Variational features of a finite exact reduction in field theory

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February 8, 2005

Abstract

A simple existence result for a class of semilinear Dirichlet problems is rediscovered by means of variational techniques applied in conjunction with an exact finite dimensional reduction.

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Introduction

We briefly present our program. Consider the following semilinear Dirichlet boundary value problem:

$$\begin{cases} N(u) := -Lu - \lambda u - \varepsilon F(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a piecewise smooth subset of $\mathbb{R}^n$, $u \in H := H_0^1(\Omega)$, L is an elliptic operator on H, $\lambda \in \mathbb{R}$ and $F : H \to H$ is a nonlinear operator.

If $N'(u)$ is symmetric with respect to the L^2 scalar product, ($F'(u)$ actually) Volterra–Vainberg theorem applies, and one can write the energy functional:

$$E(u) = \int_0^1 \langle N(tu), u \rangle_{L^2} \, dt, \tag{2}$$

and obtain a equivalent variational formulation for (1).

Asking for the Lipschitz condition on N, one can apply a reduction technique, devised by Amann, Conley and Zehnder (see [AZ80, CZ86]) and translated in field theory by Franco Cardin (see [Car03]), to the Dirichlet problem (1). This way, the original problem is reduced to a finite dimensional system of algebraic equations. At the same time, the energy functional (2) is reduced to a m variables function $\widetilde{E} : \mathbb{R}^m \to \mathbb{R}$, which critical points correspond exactly to the solutions of (1).

We will introduce some further conditions on F in order to prove $\widetilde{E}$ to be a Generating Function Quasi Quadratic at infinity, *i.e.,* to finitely differ from a non-degenerate quadratic form $\langle Qx, x \rangle$ in C^1 norm.

Proving then that the sublevel sets

$$Q^c = \left\{ x \,\middle|\, \langle Qx, x \rangle \leqslant c \right\}, \qquad \widetilde{E}^c = \left\{ x \,\middle|\, \widetilde{E}(x) \leqslant c \right\},$$

are diffeomorphic provided $c \gg 0$ is suitably large, we could apply the deformation lemma and make correspond a critical point of $\widetilde{E}$ to the critical point (0, of course) of Q. Note that this critical point is not a minimum in general.

Our existence result could be easily obtained by means of Leray–Shauder degree theory (see [**?**]). Nevertheless, we want to emphasize the applicability of the reduction technique also to existence problems. Furthermore, let us remark that this seems the first application in PDE of the theory of generating functions quadratic at infinity, originally developed by Chaperon, Sikorav and Viterbo studying Hamiltonian dynamics. (See [Cha84, Cha91, Sik86, Sik87, Vit90, Thé99])

1 A semilinear Dirichlet problem

As scalar product on H, consider

$$\begin{aligned} \langle \cdot, \cdot \rangle : H \times H &\longrightarrow \mathbb{R}, \\ (u, v) &\longmapsto \langle u, v \rangle := \int_\Omega -Lu \cdot v dx. \end{aligned} \tag{3}$$

As a result, we will take $\|u\| := \langle u, u\rangle$ as the norm on H.

The eigenfunctions associated to the eigenvalues of the elliptic operator L form a basis for the space H,

$$\begin{aligned} -L\hat{u}_j &= \lambda_j \hat{u}_j, \\ 0 &= \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots, \\ \|\hat{u}_j\| &= 1. \end{aligned}$$

For every $v \in H$ we can write

$$v = \sum_{j=1}^{\infty} \langle v, \hat{u}_j\rangle \hat{u}_j = \sum_{j=1}^{\infty} v_j \hat{u}_j.$$

We are able now to define the Green operator of L, $g : H \to H$, $g = (-L)^{-1}$,

$$gv = g\left(\sum_{j=1}^{\infty} v_j \hat{u}_j\right) = \sum_{j=1}^{\infty} \frac{v_j}{\lambda_j} \hat{u}_j.$$

It is clear that

$$-Lgv = g(-L)v = v, \qquad \forall v \in H.$$

For every fixed $m \in \mathbb{N}$, we can consider the following decomposition of H:

$$\begin{gathered} H = \mathbb{P}_m H \oplus \mathbb{Q}_m H, \\ v = \mathbb{P}_m v + \mathbb{Q}_m v = \sum_{j=1}^{m} v_j \hat{u}_j + \sum_{j=m+1}^{\infty} v_j \hat{u}_j. \end{gathered}$$

We will briefly write

$$v = \mu + \eta = \sum_{j=1}^{m} \mu_j \hat{u}_j + \sum_{j=m+1}^{\infty} \eta_j \hat{u}_j$$

we will refer to $\mu \in \mathbb{P}_m H$ as to the *finite* head of v, whereas to $\eta \in \mathbb{Q}_m H$ as to the *infinite* tail of v.

