FUNDAMENTAL GROUPS IN ALGEBRAIC GEOMETRY

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Let C be a projective, non-singular curve defined over $\mathbb{Q}$. Then $C(\mathbb{C})$ is a compact Riemann surface, topologically classified by its genus g.

(1) If $g = 0$ then topologically we get the sphere S^2.
(2) If $g = 1$ then we get the torus T.
(3) For larger g, we get a surface that looks like g-holed torus.

Theorem (Faltings). *If $g \geq 2$ then $\#C(\mathbb{Q})$ is finite.*

So the topology of $C(\mathbb{C})$ is controlling the *arithmetic* properties of C.

Question. What happens if we work over fields which don't embed into $\mathbb{C}$, for example finite fields $\mathbb{F}_q$?

The genus is simple enough to be captured algebraically, but a study of the 'topology' of varieties over $\mathbb{F}_q$ can lead to very deep results, for example the Weil conjectures. The motivating question for these lectures is that of how we might be able to define the fundamental group algebraically.

1. Fundamental groups and covering spaces

For us, a topological space will be a locally path connected, locally simply connected.

Example. manifolds, CW complexes, $X(\mathbb{C})$ for X an algebraic variety over $\mathbb{C}$.

For a connected, pointed topological space (X, x) recall that we define the fundamental group $\pi_1(X, x) = \{\text{loops based at } x\}/\text{homotopy}$. A *loop* is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x$, and a *homotopy* from γ_0 to γ_1 is a continuous family of paths $H : [0,1] \times [0,1] \to X$ such that $\gamma_0 = H(0, *)$ and $\gamma_1 = H(1, *)$. The fundamental group $\pi_1(X, x)$ is a group under concatenation of paths.

Note that algebraic varieties (with the Zariski topology) are not locally path connected or locally simply connected in general, so we will need a different approach to define π_1.

Example. How do we calculate $\pi_1(S^1, 1)$. We use the map $p : \mathbb{R} \to S^1$, $x \mapsto \exp(2\pi i x)$.

Fact. There exists an open cover $\{U_i\}$ of S^1 such that $p^{-1}(U_i) \cong \coprod_{n \in \mathbb{Z}} U_i$, each mapped homeomorphically onto U_i by p.

This is the key ingredient in the proof of the following.

Proposition. Let $f : Y \times [0,1] \to S^1$ be continuous and $\tilde{f}_0 : Y \times \{0\} \to \mathbb{R}$ a lift of $f|_{Y \times \{0\}}$. The $\exists!$ map $\tilde{f} : Y \times [0,1] \to \mathbb{R}$ such that $\tilde{f}|_{Y \times \{0\}} = \tilde{f}_0$ and $f = p\tilde{f}$.

Proof. Divide $Y \times I$ into small pieces V_α such that $f(V_\alpha) \subset U_{i(\alpha)}$. Then we may lift uniquely to some copy of $U_{i(\alpha)}$ in $p^{-1}(U_{i(\alpha)})$ and glue together to give a map $\tilde{f} : Y \times [0,1] \to \mathbb{R}$. □

Corollary.
(1) *For any loop $\gamma : [0,1] \to S^1$ based at 1 there exists a unique path $\tilde{\gamma} : [0,1] \to \mathbb{R}$ lifting γ with $\tilde{\gamma}(0) = 0$.*
(2) *For any homotopy $H : [0,1] \times [0,1] \to \S^1$ from γ_0 to γ_1 there exists a unique homotopy $\tilde{H}$ lifting H from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$.*

Corollary. The map $\mathbb{Z} \to \pi_1(S^1, 1)$ *defined by* $n \mapsto \gamma_n := p\tilde{\gamma}_n$*, where* $\tilde{\gamma}_n : [0,1] \to \mathbb{R}$ *is defined by* $\tilde{\gamma}_n(t) = nt$*, is an isomorphism.*

Inspired by this example, we make the following definition.

Definition. (1) A map $p : Y \to X$ is a covering space if there exists an open cover $X = \cup_i U_i$ such that $p^{-1}(U_i) \cong \coprod U_i$, each mapped homeomorphically onto U_i by p.
(2) A covering space is *universal* if Y is simply connected.

Example. (1) $\mathbb{R} \to S^1$ is a universal covering space.
(2) $S^1 \to S^1$ defined by $z \mapsto z^n$ is a covering space.
(3) $\mathbb{R}^2 \to T := R^2/\mathbb{Z}^2$ is a universal covering space.
(4) $T \to T$ defined by $(x, y) \mapsto (nx, my)$ is a covering space.

