AN INTRODUCTION TO RIGID COHOMOLOGY

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1. Motivation: search for a p-adic cohomology theory

Let k be a perfect field of characteristic $p>0$, $W=W(k)$ the ring of Witt vectors of k, and $K=W(k)[1/p]$ its fraction field. Let σ denote the Frobenius automorphism of K. Then for any variety X/k and any prime $\ell\neq p$, we can consider the ℓ-adic étale cohomology groups

$$H^i_{\text{ét}}(X_{\overline{k}},\mathbb{Q}_\ell)$$

which are finite dimensional over $\mathbb{Q}_\ell$ and in fact form a Weil cohomology theory. Tshe basic aim of p-adic cohomology is to fill the gap in this family at $\ell=p$, in other words to produce a well behaved cohomology theory with coefficients in a p-adic field.

The first attempt to do this was the theory of crystalline cohomology, which works well for smooth and proper schemes over k. The fundamental comparison theorem making this cohomology computable is the following.

Theorem. *Let $\mathcal{X}/W$ be a smooth and proper scheme, with special fibre X. Then there is a canonical isomorphism*

$$H^i_{\text{cris}}(X/W)\otimes_W K\cong H^i_{\text{dR}}(\mathcal{X}_K).$$

This result was the start of the general philosophy

$$\{p\text{-adic cohomology in char } p\}\leftrightarrow\{\text{de Rham cohomology of a lift to char } 0\}$$

Crystalline cohomology works well for smooth and proper varieties, what about non-proper ones? For example, what can we do for smooth affine varieties?

By a result of Elkik, smooth affine varieties always lift to char 0, that is given such an X, there always exists a flat, smooth affine scheme $\mathcal{X}\to\operatorname{Spec}(W)$ whose special fibre is X. One might then guess that we should take the de Rham cohomology $H^i_{\text{dR}}(\mathcal{X}_K)$ as the p-adic cohomology of X.

Problem. This depends on the lift $\mathcal{X}$!

Example. The affine schemes

$$\mathcal{X}=\operatorname{Spec}\left(W[z,z^{-1}]\right),\ \mathcal{Y}=\operatorname{Spec}\left(W[z,z^{-1},(z-p)^{-1}]\right)$$

both have the same special fibre, but generic fibres with different de Rham cohomology.

To get around this problem, we could try to complete. That is we replace $\mathcal{X}$ by it's p-adic completion, which is a formal scheme $\widehat{\mathcal{X}}$ over $\operatorname{Spf}(W)$. Then the de Rham cohomology $H^i_{\text{dR}}(\widehat{\mathcal{X}}_K)$ of the rigid analytic space $\widehat{\mathcal{X}}_K$ associated to $\widehat{\mathcal{X}}$ turns out to be functorial in X, however, we have introduced another problem.

Problem. The spaces $H^i_{\mathrm{dR}}(\widehat{\mathcal{X}}_K)$ are not finite dimensional in general.

Example. If $\mathcal{X} = \mathbb{A}^1_W$ then $\widehat{\mathcal{X}}_K \cong \mathbb{D}_K(0,1) = \mathrm{Sp}(K\langle z\rangle)$, the closed unit disc over K. Then one can check that $H^1_{\mathrm{dR}}(\mathbb{D}_K(0.1))$ is infinite dimensional.

To get around this problem, we introduce the notion of overconvergence. Instead of considering the de Rham cohomology of $K\langle z\rangle$, we look instead at the subring $K\langle z\rangle^\dagger$ consisting of series which converge on some strictly larger disc $|z| \leq \rho$ for $\rho > 1$. Check: the de Rham cohomology of this ring is trivial i.e. isomorphic to K is degree 0 and 0 in all other degrees.

This general method of considering 'overconvergent function' turns out to give a good theory for smooth affine schemes over k, although the eventual proof of finite dimensionality was only achieved through the theory of rigid cohomology.

So, what about singular schemes? Here we take our cue from the theory of algebraic de Rham cohomology in characteristic 0, which works well in the 'naive' way for smooth schemes, but for singular schemes X we must first taking an embedding $X \hookrightarrow P$ into a smooth scheme, and consider de Rham cohomology of the formal completion of X in P.

So to construct a good p-adic theory, there are three core ingredients that we need to incorporate into our definition.

- should be related to de Rham cohomology of a lift to characteristic 0;
- need to take account of 'overconvergence';
- for singular schemes, we should embed in smooth schemes that lift.

Berthelot's definition of de Rham cohomology fuses all these ingredients into a single construction.

2. Rigid cohomology: definitions and basic facts

Definition 2.1. A *frame* over W is a triple $T := (X, Y, \mathfrak{P})$ where $X \hookrightarrow Y$ is an open embedding of k-varieties, and $Y \hookrightarrow \mathfrak{P}$ is a closed embedding of p-adic formal W-schemes. We say that T is proper if Y is proper over k, and smooth if there exists an open subscheme $\mathfrak{U}$ of $\mathfrak{P}$ containing X which is smooth over W.