If we substitute the relation $u = gv$ the Dirichlet problem 1, we obtain

$$\begin{gathered} v - \lambda g v = \varepsilon F(gv), \\ v = (\mathbb{I} - \lambda g)^{-1} \varepsilon F(gv), \end{gathered}$$

which solutions, in other words, are just the fixed points of the map:

$$\begin{aligned} H &\longrightarrow H, \\ v &\longmapsto (\mathbb{I}-\lambda g)^{-1}F(gv). \end{aligned}$$

Note that we can write easily the map $(\mathbb{I}-\lambda g)^{-1}$ as

$$(\mathbb{I}-\lambda g)^{-1}v = (\mathbb{I}-\lambda g)^{-1}\left(\{v_j\}_{j=1}^{\infty}\right) = \left\{\frac{\lambda_j}{\lambda_j-\lambda}v_j\right\}_{j=1}^{\infty},$$

which of course is well defined when λ is different from every λ_j.

The application of the cut-off decomposition splits the original problem into a finite and an infinite part:

$$v = \mathbb{P}_m v + \mathbb{Q}_m v = \mu + \eta, \tag{4}$$

$$\mu = \mathbb{P}_m(\mathbb{I}-\lambda g)^{-1}\varepsilon F(g(\mu+\eta)), \qquad \text{“finite”} \tag{5}$$

$$\eta = \mathbb{Q}_m(\mathbb{I}-\lambda g)^{-1}\varepsilon F(g(\mu+\eta)). \qquad \text{“infinite”} \tag{6}$$

These equations can also be equivalently rewritten as follows,

$$\mathbb{P}_m N(g(\mu+\eta)) = 0, \qquad \text{“finite”} \tag{7}$$

$$\mathbb{Q}_m N(g(\mu+\eta)) = 0. \qquad \text{“infinite”} \tag{8}$$

Maybe, providing some appropriate hypotheses on F, it would be possible to uniquely solve for η with respect to μ in the infinite part of the equation. In fact, we can prove that

PROPOSITION 1. *If $F : H \to H$ is Lipschitz, for every fixed $\varepsilon > 0$, there exists $m \in \mathbb{N}$, such that*

$$\eta \longmapsto (\mathbb{I}-\lambda g)^{-1}\varepsilon \mathbb{Q}_m F(g(\mu+\eta)), \tag{9}$$

is a contraction, for every fixed $\mu \in \mathbb{P}_m H$.

Proof. First, we trivially find that $\eta \mapsto g(\mu+\eta)$ is a Lipschitz map with constant $\frac{1}{\lambda_{m+1}}$. Next, we can deduce that $(\mathbb{I}-\lambda g)^{-1}$ is bounded, indeed

$$\left\|(\mathbb{I}-\lambda g)^{-1}v\right\| = \left\|\left(\frac{\lambda_j}{\lambda_j-\lambda}v_j\right)\right\| \leqslant \sup_{j\in N}\left|\frac{\lambda_j}{\lambda_j-\lambda}\right| \|v\|,$$

and clearly $\sup\left|\frac{\lambda_j}{\lambda_j-\lambda}\right| := C_1 < +\infty$, because the sequence $\frac{\lambda_j}{\lambda_j-\lambda} \to 1$. By assumption, F is Lipschitz, so let us denote by C_2 the constant such that

$$\|F(u_1) - F(u_2)\| \leqslant C_2 \|u_1 - u_2\|.$$

Finally, the map (9) is Lipschitz:

$$\left\|(\mathbb{I}-\lambda g)^{-1}\varepsilon\mathbb{Q}_m F(g(\mu+\eta_1)) - (\mathbb{I}-\lambda g)^{-1}\varepsilon\mathbb{Q}_m F(g(\mu+\eta_2))\right\| \leqslant \\ \leqslant \frac{\varepsilon C_1 C_2}{\lambda_{m+1}}\left\|\eta_1-\eta_2\right\|.$$

