NOTES ON MONSKY–WASHNITZER COHOMOLOGY

1. De Rham cohomology of smooth affine varieties

Let K be a field, and A a finitely generated K-algebra.

Definition 1.1. $\Omega_{A/K}$ is the A-module of Kähler differentials. This is the A-module generated by symbols da for $a \in A$ and relations:

- $dk = 0$ for $k \in K$
- $d(a+b) = da + db$ and $d(ab) = a \cdot db + b \cdot da$ for all $a, b \in A$.

More explicitly, if we have
$$A = K[x_1, \dots, x_n]/(f_1, \dots, f_r)$$
then $\Omega_{A/K}$ has a presentation:
$$\Omega_{A/K} = Adx_1 \oplus \cdots Adx_n/(df_i : i = 1, \dots, r)$$
where df is the total derivative $\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$.

There is a natural derivation
$$d : A \to \Omega_{A/K}$$
sending a to da.

Remark 1.2. $\Omega_{A/K}$ is characterised by a universal property: any K-linear derivation $D : A \to M$ factors uniquely as
$$A \xrightarrow{d} \Omega_{A/K} \to M.$$

If A/K is smooth [1] then $\Omega_{A/K}$ is locally free of rank $\dim(A)$.

Definition 1.3. Set $\Omega^i_{A/K} = \bigwedge^i \Omega_{A/K}$. In particular $\Omega^0_{A/K} = A$ and $\Omega^1_{A/K} = \Omega_{A/K}$.

Abusing notation, we write d for each of the K-linear maps
$$d : \Omega^i_{A/K} \to \Omega^{i+1}_{A/K}$$
given by mapping $f_0 df_1 \wedge \dots df_i$ to $df_0 \wedge df_1 \wedge \dots df_i$.

Denote by $\Omega^\bullet_{A/K}$ the resulting complex. This is the *de Rham complex* for A/K.

Finally, set $H^i_{dR}(A/K) = H^i(\Omega^\bullet_{A/K})$.

Note that if $i > \dim(A)$ then $\Omega^i_{A/K} = 0$ and so $H^i_{dR}(A/K) = 0$.

Remark 1.4. It would be better to write $H^i_{dR}(A/K)$ as $H^i_{dR}(X/K)$, where $X = \mathrm{Spec}(A)$. The definition of $\Omega^\bullet_{A/K}$ globalises to the case of finite type schemes X/K, but then to define $H^i_{dR}(X/K)$ one needs to take hypercohomology of the resulting complex of coherent sheaves on X. Things simplify in the affine case because $H^i(U, \mathscr{F}) = 0$ for $i > 0$ for coherent sheaves $\mathscr{F}$ on affine schemes U.

Example 1.5. One very simple example: suppose K has characteristic 0 and let $A = K[x_1, \dots, x_n]$ (i.e. $X = \mathbb{A}^n_K$). Then $H^0_{dR}(A/K) = K$ and $H^i_{dR}(A/K) = 0$ for $i > 0$.

For $K = \mathbb{C}$ there is a natural isomorphism $H^i_{dR}(X/\mathbb{C}) = H^i_{dR}(X^{an}/\mathbb{C})$ where X^{an} is the complex manifold associated to $X = \mathrm{Spec}(A)$.

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[1] Here is a definition of smooth: there are elements $a_1, \dots, a_n$ generating A as an ideal, such that for each i the localisation $A_{(a_i)}/K$ can be written as $K[x_1, \dots, x_n]/(f_1, \dots, f_r)$ with Jacobian matrix $(\frac{\partial f_i}{\partial x_j})_{i,j=1,\dots,r}$ having non-zero determinant in $A_{(a_i)}$. In other words, A Zariski locally satisfies the Jacobian criterion for smoothness.

2. Cohomology in characteristic p

From now on we fix a finite field k of characteristic p and cardinality $q = p^f$. We would like to define some sort of cohomology theory for varieties over k. Today we will define a cohomology theory, with coefficients in a characteristic 0 field, for smooth affine varieties over k. First let's consider the de Rham cohomology of such a variety.

Suppose $A = k[x]$. Then $H^1_{dR}(A/k)$ is not a finite-dimensional k-vector space: if n is a positive integer divisible by p, then $x^{n-1}dx \in \Omega_{A/k}$ is not in the image of d. This suggests we should stick to characteristic 0 coefficients.