Exactly the same proof as before gives:

Proposition. *Let* $p : Y \to X$ *be a covering space,* $f : Z \times I \to X$ *continuous,* $\tilde{f}_0$ *a lift of* $f|_{Z\times\{0\}}$*. Then there exists a unique lfit* $\tilde{f}$ *of* f *such that* $\tilde{f}|_{Z\times\{0\}} = \tilde{f}_0$*.*

Now fix a universal covering space $p : \tilde{X} \to X$, choose a point $\tilde{x} \in \tilde{X}$ and set $p(\tilde{x}) = x$. Let $\mathrm{Aut}(\tilde{X}/X)$ denote the group of automorphisms of $\tilde{X}$ which preserve p. Given any $g \in \mathrm{Aut}(\tilde{X}/X)$ choose a path $\tilde{\gamma}_g$ from $\tilde{x}$ to $g(\tilde{x})$, so that $\gamma_g := p\tilde{\gamma}_g$ is a loop in X based at x. As $\tilde{X}$ is simply connected, this is well defined up to homotopy, and we get a homomorphism

$$\mathrm{Aut}(\tilde{X}/X) \to \pi_1(X, x)$$

Theorem. *This map is an isomorphism.*

Example. (1) $\mathrm{Aut}(\mathbb{R}/S^1) \cong \mathbb{Z}$ via $g_n(x) = x + n$.
(2) $\mathrm{Aut}(R^2/T) \cong \mathbb{Z}^2$ via $g_{n,m}(x, y) = (x + n, y + m)$.

Example. IF we want something to work for algebraic varieties, we need something more that the link between universal covering spaces and fundamental groups. For example, if $X = \mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^*$, the universal cover of X is $\exp(2\pi i \cdot) : \mathbb{C} \to \mathbb{C}^*$ is *not* an algebraic map. However, the 'finite level' maps $\mathbb{C}^* \xrightarrow{z \mapsto x^n} \mathbb{C}^*$ *are* algebraic.

Question. What can we deduce about π_1 using only 'finite level' covering spaces?

Fix a pointed, connected covering space $p : (Y, y) \to (X, x)$. Then $\pi_1(X, x)$ acts on the fibre $p^{-1}(x)$ via $(\gamma, y') \mapsto \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the unique lift of γ starting at y'. We therefore get a subgroup $H = \mathrm{Stab}_{\pi_1(X,x)}(y) = \{\gamma \mid \tilde{\gamma}(0) = \tilde{\gamma}(1) = y\}$. Conversely, given $H \subset \pi_1(X, x) = \mathrm{Aut}(\tilde{X}/X)$ we get a covering space $(Y, y) = (\tilde{X}/H, \text{image of } \tilde{x})$.

Theorem. *This sets up a bijection*

$$\{\text{pointed connected cov. spaces } (Y, y) \to (X, x)\}/\simeq \;\leftrightarrow\; \{\text{subgroups } H \subset \pi_1(X, x)\}.$$

Example. (1) The covering space $S^1 \xrightarrow{z \mapsto z^n} S^1$ corresponds to the subgroup $n\mathbb{Z} \subset \mathbb{Z}$.
(2) The map $T \to T$ defined by $(x, y) \mapsto (nx, my)$ is a covering space, and corresponds to the subgroup $n\mathbb{Z} \times m\mathbb{Z} \subset \mathbb{Z}^2$.

Definition. (1) Say that $p : (Y, y) \to (X, x)$ is *finite* if $p^{-1}(x)$ is finite.
(2) Say that $p : (Y, y) \to (X, x)$ is *Galois* if $\mathrm{Aut}(Y/X)$ acts transitively on $p^{-1}(x)$.

Proposition. (1) *p is finite iff $H \subset \pi_1(X, x)$ is of finite index.*
(2) *p is Galois iff H is a normal subgroup of $\pi_1(X, x)$, in this case we have* $\mathrm{Aut}(Y/X) \cong \pi_1(X, x)/H$*.*

In algebraic geometry, it is exactly the *finite* covering spaces that we expect to see. Therefore, we expect to be able to see all finite quotients of π_1.