We can conclude that for every $\varepsilon > 0$ there exists a sufficiently large m, such that the Lipschitz constant $\frac{\varepsilon C_1 C_2}{\lambda_{m+1}}$ results in something smaller than 1, *i.e.,* the map (9) is contractive as claimed. □

Let us denote by $\tilde{\eta}(m,\mu)$, or simply $\tilde{\eta}(\mu)$ if there is no ambiguity, the unique fixed point of the map (9). Substituting it into equation (5), we obtain

$$\mu = (\mathbb{I}-\lambda g)^{-1}\varepsilon\mathbb{P}_m F(g(\mu+\tilde{\eta}(m,\mu))), \tag{10}$$

which indeterminate is merely $\mu = (\mu_1,\ldots,\mu_m) \in \mathbb{R}^m$. The problem has reached a completely finite (say, algebraic) formulation, though passing through a contraction in the infinite dimensional space $\mathbb{Q}_m H$.

To every solution $\tilde{\mu}$ of this last equation corresponds a solution of the original Dirichlet Problem (1) by means of the formula:

$$\tilde{u} = g(\tilde{\mu}+\tilde{\eta}(m,\tilde{\mu})),$$

and clearly viceversa,

$$\tilde{\mu} = \mathbb{P}_m(-L)\tilde{u},$$

to every solution of (1) corresponds a solution of (10).

2 Infinite dimensional and reduced finite dimensional variational formulation.

If we assume the Gateaux derivative $N'(u)$ of the non linear operator considered (1) to be *symmetric* with respect to the L^2-scalar product, *i.e.,*

$$(u,v) := \int_\Omega uv\,dx,$$
$$\big(N'(u)h,k\big) = \big(N'(u)k,h\big), \qquad \forall h,k \in H,$$

the Volterra–Vainberg theorem admits us to write a variational principle,

$$\begin{aligned} E : H &\longrightarrow \mathbb{R}, \\ u \longmapsto E(u) &:= \int_{t=0}^{t=1} (N(tu),u)\,dt, \end{aligned} \tag{11}$$

which is equivalent to the original Dirichlet Problem, More precisely, THEOREM 1. *Every critical point of E is a solution of (1), and viceversa,* i.e.,

$$dE(u)h = 0 \quad \forall h \in H \qquad \Leftrightarrow \qquad \begin{cases} N(u) = 0, \\ u\Big|_{\partial\Omega} = 0. \end{cases} \tag{12}$$

Proof.

$$dE[u] \cdot h = \frac{d}{d\lambda} E[u + \lambda h]\Big|_{\lambda=0} = \int_0^1 \big(N'(tu)th, u\big) + (N(tu), h)\, dt =$$

$$= \int_0^1 \big(N'(tu)u, th\big) + (N(tu), h)\, dt = \int_0^1 \frac{d}{dt}\,(N(tu), th)\, dt = (N(u), h)\,.$$

So $dE[u] = 0$ if and only if $N(u) = 0$ as claimed. ☐

The finite parameters reduction of the preceeding section can be applied to the functional $E(u)$ to obtain a finite parameters variational principle, which is equivalent to the infinite dimensional one, *i.e.,* every critical point of E is also a critical point for

$$\begin{aligned} \widetilde{E} &: \mathbb{R}^m \longrightarrow \mathbb{R}, \\ \widetilde{E}(\mu) &:= E(g(\mu + \tilde{\eta}(m, \mu))). \end{aligned} \tag{13}$$

We will show how this works proving the equivalence between the finite parameters variational principle $\widetilde{E}(\mu)$ and the original Dirichlet problem (1).

THEOREM 2. *To every critical point $\tilde{\mu}$ of $\widetilde{E}(\mu)$ corresponds a solution $\tilde{u} = g(\mu + \tilde{\eta}(\mu))$ of the problem (1) and viceversa to every solution $\tilde{u}$ of (1) corresponds a critical point $\tilde{\mu} = \mathbb{P}_m(-Lu)$ of the variational principle $\widetilde{E}(\mu)$.*

