RIGID COHOMOLOGY OVER LAURENT SERIES FIELDS I

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Last time, I talked about the construction of rigid cohomology for varieties over a perfect field k, taking values in vector spaces over $K = W(k)[1/p]$. Actually the construction works in much more generality than this. We can actually take any field k of characteristic p, and any complete, discretely valued field K of characteristic 0 whose residue field is k, and we still have a good theory

$$X/k \mapsto H^i_{\text{rig}}(X/K)$$

which satisfies all the same properties as we saw last time. However, when k is not perfect there are good reasons to expect that this is not the whole story of p-adic cohomology, and that actually there should be refinements of this theory that reflect better arithmetic properties of X/k. Today I'm going to talk about what this story looks like for varieties over a Laurent series field $k(\!(t)\!)$ where k is a perfect field.

1. ℓ-adic cohomology over local function fields and the monodromy theorem

To explain what we really expect from our p-adic theory, let us go back and review what happens in the ℓ-adic situation, $\ell \neq p$. Fix a separable closure $k(\!(t)\!)^{\text{sep}}$ of $k(\!(t)\!)$, and let $G := \text{Gal}(k(\!(t)\!)^{\text{sep}}/k(\!(t)\!))$ denote the corresponding absolute Galois group. Then for any variety $X/k(\!(t)\!)$ the ℓ-adic cohomology groups $H^i_{\text{ét}}(X_{k(\!(t)\!)^{\text{sep}}}, \mathbb{Q}_\ell)$ are naturally Galois representations, that is finite dimensional $\mathbb{Q}_\ell$-vector spaces together with a continuous action of G. Up to a finite amount of error, these can be captured by quite simple linear algebra data thanks to Grothendieck's ℓ-adic local monodromy theorem.

Theorem (Grothendieck)**.** *Let V be an ℓ-adic representation of G. Then V is quasi-unipotent, that is there exists a finite separable extension $F/k(\!(t)\!)$ such that the corresponding inertia group $I_F \subset G$ acts unipotently on V.*

In fact, this can be viewed as a cohomological version of the 'semistable reduction conjecture'.

$$X \text{ has semistable reduction} \leftrightarrow H^i_{\text{ét}}(X_{k(\!(t)\!)^{\text{sep}}}, \mathbb{Q}_\ell) \text{ is unipotent}$$

potentially semistable reduction for varieties $\leftrightarrow$ Galois representations are quasi-unipotent

We can also use the monodromy theorem to compare the various cohomologies $H^i_{\text{ét}}(X_{k(\!(t)\!)^{\text{sep}}}, \mathbb{Q}_\ell)$ for different values of ℓ. The 'simple linear algebra data' that we can use to describe $H^i_{\text{ét}}(X_{k(\!(t)\!)^{\text{sep}}}, \mathbb{Q}_\ell)$ is what is called a Weil–Deligne representation of G (with values in $\mathbb{Q}_\ell$). These objects are then algebraic enough (in particular, they do not use the topology of $\mathbb{Q}_\ell$) that it makes sense to compare them as ℓ varies.

2. The monodromy theorem in p-adic cohomology

What are the analogues of the Galois representations that are produced in p-adic cohomology over $k(\!(t)\!)$? To see, let us look more closely at how rigid cohomology works for varieties over $k(\!(t)\!)$.

Consider the ring

$$\mathcal{E}_K := \left\{ \sum_{i=-\infty}^{\infty} a_i t^i \,\middle|\, \sup_i |a_i| < \infty,\ a_i \to 0 \text{ as } i \to -\infty \right\}$$

Then it turns out that $\mathcal{E}_K$ is a complete, discretely valued field whose residue field is $k(\!(t)\!)$. Moreover, we may endow $\mathcal{E}_K$ with a 'Frobenius' lifting the absolute Frobenius of $k(\!(t)\!)$ by setting

$$\sigma(\sum_i a_i t^i) = \sum_i \sigma(a_i) t^{ip}$$

where σ is the Frobenius on K.

