semistable at c, such that $\mathcal{X} \times_C \mathrm{Spec}\,(F) \cong X$.

Proposition

Suppose that $C_{\mathrm{WD}}(X,i)_{\mathrm{faible}}$ holds for all smooth and proper F-varieties which are semistable and globally defined. Then $C_{\mathrm{WD}}(X,i)_{\mathrm{faible}}$ holds for all smooth and proper F-varieties.

Ingredients:

1. Alterations
2. Weight-monodromy conjecture
3. Cohomological descent
4. 'Spreading out' lemma
5. Uniqueness of 'geometric weight filtrations'

Lemma

Assume that $\mathfrak{X} \to \mathrm{Spec}\,(\mathcal{O}_F)$ is semistable, and $n \geq 1$. Then there exists a globally defined semistable scheme $\mathfrak{Y} \to \mathrm{Spec}\,(\mathcal{O}_F)$ such that $\mathfrak{X} \otimes \mathcal{O}_F/t^n \cong \mathfrak{Y} \otimes \mathcal{O}_F/t^n$.

The proof of $C_{\mathrm{WD}}(X,i)_{\mathrm{faible}}$ for smooth and proper F-varieties therefore reduces to the following.

Theorem

Let C/k be a smooth curve $c \in C(k)$, $U = C \setminus c$, $F = \widehat{k(C)}_c$. Suppose that $\{\mathcal{F}_\ell\}_\ell$ is a collection of local systems on U, such that for all $u \in U$

$$\mathrm{Tr}(- \mid \mathcal{F}_{\ell,\bar{u}}) : \mathrm{Frob}_u^{\mathbb{Z}} \to \mathbb{Q}'_\ell$$

takes values in $\mathbb{Q}$ and is independent of ℓ. Then for all $k \geq 0$

$$\mathrm{Tr}(- \mid Gr_k^M \mathcal{F}_{\ell,\bar{c}}) : W_F \to \mathbb{Q}'_\ell$$

takes values in $\mathbb{Q}$ and is independent of ℓ.

This was proved by Deligne for $\ell \neq p$; to include $\ell = p$ we use the theory of arithmetic $\mathcal{D}^\dagger$-modules.

What about mixed characteristic local fields? The weight monodromy conjecture is used in two key places, but the rest of the proof should work. So for smooth and proper varieties, $C(X,i)_{\mathrm{faible}}$ would follow from the weight monodromy conjecture for X.

There are also results for proper varieties.

Theorem

Let X/F be proper, and $k \in \mathbb{Z}$. Then

$$\sum_i (-1)^i \mathrm{Tr}(- \mid \mathrm{Gr}_k^M H_\ell^i(X)) : W_F \to \mathbb{Q}_\ell'$$

has values in $\mathbb{Q}$ and is independent of ℓ.

What about other invariants such as homotopy groups?

Example

Unipotent π_1 is expected to be 'motivic' $\rightsquigarrow$ should have 'ℓ-independence' results for this.

So let X be a pointed variety over F.

Definition

For $\ell \neq p$ define $\pi_1^{\ell}(X)$ to be the $\mathbb{Q}_\ell$-pro-unipotent completion of $\pi_1^{\text{ét}}(X_{\overline{F}})$. This comes with an action of G_F.

When $\ell = p$ need to use Tannakian methods.

Definition

We define $\pi_1^p(X)$ to be the Tannaka dual of the category $\mathcal{N}\mathrm{Isoc}^\dagger(X/\mathcal{E}^\dagger)$ of unipotent overconvergent isocrystals on $X/\mathcal{E}^\dagger$.

Thus $\pi_1^p(X)$ is a (pro-unipotent) affine group scheme over $\mathcal{E}^\dagger$.

Theorem (L.)

The group scheme $\pi_1^p(X)$ has a canonical structure as a 'non-abelian' (φ, ∇)-module over $\mathcal{E}^\dagger$.

Let ℓ be any prime. Set $L_\ell := \mathrm{Lie}(\pi_1^\ell(X))$, $\mathcal{U}_\ell := \mathcal{U}(L_\ell)$, $\mathfrak{a}_\ell :=$ augmentation ideal.

$$\Rightarrow \mathcal{U}_\ell/\mathfrak{a}_\ell^k \in \mathrm{Rep}_{\mathbb{Q}_\ell}(G_F) \;\; (\ell \neq p)$$
$$\mathcal{U}_p/\mathfrak{a}_p^k \in \underline{\mathbf{M\Phi}}_{\mathcal{E}^\dagger}^{\nabla}$$

Conjecture ($C_{\mathrm{WD}}(X, \pi_1)$)

For all $k \geq 1$ the Weil Deligne representations associated to $\mathcal{U}_\ell/\mathfrak{a}_\ell^k$ are independent of ℓ.

Over finite fields, can prove Frobenius semisimplicity for $\mathcal{U}_\ell/\mathfrak{a}_\ell^k$.

Theorem (Chiarellotto, L.)

Assume that X is smooth and proper over F with semistable reduction. Then $C_{\mathrm{WD}}(X, \pi_1)$ holds.

As before, we reduce to the 'globally defined' case, and then show that the $\mathcal{U}_\ell/\mathfrak{a}_\ell^k$ can be 'spread out' to local systems on some global model C of F.

Questions

1. Can we remove the semistable hypothesis?
2. Does the argument work for mixed characteristic local fields? (We know the weight-monodromy conjecture for H^1.)

Thank-you!