We set R to be the Witt vectors of k. This is the ring of integers in the (unique, up to isomorphism) unramified extension of $\mathbb{Q}_p$ with residue field k. We have $R/pR = k$. We set $K = \mathrm{Frac}(R)$. If $k = \mathbb{F}_p$ then R is the ring of p-adic integers $\mathbb{Z}_p$.

Suppose we have a smooth k-algebra A. Then, if we can find a smooth R-algebra $\tilde{A}$ with $\tilde{A} \otimes_R k \cong A$, we could consider the de Rham cohomology of the smooth K-algebra $\tilde{A}_K := \tilde{A} \otimes_R K$. Here is an example which shows this is not such a good idea:

Example 2.1. Set $A = k[x]$, $\tilde{A}_1 = R[x]$, $\tilde{A}_2 = R[x, y]/((1 + px)y - 1)$. Then $H^1_{dR}(\tilde{A}_{1,K}/K)$ is zero, whilst $H^1_{dR}(\tilde{A}_{2,K}/K)$ is one-dimensional.

So there are many lifts of A to smooth R-algebras, which can give different cohomology groups. In fact, there is a way to get a unique lift of A, but we need to work with R-algebras which are only topologically finitely generated (for the p-adic topology).

Proposition 2.2. *Suppose A is a smooth k-algebra. Then there is a unique (up to isomorphism) R-algebra $\hat{A}$ which is complete in the p-adic topology, flat over R, and with*

$$\hat{A} \otimes_R k \cong A.$$

Proof. For each $n \geq 1$ there is a unique lift of A to a flat R/p^nR-algebra A_n, since obstructions to lifting and the set of lifts are classified by H^2 and H^1 of a coherent sheaf on $\mathrm{Spec}(A)$ (see Hartshorne, Deformation Theory, Section 10), which vanish. We then have $\hat{A} = \varprojlim_n A_n$. □

Example 2.3. For $A = k[x]$ we can take

$$\hat{A} = R\langle x\rangle := \{\sum_{n=0}^{\infty} a_n x^n : |a_n|_p \to 0 \text{ as } n \to \infty\}.$$

For rings like $R\langle x\rangle$ and $R\langle x\rangle \otimes_R K$ which are only *topologically* finitely generated, we need to work with continuous differentials, for example $\Omega_{R\langle x\rangle/R} = R\langle x\rangle dx$. These are universal with respect to continuous R-linear derivations $A \to M$ from A to a topological A-module M.

But now we still have a problem: we have $(\sum_{n\geq 0} p^n x^{p^n - 1})dx \in \Omega_{\hat{A}_K/K}$ which *should* integrate to $\sum_{n\geq 0} x^{p^n}$, but this is not in $\hat{A}$. The idea of Monsky and Washnitzer is to use a subring $A^\dagger \subset \hat{A}$ defined by

$$A^\dagger = \{\sum a_n x^n : \exists C > 0, \rho \in (0, 1) \text{ with } |a_n|_p \leq C\rho^n \forall n\}.$$

Remark 2.4. Elements of $f \in \hat{A}$ are functions on the closed unit disc $|x| \leq 1$. If we have $f = \sum a_n x^n$ with $|a_n|_p \leq C\rho^n$, then f extends to a function on the open disc $|x| < \rho^{-1}$.

This means that $A^\dagger$ consists of functions on the closed unit disc which in fact converge on some bigger disc. Therefore $A^\dagger$ is sometimes referred to as a ring of overconvergent functions on the closed unit disc.

You can check that the coefficients of $\sum_{n\geq 0} p^n x^{p^n - 1}$ tend to zero too slowly for this element to lie in $A^\dagger$.

Exercise 2.5. The map $d : A^\dagger_K \to \Omega_{A^\dagger_K/K}$ is surjective.

The lifting from $A = k[x]$ to $A^\dagger_K$ is a model for what we are going to do in general.

3. Weakly complete lifts

Definition 3.1. We define the weakly complete power series ring

$$W_n := R\langle x_1, \ldots, x_n\rangle^\dagger = \left\{\sum a_\alpha x^\alpha : \exists C > 0, \rho \in (0,1) \text{ with } |a_\alpha|_p \leq C\rho^{|\alpha|} \forall \alpha\right\}.$$

In the above, α runs over tuples $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$.