For a frame $T = (X, Y, \mathfrak{P})$ we will usually write $Z = Y \setminus X$. We can consider the generic fibre $\mathfrak{P}_K$, which is a rigid analytic space over K. There is a specialisation map

$$\mathrm{sp} : \mathfrak{P}_K \to \mathfrak{P}$$

and we define the 'tubes'

$$]Y[_{\mathfrak{P}} := \mathrm{sp}^{-1}(X), \quad]X[_{\mathfrak{P}} := \mathrm{sp}^{-1}(X), \quad]Z[_{\mathfrak{P}} = \mathrm{sp}^{-1}(Z).$$

Example.
i) Let $(X, Y, \mathfrak{P}) = (\mathbb{A}^1_k, \mathbb{P}^1_k, \widehat{\mathbb{P}}^1_W)$. Then $]Y[_{\mathfrak{P}} = \mathbb{P}^{1,\mathrm{an}}_K$ is the analytic projective line, and $]X[_{\mathfrak{P}} = \mathbb{D}_K(0,1)$ is the closed unit disc over K.
ii) Let $(X, Y, \mathfrak{P}) = (\mathrm{Spec}\,(k), \mathrm{Spec}\,(k), \widehat{\mathbb{A}}^1_W)$. Then $]X[_{\mathfrak{P}} =]Y[_{\mathfrak{P}}$ is the open unit disc $\mathbb{D}_K(0,1^-)$ over K.

The systematic way of working 'overconvergence' into the definition is via Berthelot's $j^\dagger$ construction. Note that if $(X, Y, \mathfrak{P})$ is a frame, then we have $]Y[_{\mathfrak{P}} =]X[_{\mathfrak{P}} \cup]Z[_{\mathfrak{P}}$, but this covering is not admissible.

Example. If $(X, Y, \mathfrak{P}) = (\mathrm{Spec}\,(k), \mathbb{A}^1_k, \widehat{\mathbb{A}}^1_W)$ then $]Y[_{\mathfrak{P}} = \mathbb{D}_K(0,1)$, $]X[= \{z \in \mathbb{D}^1_K \mid |z| < 1\}$ and $]Z[_{\mathfrak{P}} = \{z \in \mathbb{D}_K(0,1) \mid |z| = 1\}$.

Definition 2.2. A strict neighbourhood of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$ is an open subset $]X[_{\mathfrak{P}} \subset V \subset]Y[_{\mathfrak{P}}$ such that the covering $]Y[_{\mathfrak{P}} = V \cup]Z[_{\mathfrak{P}}$ is admissible.

Example. Let $(X, Y, \mathfrak{P}) = (\mathbb{A}^1_k, \mathbb{P}^1_k, \widehat{\mathbb{P}}^1_W)$. Then the closed disc $\{z \in \mathbb{A}^{1,\mathrm{an}}_K \mid |z| \leq \rho\}$ of radius $\rho >$ is a strict neighbourhood of $]X[_{\mathfrak{P}}$ inside $\mathbb{P}^{1,\mathrm{an}}_K$. In fact, these form a cofinal system of such strict neighbourhoods.

Definition 2.3. Let $\mathcal{F}$ be a sheaf on $]Y[_{\mathfrak{P}}$. Then we define

$$j^\dagger_X \mathcal{F} := \mathrm{colim}_V \, j_{V*} j_V^{-1} \mathcal{F}$$

where the colimit is over all strict neighbourhoods $j_V : V \to]Y[_{\mathfrak{P}}$ of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$.

Example. Let $(X, Y, \mathfrak{P}) = (\mathbb{A}^1_k, \mathbb{P}^1_k, \widehat{\mathbb{P}}^1_W)$. Then $\Gamma(]Y[_{\mathfrak{P}}, j^\dagger_X \mathcal{O}_{]Y[_{\mathfrak{P}}}) = K\langle z \rangle^\dagger$.

We are now in a position to define rigid cohomology.

Definition 2.4. Let X be a k-variety and choose a smooth and proper frame $(X, Y, \mathfrak{P})$. Then we define the rigid cohomology of X to be

$$H^i_{\mathrm{rig}}(X/K) := H^i(]Y[_{\mathfrak{P}}, j^\dagger_X \Omega^*_{]Y[_{\mathfrak{P}}/K}).$$

Of course, we must show that this does not depend on the choice of frame $(X, Y, \mathfrak{P})$. The key ingredient in the proof of this fact are the following.

Theorem (Strong Fibration Theorem)**.** *Let*

$$\begin{array}{ccccc} X & \longrightarrow & Y' & \longrightarrow & \mathfrak{P}' \\ \| & & \downarrow & & \downarrow f \\ X & \longrightarrow & Y & \longrightarrow & \mathfrak{P} \end{array}$$

be a morphism of frames such that $Y' \to Y$ is proper, and $\mathfrak{P}' \to \mathfrak{P}$ is étale in a neighbourhood of X. Then for all sufficiently small strict neighbourhoods V of $]X[_{\mathfrak{P}}$ in $]Y[_{\mathfrak{P}}$, the induced map $f^{-1}(V) \to V$ is an isomorphism.