Example. (1) Since every finite index subgroup of $\mathbb{Z}$ is $n\mathbb{Z}$ for some $n \geq 1$, the maps $\mathbb{C}^* \xrightarrow{Z \mapsto Z^n}$ give us all possible finite covering spaces of $\mathbb{C}^*$, each with automorphism group $\mathbb{Z}/n\mathbb{Z}$. So algebraically, while we can't see $\mathbb{Z}$, we can see the collection $\{\mathbb{Z}/n\mathbb{Z}\}$.

(2) If we identify T with $\mathbb{C}/\mathbb{Z}^2$, then there exists a unique projective non-singular curve $E/\mathbb{C}$ (an elliptic curve) such that $T \cong E(\mathbb{C})$. The maps $p_n : T \to T$ given by $(x, y) \mapsto (nx, ny)$ are determined by holomorphic (regular) maps $p_n : E \to E$. We have $\mathrm{Aut}(E \xrightarrow{p_n} E) \cong (\mathbb{Z}/n\mathbb{Z})^2$, so algebraically we can recover the collection $\{(\mathbb{Z}/n\mathbb{Z})^2\}$.

Remark. There exists an algebraic way to 'bundle together' the finite quotients $\{G/N\}_N$ of a given group G to obtain a new group of a particular flavour - a 'pro-finite' group. This retains a lot of information about G, but not all. If we let $\hat{\pi}_1(X, x)$ be this new group, then for algebraic varieties over $k \subset \mathbb{C}$ we will be able to capture $\hat{\pi}_1(X(\mathbb{C}), x)$ algebraically.

2. Local isomorphisms and étale maps

How can we transport the notion of a covering space into algebraic geometry? In other words when should a map $Y \to X$ be considered a covering space? We'll start with a slightly weaker notion.

Definition. A map $f : Y \to X$ of topological spaces is a local isomorphism if $\forall y \in Y$ there exist $y \in V \subset Y$ and $f(y) \in U \subset X$ open such that $f : V \xrightarrow{\sim} U$.

Example. $p : \mathbb{R} \setminus \{0\} \to \mathbb{S}^1$ is a local isomorphism, but is not a covering space.

Now fix an algebraically closed field k, and a morphism of algebraic varieties $f : Y \to X$ over k. Then the above notion of local isomorphism os far too rigid to transport naïvely. If we say that f is a 'naïve' local isomorphism if $\forall y \in Y$ there exists $y \in V \subset Y$ and $f(y) \in U \subset X$ Zariski opens with $f : V \xrightarrow{\sim} U$, then f is in fact a birational map.

Example. The map $\mathbb{G}_m \xrightarrow{z \mapsto z^n} \mathbb{G}_m$ is 'topologically' a covering space (i.e. is one over $\mathbb{C}$) but is not birational.

The problem is that Zariski neighbourhoods are far too 'large'. The solution is to work with formal neighbourhoods instead.

Recall that if $x \in X$ is a point on an algebraic variety, we define the local ring of X at x to be $\mathcal{O}_{X,x} := \lim_U \Gamma(U, \mathcal{O}_X)$, the limit being taken over all Zariski open neighbourhoods of x. Concretely, elements of $\mathcal{O}_{X,x}$ are represented by pairs (U, f) where $x \in U$ and f is a function on U. There is a canonical ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ which consists of those 'germs' (U, f) such that $f(x) = 0$.

Fact. $\mathcal{O}_{X,x}$ is a Noetherian ring with maximal ideal $\mathfrak{m}_x$.

We can use $\mathfrak{m}_x$ to define a topology on $\mathcal{O}_{X,x}$, where a basis for the topology is given by the 'open sets' $V_{f,n} = f + \mathfrak{m}_x^n$ for $f \in \mathcal{O}_{X,x}$. We can complete $\mathcal{O}_{X,x}$ with respect to this topology, and define

$$\widehat{\mathcal{O}}_{X,x} = \varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^n$$

to be this '$\mathfrak{m}_x$-adic' completion. The intuition is that $\widehat{\mathcal{O}}_{X,x}$ captures information about an 'infinitesimally small' neighbourhood of x.

Proposition. *Let X be a non-singular variety of dimension n, and $x \in X$. Then $\widehat{\mathcal{O}}_{X,x} \cong k[\![t_1, \dots, t_n]\!]$ is a power series ring over k.*

Thus the differential geometric statement "all points on smooth manifolds have isomorphic local neighbourhoods" becomes the algebro-geometric statement "all points on non-singular varieties have isomorphic *formal* neighbourhoods". This motivates the following.