Then for $X/k(\!(t)\!)$ a variety, we can consider its rigid cohomology $H^i_{\mathrm{rig}}(X/\mathcal{E}_K)$, these are finite dimensional vector spaces over $\mathcal{E}_K$, and come endowed with a Frobenius structure, that is an isomorphism

$$\varphi : H^i_{\mathrm{rig}}(X/\mathcal{E}_K) \otimes_{\mathcal{E}_K,\sigma} \mathcal{E}_K \to H^i_{\mathrm{rig}}(X/\mathcal{E}_K).$$

But these objects also come with extra structure: we may 'differentiate with respect to t' to put a *connection* on these vector space, which is compatible with the Frobenius structure in a suitable sense. This is formalised in the notion of a (φ, ∇)-module over $\mathcal{E}_K$.

Definition. Let $\partial_t : \mathcal{E}_K \to \mathcal{E}_K$ denote the derivation given by 'differentiation with respect to t'. A (φ, ∇)-module over $\mathcal{E}_K$ is a finite dimensional vector space M together with:

- a connection, that is a K-linear map $\nabla : M \to M$ such that $\nabla(fm) = f\nabla(m) = \partial_t(f)m$ for all $f \in \mathcal{E}_K, m \in M$;
- a horizontal isomorphism $\sigma^* M := M \otimes_{\mathcal{E}_K,\sigma} \mathcal{E}_K \to M$.

So for $X/k(\!(t)\!)$, the vector spaces $H^i_{\mathrm{rig}}(X/\mathcal{E}_K)$ are naturally (φ, ∇)-modules over $\mathcal{E}_K$. It is these (φ, ∇)-modules over $\mathcal{E}_K$ that are our first candidate for the p-adic analogues of Galois representations, however, the category of such objects turn out to be not well behaved, in particular there is no analogue of the local monodromy theorem in this context.

There is a version of the p-adic local monodromy theorem, but it does not apply to (φ, ∇)-modules over $\mathcal{E}_K$, but instead to (φ, ∇)-modules over a closely related ring, the Robba ring $\mathcal{R}_K$.

$$\mathcal{R}_K = \left\{ \sum_i a_i t^i \in K[\![t, t^{-1}]\!] \,\middle|\, \begin{array}{c} \exists \eta < 1 \text{ s.t. } |a_i|\,\eta^i \to 0 \text{ as } i \to -\infty \\ \forall \rho < 1,\ |a_i|\,\rho^i \to 0 \text{ as } i \to \infty \end{array} \right\}$$

Then exactly as before, we may define a Frobenius σ on $\mathcal{R}_K$ and a derivation ∂_t, and we have the notion of a (φ, ∇)-module over $\mathcal{R}_K$. The analogue of the monodromy theorem is then the following.

Theorem (André–Mebkhout, Kedlaya)**.** *Let M be a (φ, ∇)-module over $\mathcal{R}_K$. Then M is quasi-unipotent, that is after making a finite étale extension of $\mathcal{R}_K$, the connection ∇ acts via a unipotent matrix.*

The problem if we want to relate this to the rigid cohomology $H^i_{\mathrm{rig}}(X/\mathcal{E}_K)$ is that we have $\mathcal{E}_K \not\subset \mathcal{R}_K$ and $\mathcal{R}_K \not\subset \mathcal{E}_K$, so there is no obvious way to pass back and forth between (φ, ∇)-modules over one and (φ, ∇)-modules over the other.

Dumb Idea. try to work instead with the intersection $\mathcal{E}_K^\dagger = \mathcal{R}_K \cap \mathcal{E}_K$, known as the bounded Robba ring

This is concretely described as follows.

$$\mathcal{E}_K^\dagger = \left\{ \sum a_i t^i \middle| \sup_i |a_i| < \infty,\ \exists \eta < 1 \text{ s.t. } |a_i| \eta^i \to 0 \text{ as } i \to -\infty \right\},$$

This is a henselian discrete valuation field with residue field $k(\!(t)\!)$, but it is not complete (its completion is $\mathcal{E}_K$). It is stable under Frobenius σ and the derivation ∂_t, and we may therefore speak of (φ, ∇)-modules over $\mathcal{E}_K$.