Theorem 3.2. *W_n is Noetherian.*

Proof. One way to show Noetherianity is to use the Weierstrass division and preparation theorems. See [2] Theorems 3.1.1 and 3.2.1 for the case of $K\langle x_1, \ldots, x_n\rangle$, which can be adapted to the weakly complete situation.

Alternatively, the note of Fulton [3] gives the original proof this lemma. □

Definition 3.3. A WCFG (weakly complete, finitely generated) R-algebra is a quotient of W_n.

Suppose A is a finitely generated k-algebra. A WCFG R-algebra $\tilde{A}^\dagger$ is a *lift* of A if it is flat over R and $\tilde{A}^\dagger \otimes_R k \cong A$.

Theorem 3.4 (Elkik, Theorem 6 [1])**.** *Suppose A is a smooth finitely generated k-algebra. Then there exists a finitely generated smooth R-algebra $\tilde{A}$ with $\tilde{A}/p\tilde{A} = \tilde{A} \otimes_R k \cong A$.*

The above theorem is easy to see when $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ satisfies the Jacobian criterion. Choose some lifts $\tilde{f}_1, \ldots, \tilde{f}_r$ of the f_i to $R[x_1, \ldots, x_n]$ and consider $R[x_1, \ldots, x_n]/(\tilde{f}_1, \ldots, \tilde{f}_r)$. This R-algebra is smooth in a Zariski open neighbourhood of $V(pR) \subset \operatorname{Spec}(R)$ (since the determinant of the Jacobian matrix is non-zero modulo p), so some localisation gives a smooth lift of A.

Corollary 3.5. *Suppose A is a smooth finitely generated k-algebra. Then there is a lift of A to a WCFG R-algebra.*

Proof. Take $\tilde{A}$ as in the above Theorem and write $\tilde{A} = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Set

$$\tilde{A}^\dagger = R\langle x_1, \ldots, x_n\rangle^\dagger/(f_1, \ldots, f_r).$$

Then we claim $\tilde{A}^\dagger$ is a lift of A. It is clear that $\tilde{A}^\dagger \otimes_R k = \tilde{A} \otimes_R k$, so we just need to check that $\tilde{A}^\dagger$ is flat over R. But $\tilde{A}^\dagger/p^n\tilde{A}^\dagger = \tilde{A}/p^n\tilde{A}$ is flat over R/p^nR for all n, so this follows from [4, Theorem 2.4] (see also [5, Tag 0523]). □

4. Monsky–Washnitzer cohomology

We will now simplify our notation a bit. From now on $\overline{A}$ will be a smooth k-algebra and A will be a lift to a WCFG algebra over R. As mentioned before, when we discuss differentials of A we will want to take *continuous* differentials.

Definition 4.1. For $A = R\langle x_1, \ldots, x_n\rangle/(f_1, \ldots, f_r)$, set $D^1(A/R)$ to be the A-module

$$D^1(A/R) := Adx_1 \oplus \cdots \oplus Adx_n/(df_i : i = 1, \ldots, r)$$

The more abstract description of $D^1(A/R)$ as continuous differentials shows that it is well-defined, independently of the presentation of A.

Lemma 4.2. *$D^1(A/R)$ is a locally free A-module of rank* $\dim(\overline{A})$.

Proof. Since A/R is flat and and $A/pA = \overline{A}$ is smooth over k, A/p^nA is smooth over R/p^nR for all $n \geq 1$ [5, Tag 031L]. Therefore $D^1(A/R) \otimes_A A/p^nA = \Omega_{(A/p^nA)/(R/p^nR)}$ is flat over A/p^nA for all $n \geq 1$. Now [4, Lemma 2.1] (see also [5, Tag 0523]) shows that $D^1(A/R)$ is flat over A. Since it is also finitely generated, it is locally free. The statement about the rank follows from $D^1(A/R) \otimes_A \overline{A} \cong \Omega_{\overline{A}/k}$. □

Definition 4.3. Set $D^i(A/R) = \bigwedge_A^i D^1(A/R)$, and $D^i(A/K) = D^i(A/R) \otimes_R K = \bigwedge_{A_K}^i D^1(A/K)$.

Abusing notation, we write d for each of the K-linear maps

$$d : D^i(A/K) \to D^{i+1}(A/K)$$

given by mapping $f_0 df_1 \wedge \ldots df_i$ to $df_0 \wedge df_1 \wedge \ldots df_i$.