Definition. A morphism $f : Y \to X$ is said to be étale at $y \in Y$ (or a local isomorphism) if the induced map $f^* : \widehat{\mathcal{O}}_{X,f(y)} \to \widehat{\mathcal{O}}_{Y,y}$ of completed local rings is an isomorphism.

If we want to capture the notion of a covering space, we need to impose an addition condition to rule out examples like $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$.

Definition. A map $f : Y \to X$ is said to be *finite* if for all $U \subset X$ Zariski open, $\Gamma(f^{-1}(U), \mathcal{O}_Y)$ is a finitely generated $\Gamma(U, \mathcal{O}_X)$-module. (When f is a map between projective varieties, this boils down to simply having finite fibres.)

Then a finite étale map $f : Y \to X$ is our sought after algebro-geometric analogue of a covering space map in topology. For a variety X, then, the "fundamental group" of X is defined to be the collection of groups $\{\mathrm{Aut}(Y/X)\}$ as Y ranges over all finite étale maps $Y \to X$. As before, there exists an algebraic way to package these all together into a single group $\pi_1^{\text{ét}}(X,x)$.

Theorem. *Let $X/\mathbb{C}$ be an algebraic variety. Then every finite covering space of $X(\mathbb{C})$ arises via a finite étale map of algebraic varieties $Y \to X$. Hence $\pi_1^{\text{ét}}(X,x) \cong \hat{\pi}_1(X(\mathbb{C}),x)$.*

3. Algebraic curves, ramification and étale covers

Now let us fix an algebraically closed field k, a curve will ben a non-singular variety over k of dimension 1, not necessarily projective (i.e. we will allow affine curves).

Fact. Let C be a curve, and $P \in C$. Then the local ring $\mathcal{O}_{C,P}$ of C at P is a discrete valuation ring, i.e. a Noetherian ring with a unique non-zero prime ideal.

We call an element $t_P \in \mathcal{O}_C$ sucht hat $t_P \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$ a *local parameter* at P. It follows that $\mathfrak{m}_P = (t_P)$, and we define a discrete valuation $v_P : k(C)^* \to \mathbb{Z}$ by extending multiplicatively the map $v_P : \mathcal{O}_{C,P} \to \mathbb{Z}$ defined by $v_P(f) := \max\{n \mid f \in \mathfrak{m}_P^n\}$.

Example.
1. Let $C = V(f) \subset \mathbb{A}_k^2$ be a plane curve defined by some $f(x,y) \in k[x,y]$. Then the non-singularity of C is equivalent to the fact that at any $P \in C$ either $\frac{\partial f}{\partial x}\big|_P \neq 0$ or $\frac{\partial f}{\partial y}\big|_P \neq 0$.Write $P = (a,b)$, then $\frac{\partial f}{\partial x}\big|_P \neq 0 \Rightarrow (y-b)$ is a local parameter at P, and $\frac{\partial f}{\partial y}\big|_P \neq 0 \Rightarrow (x-a)$ is a local parameter at P.
2. $P = \lambda \in \mathbb{A}_k^1 = k$. Then $(x-\lambda)$ is a local parameter at P.
3. Let $C = V(y^2 - x^3 + x) \subset \mathbb{A}_k^2$. Then $\frac{\partial f}{\partial x} = 1 - 3x^2$ and $\frac{\partial f}{\partial y} = 2y$. Hence if $y \neq 0$ then $x - a$ is a local parameter, and y is a local parameter whenever $y = 0$. Near $P = (0,0)$ we have $y^2 = x(x^2-1)$, and (x^2-1) is non-zero at P, hence invertible there. Therefore $v_P(x) = 2$.

Now let $f : C \to C'$ be a non-constant morphism of curves, $P \in C$ and $Q = f(P)$. Then we get an induced map $f^* : \mathcal{O}_{C',Q} \to \mathcal{O}_{C,P}$.

Definition. The ramification index of f at P is defined to be $e_P := v_P(f^*(t_Q))$ where t_Q is a local parameter at Q. This is the order of vanishing of $f^*(t_Q)$ at P.