Aim. Show that the (φ, ∇)-modules $H^i_{\mathrm{rig}}(X/\mathcal{E}_K)$ 'descend' to (φ, ∇)-modules over $\mathcal{E}_K^\dagger$.

Since these (φ, ∇)-modules over $\mathcal{E}_K$ can be base changed to $\mathcal{R}_K$, they satisfy a certain version of a monodromy theorem, although it is slightly complicated to state in a precise form.

Here's another analogy for what we're trying to do: the p-adic cohomology of varieties over local fields in mixed characteristic are naturally p-adic Galois representations, and in general this category contains far too many objects to be easily amenable to study. In p-adic Hodge theory one isolates a particular class of p-adic representations (those that are 'de Rham') that are much better behaved. In our case, we view the category of (φ, ∇)-modules over $\mathcal{E}_K$ as the analogue of the category of *all* p-adic Galois representations, and the category of (φ, ∇)-modules over $\mathcal{E}_K^\dagger$ as the analogue of category of de Rham representations. Note that by a result of Kedlaya, the base extension functor from (φ, ∇)-modules over $\mathcal{E}_K^\dagger$ to those over $\mathcal{E}_K$ is fully faithful, so this analogy does make sense! Showing that $H^i_{\mathrm{rig}}(X/\mathcal{E}_K)$ descends to $\mathcal{E}_K^\dagger$ can therefore be viewed as an equicharacteristic analogue of the theorem in p-adic Hodge theory saying that all Galois representations coming from geometry are de Rham.

3. $\mathcal{E}_K^\dagger$-valued rigid cohomology

The way that we going to show that the (φ, ∇)-modules $H^i_{\mathrm{rig}}(X/\mathcal{E}_K)$ descend to $\mathcal{E}_K^\dagger$ is to construct new rigid cohomology groups $H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger)$ taking values in vector spaces over $\mathcal{E}_K^\dagger$, these will naturally be endowed with (φ, ∇)-module structures. The key result with then be the existence of a base change isomorphism

$$H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger) \otimes_{\mathcal{E}_K^\dagger} \mathcal{E}_K \to H^i_{\mathrm{rig}}(X/\mathcal{E}_K).$$

Remember that rigid cohomology is defined as the de Rham cohomology of certain 'overconvergent' structure sheaves $j_X^\dagger \mathcal{O}_{]Y[_\mathfrak{P}}$. The way that we will construct our new theory over $\mathcal{E}_K^\dagger$ is by interpreting $\mathcal{E}_K^\dagger$ as $j_X^\dagger \mathcal{O}_{]Y[_\mathfrak{P}}$ for a suitable frame $(X, Y, \mathfrak{P})$, then the cohomology groups we seek will be expressed using a version of rigid cohomology relative to this base frame.

Recall that $\mathcal{E}_K^\dagger$ is defined to be the set of series $\sum_i a_i t^i$ such that $\sup_i |a|_i < \infty$ and $|a_i| \eta^i \to 0$ as $i \to -\infty$ for some $\eta < 1$. More geometrically, these consist of analytic functions, convergent and bounded on some half open annulus $\eta \leq |t| < 1$.

Thinking about Berthelot's $j_X^\dagger$ construction, we want to view these half open annuli as 'strict neighbourhoods' of the 'non-existent' boundary of the open unit disc. So we are looking for a rigid analytic space $\mathscr{X}$ such that:

- $\mathscr{X}$ looks like the open unit disc but with some extra boundary points;
- strict neighbourhoods of these boundary points correspond to half open annuli $\eta \leq |t| < 1$;
- functions on these strict neighbourhoods are exactly the bounded functions on $\eta \leq |t| < 1$.

Define $S_K = K \otimes_W W[\![t]\!]$, the ring of bounded power series, or equivalently the ring of bounded analytic functions on the open unit disc. Then the space we want to consider should then be something like $\mathrm{Sp}(S_K)$, except that this doesn't make sense in the world of Tate's rigid analytic spaces. Instead we move to the world of adic spaces.