Denote the resulting complex by $D^\bullet(A/K)$. Finally, set $H^i_{MW}(A/K) = H^i(D^\bullet_{A/K})$.

In the next section, we will show that $H^i_{MW}(A/K)$ only depends on the smooth k-algebra $\overline{A}$, so we may instead write $H^i_{MW}(\overline{A}/K)$.

5. Functoriality and uniqueness of lifting

Theorem 5.1. *Suppose $\overline{A}, \overline{B}$ are smooth k-algebras. Let A/R and B/R be WCFG lifts of $\overline{A}, \overline{B}$ respectively (which exist by Corollary 3.5). Then:*

(1) *Every lift of $\overline{A}$ is isomorphic (over R) to A.*
(2) *Suppose*
$$\overline{f} : \overline{A} \to \overline{B}$$
is a k-algebra map. Then there exists an R-algebra map
$$f : A \to B$$
lifting $\overline{f}$.

Proof. We just explain part (1). The second part is proved by the same argument. First we show that A/p^nA is smooth over R/p^nR for all $n \geq 1$. Indeed, since A/R is flat, and $A/pA = \overline{A}$ is smooth over $R/pR = k$, it follows that A/p^nA is smooth ([5, Tag 031L]).

In particular A/p^nA is formally smooth over R/p^nR, so we have the following lifting property:

Suppose C is another R-algebra. Then any R-algebra map $A/p^nA \to C/p^nC$ lifts (not necessarily uniquely) to an R-algebra map $A/p^{n+1}A \to C/p^{n+1}C$.

Now we assume that C is another lift of $\overline{A}$. Denoting the p-adic completions of A and C by $\hat{A}$ and $\hat{C}$, we apply the above lifting property to get a map $\hat{f} : \hat{A} \to \hat{C}$, lifting the identity map mod p. Applying a form of Artin approximation (see [6, 2.4.1]) gives a map $f : A \to C$ which is an isomorphism mod p. Finally, the map f must be an isomorphism. Indeed, A and C are flat R-algebras, so there are isomorphisms $A/pA \cong p^nA/p^{n+1}A$ and $C/pC \cong p^nC/p^{n+1}C$. Therefore f induces isomorphisms $p^nA/p^{n+1}A \cong p^nC/p^{n+1}C$ so $\ker(f) \subset \cap_{n\geq 1} p^nA = \{0\}$. Hence f is injective. Surjectivity is given by the following lemma. □

Lemma 5.2. *Suppose $f : A \to B$ is a map between WCFG R-algebras, which is surjective mod p. Then f is surjective.*

Proof. First note that we may assume that $A = W_n$ for some n (just write A as a quotient of W_n).

For some m, we have a surjective map $f' : A' = A\langle x_{n+1}, \dots x_{n+m}\rangle^\dagger \to B$. Suppose m is minimal. If $m = 0$ we are done, so assume $m > 0$. There is an $a \in A$ such that $x_{n+m} - a$ is in the kernel of $\overline{f}$. Now we have $x_{n+m} - a = a' + pr$ for some $a' \in \ker(f)$ and $r \in A'$. Weierstrass division implies that $A\langle x_{n+1}, \dots x_{n+m-1}\rangle^\dagger \to A'/(a')$ is an isomorphism, so $A\langle x_{n+1}, \dots x_{n+m-1}\rangle^\dagger$ surjects onto B, contradicting minimality of m. □

Finally, we want to show that the map on Monsky-Washnitzer cohomology induced by lifts $f : A \to B$ of a map $\overline{f} : A \to B$ depends only on $\overline{f}$ (i.e. it does not depend on the choice of lifting f, which is not unique).

Theorem 5.3. *Suppose A, B are lifts of two smooth k-algebras $\overline{A}$, $\overline{B}$, and f_0, f_1 are two maps from A to B lifting a single map $\overline{f} : \overline{A} \to \overline{B}$. Then the induced maps*
$$f_{0,*}, f_{1,*} : D^\bullet(A/K) \to D^\bullet(B/K)$$
are homotopic. In other words, for $q \geq 0$ there exist K-linear maps $\delta_q : D^{q+1}(A/K) \to D^q(B/K)$ such that (with $\delta_{-1} = 0$)
$$f_{1,*} - f_{0,*} = d\delta_{q-1} + \delta_q d : D^q(A/K) \to D^q(B/K).$$
In particular, f_0 and f_1 induce the same map $H^i_{MW}(A/K) \to H^i_{MW}(B/K)$.