Example.
1. Let $C = V(f) \subset \mathbb{A}_k^2$ be a plane curve, $P = (a,b)$. Then $\frac{\partial f}{\partial y}\big|_P \neq 0 \Rightarrow (x-a)$ is a local parameter at P, and hence the projection map $f : C \to \mathbb{A}_k^1$ defined by $(x,y) \mapsto x$ has $e_P = 1$, since $f^*(x-a) = x-a$.
2. Let $C = V(y^2 - x^3 + x) \subset \mathbb{A}_k^2$, $P = (0,0)$ and $f : C \to \mathbb{A}_k^1$ the projection $(x,y) \mapsto x$. Then y is a local parameter ar p, x is a local parameter at $Q = f(P)$ and $y_v(x) = 2$. Hence $e_P = 2$. Note that for $Q \neq 0,1,-1$ there are exactly 2 points P, P' mapping to Q each with ramification index 1, but for $Q = 0,1,-1$ there is only 1, with ramification index 2.
3. Consider the map $f : \mathbb{P}_k^1 \to \mathbb{P}_k^1$ defined by $[X : Y] \mapsto [X^3(X-Y)^2 : Y^5]$. Let $P = [0:1]$, $P' = [1,1]$ so that $f(P) = f(P') = Q = [0:1]$. Restricting f to $\mathbb{A}_k^1$ gives the map $x \mapsto x^3(x-1)^2$, so $e_P = 3$, $e_{P'} = 2$. Notice that $e_P + e_{P'} = 5$.
4. $C = \mathbb{A}_k^1$, $\mathrm{char}(k) = p$, and $F : \mathbb{A}_k^1 \to \mathbb{A}_k^1$ the Frobenius map $x \mapsto x^p$. Let $P = \lambda \in \mathbb{A}_k^1$, $Q = F(P) = \lambda^p$. Then $t_Q = x - \lambda^p$ and $F^*(t_Q) = x^p - \lambda^p = (x_\lambda)^p$. Hence $e_P = p$ for all P.

Say a non-constant map $f: C \to C'$ of curves has degree $d = [k(C):k(C')]$.

Theorem. *Let $f: C \to C'$ be a finite morphism of curves, and $Q \in D$. Then $\sum_{P \in f^{-1}(Q)} e_P = \deg(f)$.*

To see the relation between ramification and the notion of an étale morphism, suppose that we have $P \in C$ and t_P a local parameter at P. Then we can identify $\widehat{\mathscr{O}}_{C,P}$ with the power series ring $k[\![t]\!]$ with the variable t_P. SO if $f: C \to C'$ is unramified at P, $Q = f(P)$, then $v_P(f^*(t_Q)) = 1 \Rightarrow f^*(t_Q)$ *is* a local parameter for C at P. Hence $\widehat{\mathscr{O}}_{C,P} \cong k[\![f^*(t_Q)]\!]$ and $f^*: \widehat{\mathscr{O}}_{C',Q} \to \widehat{\mathscr{O}}_{C,P}$ is an isomorphism, i.e. f is étale at P.

Conversely, if $e_P > 1$ then $f^*(t_Q) = t_P^e u$ for some $e \geq 1$ and $u \in \mathscr{O}_{C,P}^*$ and so t_P is not in the image of $f^*: \widehat{\mathscr{O}}_{C',Q} \to \widehat{\mathscr{O}}_{C,P}$. Hence f is not étale at P.

Proposition. *$f: C \to C'$ a non-constant morphism of curves. Then f is étale at $P \in C \Leftrightarrow$ it is unramified at P.*

Example.
(1) A polynomial $f(x) \in k[x]$ defines a map $f: \mathbb{A}_k^1 \to \mathbb{A}^1$. If $\lambda \in \mathbb{A}_k^1$ then $x - f(\lambda)$ is a local parameter at $f(\lambda)$, and $f^*(x - f(\lambda)) = f(x) - f(\lambda)$. Hence f is unramified at $\lambda \Leftrightarrow f(x) - f(\lambda)$ has a simply root at $\lambda \Leftrightarrow f'(\lambda) \neq 0$.
(2) Suppose $\mathrm{char}(k) = 0$ and let $f: \mathbb{G}_m \to \mathbb{G}_m$ be defined by $z \mapsto f(z) = z^n$. Since $f'(z) = nz^{n-1} \neq 0$ for any $z \in \mathbb{G}_m$, it follows that f is everywhere étale. It is much harder to see that f is finite and moreover every finite étale map $X \to \mathbb{G}_m$ is of this form. Hence $\pi_1^{\text{ét}}(\mathbb{G}_m, 1)$ is determined by the collection of finite groups $\{\mathbb{Z}/n\mathbb{Z}\}_{n \geq 1}$.
(3) Suppose $\mathrm{char}(k) = p$ and let $f: \mathbb{A}_k^1 \to \mathbb{A}_k^1$ be defined by $f(x) = x^p - x$. Then $f'(x) = -1 \neq 0$, so f is everywhere étale. The affine line in characteristic p has *lots* of finite étale covers!