Definition. $\mathbb{D}_K^b := \mathrm{Spa}(S_K, W[\![t]\!])$, the bounded open unit disc over K.

Then $\mathbb{D}_K^b = \mathbb{D}_K(0,1^-) \cup \{\xi, \xi_-\}$ looks like the open unit disc with a couple of extra points added. ξ is an open point, and its closure is the 'boundary' $\partial = \{\xi, \xi_-\}$.

If we let U_η denote $\{t \in \mathbb{D}_K(0,1^-) \mid |t| \geq \eta\}$, and $\overline{U}_\eta = U_\eta \cup \partial$, then the $\overline{U}_\eta$ are a cofinal system of neighbourhoods of ∂ inside $\mathbb{D}_K^b$, and we have

$$\Gamma(\overline{U}_\eta, \mathcal{O}_{\overline{U}_\eta}) = \{f \in \Gamma(U_\eta, \mathcal{O}_{U_\eta}) \mid f \text{ bounded}\}.$$

So if we endow $W[\![t]\!]$ with the p-adic topology, and look at the frame

$$(X, Y, \mathfrak{P}) = \left(\mathrm{Spec}\left(k(\!(t)\!)\right), \mathrm{Spec}\left(k[\![t]\!]\right), \mathrm{Spf}\left(W[\![t]\!]\right)\right)$$

then $\mathfrak{P}_K = \mathbb{D}_K^b$ and the above considerations show that

$$\Gamma(\mathfrak{P}_K, j_X^\dagger \mathcal{O}_{\mathfrak{P}_K}) \cong \mathcal{E}_K^\dagger.$$

This therefore motivates the following definition of a frame.

Definition. A frame over $\mathcal{V}[\![t]\!]$ is a triple $T = (X, Y, \mathfrak{P})$ where $X \hookrightarrow Y$ is an open immersion of a $k(\!(t)\!)$ variety into a proper $k[\![t]\!]$-scheme, and $Y \hookrightarrow \mathfrak{P}$ is a closed immersion of Y into a smooth, p-adic, formal $\mathcal{V}[\![t]\!]$-scheme. We say that T is proper if Y is proper over $k[\![t]\!]$, and smooth if $\mathfrak{P}$ is smooth over $\mathcal{V}[\![t]\!]$ in a neighbourhood of X.

If $(X, Y, \mathfrak{P})$ is a frame, then the generic fibre $\mathfrak{P}_K$ is an adic space, locally of finite type over $\mathbb{D}_K^b$, and we have a specialisation map

$$\mathrm{sp} : \mathfrak{P}_K \to \mathfrak{P}.$$

We may therefore define the tubes

$$]Y[_{\mathfrak{P}} := \mathrm{sp}^{-1}(Y)^\circ, \quad]X[_{\mathfrak{P}} = \overline{\mathrm{sp}^{-1}(X)}$$

and we have $j :]X[_{\mathfrak{P}} \to]Y[_{\mathfrak{P}}$. We may therefore define

$$j_X^\dagger \mathscr{F} := j_* j^{-1} \mathcal{F}$$

for any sheaf $\mathscr{F}$ on $]Y[_{\mathfrak{P}}$.

Definition. Let $X/k((t))$ be a variety, and choose a smooth and proper frame $(X, Y, \mathfrak{P})$. Then we define

$$H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger) = H^i(]Y[_{\mathfrak{P}}, j_X^\dagger \Omega^*_{]Y[/S_K}).$$

As before, this turns out to be independent of the choice of frame $(X, Y, \mathfrak{P})$. There are natural Frobenius structures and connections on these cohomology groups, as well as a natural base change map

$$H^i_{\mathrm{rig}}(X/\mathcal{E}_K^\dagger) \otimes_{\mathcal{E}_K^\dagger} \mathcal{E}_K \to H^i_{\mathrm{rig}}(X/\mathcal{E}_K).$$

Theorem (L., Pál)**.** *This map is an isomorphism.*

In the next talk, Dr. Pál will explain the proof of this theorem, as well as hopefully discussing some nice arithmetic consequences/applications.