Theorem (Poincaré Lemma)**.** *Let $(X, Y, \mathfrak{P})$ be a frame, and consider the natural morphism of frames*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & \widehat{\mathbb{A}}^1_{\mathfrak{P}} \\ \| & & \| & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & \mathfrak{P} \end{array}$$

Then the induced map

$$H^i(]Y[_{\mathfrak{P}}, j^\dagger_X \Omega^*_{]Y[_{\mathfrak{P}}/K}) \to H^i(]Y[_{\widehat{\mathbb{A}}^1_{\mathfrak{P}}}, j^\dagger_X \Omega^*_{]Y[_{\widehat{\mathbb{A}}^1_{\mathfrak{P}}}/K})$$

is an isomorphism.

The proof that $H^i_{\mathrm{rig}}(X/K)$ is independent of the frame now proceeds (roughly speaking) as follows. Suppose that we have two smooth and proper frames $(X, Y, \mathfrak{P})$

and $(X, Y', \mathfrak{P}')$. Then after replacing $\mathfrak{P}'$ by $\mathfrak{P} \times_W \mathfrak{P}'$ we may assume that we have a morphism

$$\begin{array}{ccccc} X & \longrightarrow & Y' & \longrightarrow & \mathfrak{P}' \\ \| & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & \mathfrak{P} \end{array}$$

such that $Y' \to Y$ is proper, and $\mathfrak{P}' \to \mathfrak{P}$ is smooth in a neighbourhood of X. Since a smooth morphism $\mathfrak{P}' \to \mathfrak{P}$ locally factors through an étale map $\mathfrak{P}' \to \widehat{\mathbb{A}}^n_{\mathfrak{P}}$, for some n, we can repeatedly apply the previous results to conclude that

$$H^i(]Y[_{\mathfrak{P}}, j_X^\dagger \Omega^*_{]Y[_{\mathfrak{P}}/K}) \to H^i(]Y'[_{\mathfrak{P}'}, j_X^\dagger \Omega^*_{]Y'[_{\mathfrak{P}'}/K})$$

is an isomorphism.

We can also use similar ideas to show that these spaces are functorial in X, in particular using this functoriality with respect to Frobenius puts a 'Frobenius structure' on $H^i_{\mathrm{rig}}(X/K)$, that is a σ-linear isomorphism

$$\varphi : H^i_{\mathrm{rig}}(X/K) \to H^i_{\mathrm{rig}}(X/K).$$

It is not known whether we can always find smooth and proper frames $(X, Y, \mathfrak{P})$ containing our variety X of interest. However, we can always do this locally on X, and then use Zariski descent to define rigid cohomology in general. The details are a bit tedious.

3. **Finite dimensionality**

To show that $H^i_{\mathrm{rig}}(X/K)$ is finite dimensional, there are two main steps:

i) prove that $H^i_{\mathrm{rig}}(X/K)$ is finite dimensional when X is smooth over k;
ii) use cohomological descent to reduce to this case.

The second of these is due to Tsuzuki and Chiarellotto–Tsuzuki, and is incredibly long and involved. I won't say anything about it. The idea behind the first of these is to reduce to the smooth and proper case by developing enough formalism of rigid cohomology. The key ingredient is the excision exact sequence.

Theorem (Berthelot)**.** *Let X be a smooth k-variety, and $Z \subset X$ a smooth closed subscheme of constant codimension c and complement $U = X \setminus Z$. Then there exists a long exact sequence*

$$\ldots \to H^{i-2c}_{\mathrm{rig}}(Z/K) \to H^i_{\mathrm{rig}}(X/K) \to H^i_{\mathrm{rig}}(U/K) \to \ldots$$

We can then use this to feed into an induction argument which reduces to the case of X smooth and proper. Now we apply Berthelot's comparison theorem.

Theorem (Berthelot)**.** *Let X/k be smooth and proper. Then*

$$H^i_{\mathrm{rig}}(X/K) \cong H^i_{\mathrm{cris}}(X/W) \otimes_W K.$$

4. **Why bother?**

The original motivation for the search for Weil cohomology theories was in order to prove the Weil conjectures. Since the ℓ-adic theory for $\ell \neq p$ acheives this, why should we even bother with constructing a p-adic theory?

Well, one reason is that p-adic cohomology is much more amenable to computer calculation, for example, there are now algorithms due to Lauder and Kedlaya that can calculate the zeta function of a variety over a finite field, using rigid cohomology. No-one has any idea how to do this using ℓ-adic étale cohomology.

Another reason is connected with one of the big open questions in p-adic cohomology theory. In the ℓ-adic world, we don't just have a rational theory $H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$, but also an integral theory $H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_\ell)$ which crucially can explain ℓ-torsion phenomena. If we are to be able to explain p-torsion, then we need an *integral* p-adic theory. Crystalline cohomology $H^i_{\text{cris}}(X/W)$ works well for smooth and proper varieties, but it is not really clear at the moment what the 'right' approach is for open or singular varieties.