4. Projective, non-singular curves

Fact. Any non-constant morphism $C \to C'$ of projective non-singular curves is finite.

Why? Because for projective varieties finite $\Leftrightarrow$ each preimage $f^{-1}(Q)$ is finite. But for curves, preimages are either finite or the whole curve. Hence in this case finite étale $\Leftrightarrow$ everywhere unramified.

Example. Let $f: \mathbb{P}_k^1 \to \mathbb{P}_k^1$ extend $\mathbb{A}_k^1 \to \mathbb{A}_k^1$ defined by $p(x) \in k[x]$, so $f(\infty) - \infty$ and a local parameter at ∞ is $u := 1/x$. Then near ∞, f is defined by $\frac{u^n}{a_n + a_{n-1}u^{n-1} + \ldots + a_0 u^n}$ where $p = a_n x^n + \ldots + a_0$. Hence f is ramified at ∞, with $e_\infty = \deg(f)$. We can beef up this example to show that even in positive characteristic, there don't exist any non-trivial finite étale maps $\mathbb{P}_k^1 \to \mathbb{P}_k^1$.

A *divisor* on a projective curve C is a finite formal linear combination $D = \sum_P n_P[P]$ of points on C. We define the degree of a divisor to be $\sum_P n_P$. Given a divisor D we define

$$\mathscr{L}(D) = \{f \in k(C) | v_P(f) \geq -n_P \ \forall P \in C\}$$

to be the k-vector space of functions only having poles at points of D, with prescribed order. We also define $\ell(D) = \dim_k \mathscr{L}(D)$.

Example.
(1) Let $C = \mathbb{P}_k^1$, $D = [0]$. Then $f = p(x)/q(x) \in \mathscr{L}(D)$ iff f is regular away from 0, and $v_0(f) \geq -1$. If f is regular away from $0, \infty$ then we must have $q(x) = x^n$ for some $n \in \mathbb{N}$, and if f is regular at ∞ we must have $\deg(p) \leq \deg(q)$. Moreover, if f has a pole of order ≤ 1 at 0 implies that $n \leq 1$. Hence $f = \lambda + \mu(1/x)$ for some $\lambda, \mu \in k$ and therefore $\ell(D) = 2$.
(2) Let $E = V(Y^2Z - X^3 + XZ^2) \subset \mathbb{P}_k^2$, $P = [0:1:0]$ and $D = 2[P]$. Then $k(E) \cong \frac{k(x)[y]}{y^2 - x^3 + x}$ where $y = Y/Z$ and $x = X/Z$. Hence any $f \in k(E)$ is uniquely of the form $p(x) + q(x)y$ with $p, q \in k(x)$. Then f being regular on the affine patch defined by $Z = 1$ is equivalent to both p and q actually being polynomials. Now let $u = X/Y$ and $v = Z/Y$, so P corresponds to the point $(u, v) = (0, 0)$ and we have $x = u/v$, $y = 1/v$. Can check that $v_P(x) = 2$ and $v_P(y) = 3$ and hence $f = p(x) + q(x)y \in \mathscr{L}(D) \Leftrightarrow f = \lambda + \mu x$ for some $\lambda, \mu \in k$. Therefore $\ell(D) = 2$.

Theorem. *Let C be a non-singular, projective curve. Then $\exists!$ integer $g \geq 0$, called the* genus *of C, such that $\ell(D) \geq \deg(D) + 1 - g$ for all divisors D.*

Example.
- The only example for $g = 0$ in $\mathbb{P}^1_k$. In the above example, we had $\deg(D) = 1$, $\ell(D) = 2$.
- Every projective curve of genus $g = 1$ is isomorphic to $V(Y^2Z - X(X-Z)(X-\lambda Z)) \subset \mathbb{P}^2_k$ for some $\lambda \in k \setminus \{0,1\}$. These are called elliptic curves, and are the algebraic analogues of complex tori $\mathbb{C}/\Lambda$, with $\Lambda \cong \mathbb{Z} \oplus \mathbb{Z}$. Above we had $\deg(D) = 2$, $\ell(D) = 2$.
- Curve of genus $g \geq 2$ are harder to classify.