Proof. We first suppose that there is a map $f : A \to B\langle T\rangle^\dagger$ such that, if $\alpha_0, \alpha_1 : B\langle T\rangle^\dagger \to B$ are defined by sending T to $0, 1$ respectively, then $\alpha_0 \circ f = f_0$, $\alpha_1 \circ f = f_1$. This is an analogue of the topological definition of homotopy. It is sufficient to prove that $\alpha_{0,*}$ and $\alpha_{1,*}$ are homotopic.

Now
$$D^{q+1}(B\langle T\rangle^\dagger/K) = B\langle T\rangle^\dagger \otimes_B D^{q+1}(B/K) \oplus B\langle T\rangle^\dagger dT \otimes_B D^q(B/K)$$

and we define δ_q to be 0 on the first factor in the direct sum, and

$$g \otimes \omega \mapsto \left(\int_0^1 g dT \right) \omega$$

on the second.

Now we just need to show the existence of f. We want to define it by sending a to $(1-T)f_0(a)+Tf_1(a)$. This will not in general be an algebra homomorphism, so instead we approximate it, slightly indirectly. Set $S = pT$ and define

$$h : A \to \hat{B}[[S]]/(S^2 - pS)$$

by

$$a \mapsto (1-T)f_0(a) + Tf_1(a) = f_0(a) + \frac{f_1(a) - f_0(a)}{p} S.$$

You can check that h is an R-algebra homomorphism.

Formal smoothness allows us to lift h to

$$\hat{h} : \hat{A} \to \hat{B}[[S]] \subset \hat{B}\langle T \rangle$$

since $\hat{B}[[S]] = \lim_n \hat{B}[[S]]/(p^n, (S^2 - pS)^n)$. Now Artin approximation gives the desired map $A \to B\langle T \rangle^\dagger$. □

The previous two theorems establish that $H^i_{MW}(A/K)$ depends only on $\overline{A}$, and is functorial in maps of smooth k-algebras $\overline{A} \to \overline{B}$.

An important result (proved by Berthelot long after Monsky and Washnitzer's work) is that the K-vector spaces $H^i_{MW}(A/K)$ are *finite-dimensional*.

6. Frobenius

Recall that k is a finite field of cardinality q. For a smooth k-algebra $\overline{A}$, denote by $F : \overline{A} \to \overline{A}$ the q-power Frobenius map.

Theorem 6.1. *F_* is bijective on $H^i_{MW}(\overline{A}/K)$.*

Proof. We just sketch the proof. Recall that F lifts to an endomorphism of A which we again call F. F induces an isomorphism $A \cong F(A)$. It is possible to define a trace map $S_{A/F(A)} : D^\bullet(A/R) \to D^\bullet(F(A)/R)$.

We define a map ψ by composing $S_{A/F(A)}$ with the inverse of the isomorphism $D^\bullet(A/R) \cong D^\bullet(F(A)/R)$ induced by F.

The composition $\psi \circ F$ induces multiplication by $[A : F(A)] = [\overline{A} : \overline{A}^q] = q^{\dim(\overline{A})}$ on $D^\bullet(A/R)$. This shows that F_* is injective on $H^i_{MW}(\overline{A}/K)$. Another argument involving the Galois closure of $\mathrm{Frac}(A)/\mathrm{Frac}(F(A))$ shows that $S_{A/F(A)}$ is injective on $H^i_{MW}(\overline{A}/K)$, which implies that ψ_* is injective and hence F_* is surjective on $H^i_{MW}(\overline{A}/K)$. □

Finally, we can state the Lefschetz fixed point formula:

Let $\overline{A}$ be a smooth k-algebra which is an integral domain of dimension n. Let $N(\overline{A})$ denote the number of k-algebra maps $\overline{A} \to k$. In other words, this is the number of k-points of $\mathrm{Spec}(\overline{A})$.

Theorem 6.2.

$$N(\overline{A}) = \sum_{i=0}^{n} (-1)^i \mathrm{Tr}\left(q^n F_*^{-1} | H^i(\overline{A}/K) \right)$$

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