We have some control over how the genus grows under a finite map of curves, given by the Riemann-Hurwitz formula. Recall that a non-constant morphism of curves $f : C \to C'$ is *separable* if the corresponding extension $k(C') \subset k(C)$ of function fields is.

Theorem. *Let $f : C \to C'$ be a finite, separable morphism of projective curves of degree n. Then $e_P = 1$ for all but finitely many P, and if $\mathrm{char}(k) \nmid e_P$ for all P, then*

$$2g(C) - 2 = n(2g(C') - 2) + \sum_{P \in C} (e_P - 1).$$

Example.
1. Let $f : C \to \mathbb{P}^1$ be a finite étale cover of degree n. Then $e_P = 1$ for all $P \in C$, and hence $2g(C) - 2 = -2n$. The only way this can happen is if $n = 1$ and $g(C) = 0$, i,e, f is an isomorphism. Hence $\mathbb{P}^1_k$ has *no* non-trivial finite étale covers, and $\pi_1^{\text{ét}}(\mathbb{P}^1_k) = \{1\}$.
2. Let $f : C \to E$ be a finite étale cover and suppose $g(E) = 1$, i.e. E is an elliptic curve. Then $2g(C) - 2 = 0 \Rightarrow g(C) = 1$. Hence C is also an elliptic curve. This is an algebraic analogue of the fact that every finite covering space of the torus $\mathbb{R}^2/\mathbb{Z}^2$ is another torus.
3. Let $E = V(Y^2Z - X^3 + XZ^2) \subset \mathbb{P}^2_k$. Let $f : E \to \mathbb{P}^1_k$ be defined by $[X : Y : Z] \mapsto [X : Z]$. On the affine patch $Z = 1$ we have ramification at $(x, y) = (0,0), (1,0)$ and $(-1,0)$ of degree 2, and on the affine patch $Y = 1$ we have additional ramification at $(u, v) = (0,0)$ of degree 2. So we have $2g(E) - 2 = 0$, $2g(\mathbb{P}^1_k) - 2 = -2$, and $\sum_{P \in E}(e_P - 1) = 4$. Hence this map is of degree 2.

5. Higher dimensional varieties

We could understand finite étale covers of curves essentially for two reasons:

1. Every non-constant morphism of curves was finite.
2. Being étale is equivalent to being everywhere unramified, which in principle (and often in practise) can be checked.

What happens in higher dimensions, for example for projective algebraic surfaces S? Now a morphism $S \to S'$ might not be finite.

Example. $S = V(y^2 - x(x-w)(x-z)) \subset \mathbb{P}^3_k$, then $\varphi : S \dashrightarrow \mathbb{P}^2_k$ defined by $[x - w : y : z - w]$ contracts the line $[s : t : s : s]$ to the point $[0 : 1 : 0]$.

Therefore finiteness really is a condition that we need to check. More seriously, if $P \in S$, then the local ring $\mathcal{O}_{S,P}$ of S at P is no longer a DVR, so 'ramification theory' for points no longer makes sense.

Definition. Let $C \subset S$ be a curve inside S (irreducible, but not necessarily non-singular). Then we define the local ring of S at C to be

$$\mathcal{O}_{S,C} := \lim_{U \cap C \neq \emptyset} \Gamma(U, \mathcal{O}_S),$$

elements of $\mathcal{O}_{S,C}$ are therefore represented by pairs (U, f) where $U \subset S$ is open, $U \cap C \neq \emptyset$ and f is a regular function on U.

It turns out that when S is non-singular, and $C \subset S$ is a curve, then $\mathcal{O}_{S,C}$ is a discrete valuation ring. Hence we obtain a theory of ramification along *curves* inside surfaces, rather than at points. The point is that if $f : S \to S'$ is finite, and $C \subset S$ is curve, then $C' := f(C) \subset S$ is also a curve. We therefore get a map $f^* : \mathcal{O}_{S',C'} \to \mathcal{O}_{S,C}$ and can define $e_C := v_C(f^* t_{C'})$ where $t_{C'} \in \mathcal{O}_{S',C'}$ is a local parameter along C, i.e. generates the maximal ideal of $\mathcal{O}_{S',C'}$. We say that f is unramified along C if $e_C = 1$.

Theorem (Very deep!)**.** *Let $f : S \to S'$ be a finite morphism of projective, non-singular algebraic surfaces over k. Then f is étale iff f is unramified along all curves $C \subset S$.*