Proof. Almost directly,

$$\begin{aligned} d\widetilde{E}(\mu) &= dE[u] \cdot \frac{d}{d\mu}\,(g(\mu + \tilde{\eta}(\mu)))\, d\mu = \\ &= \left(N(u)\Big|_{u=g(\mu+\tilde{\eta}(\mu))}, g(d\mu) + g(\tilde{\eta}'(\mu)d\mu) \right) = \\ &= \left(\mathbb{P}_m N(u) + \underbrace{\mathbb{Q}_m N(u)}_{=0 \text{ by } (8)}\Big|_{u=g(\mu+\tilde{\eta}(\mu))}, \underbrace{g(d\mu)}_{\in \mathbb{P}_m H} + \underbrace{g(\tilde{\eta}'(\mu)d\mu)}_{\in \mathbb{Q}_m H} \right) = \\ &= (\mathbb{P}_m N(g(\mu + \tilde{\eta}(\mu))), g(d\mu))\,. \end{aligned}$$

Being g acting as a diagonal isomorphism of $\mathbb{R}^m$ into itself, we can conclude that,

$$d\widetilde{E}(\mu) = 0 \qquad \Leftrightarrow \qquad \mathbb{P}_m N(g(\mu + \tilde{\eta}(\mu))) = 0,$$

i.e., the reduced variational principle is equivalent to the Dirichlet problem. □

We used in the last proof the fact that $\mathbb{P}_m H$ and $\mathbb{Q}_m H$ are still orthogonal also with respect to the L^2 scalar product, indeed:

LEMMA 1. $(\hat{u}_i, \hat{u}_j) = 0$ *whenever* $i \neq j$.

Proof. Clearly,

$$(\hat{u}_i, \hat{u}_j) = \frac{1}{\lambda_i}(-L\hat{u}_i, \hat{u}_j) = \frac{1}{\lambda_i}\langle \hat{u}_i, \hat{u}_j \rangle = 0.$$

□

This is the reason of the choices of the scalar product $\langle \cdot, \cdot \rangle$ to perform the spectral decomposition and the L^2 scalar product $(\cdot, \cdot)$ for the variational principle.

3 *Quasi-quadratic* functions and their properties

3.1 The deformation Lemma

Cohomology on a differentiable manifold M can be defined as follows

$$H^k(M) := \frac{\{\text{closed } k-\text{forms on } \ M\}}{\{\text{exact } k-\text{forms on } \ M\}},$$

while, if $N \subseteq M$ is a submanifold (also with boundary), the *relative cohomology* of M on N is defined as,

$$H^k(M, N) := \{\alpha \text{ closed } k-\text{form on } \ M, \text{ exact on } \ N\}.$$

In the theory of calculus of variations, it is often the case that the manifold in study is not compact. In such a case it may be necessary to replace compactness by some weaker condition in order to avoid the loss of some relevant features. Very often, compactness is replaced by the Palais-Smale condition on the functional in study.

DEFINITION 3.1 (Palais–Smale). *We will say $f : M \to \mathbb{R}$ is Palais–Smale, if every sequence $\{x_i\}$ such that $|f(x_i)|$ is bounded and $|f'(x_i)| \to 0$, admits a converging subsequence.*

When this condition is fulfilled, the critical levels results in compact, even when the domain M of f is not. Here follow some elementary examples of non P–S functions.

EXAMPLE. *1. $e^{-\frac{x^2}{2}}$ is not P-S. Consider the sequence $n \mapsto x_n = n$.*

2. $\sin x$ is not P-S. Consider $n \mapsto x_n = \frac{\pi}{2} + 2n\pi$.

We will prove that it is possible to associate a critical level $\gamma(\alpha, f)$ to every non zero cohomology class, $\alpha \neq 0$, *i.e.,*

$$\exists x, \quad f(x) = \gamma(\alpha, f), \qquad df(x) = 0.$$

DEFINITION 3.2. *Let*

$$M^\lambda := \{x \in M : f(x) \leqslant \lambda\}.$$

denote the sublevel set*.*

When there will be no ambiguity, the sublevel sets of $f : M \to \mathbb{R}$ will be denoted also by f^c, instead of M^c.

LEMMA 2 (Deformation). *Let $f : M \to \mathbb{R}$ be a Palais–Smale function. Let $\alpha \in H^\star(M)$, $\alpha \neq 0$. Then*

$$\gamma(\alpha, f) := \inf \left\{ \lambda : \alpha\Big|_{M^\lambda} \neq 0 \quad in \quad H^\star(M^\lambda) \right\}, \tag{14}$$

is a critical level for f.

Proof. If c is not a critical level, by Palais–Smale condition, there exists $\varepsilon > 0$ such that $f^{-1}([c - \varepsilon, c + \varepsilon])$ does not contain critical points. It is possible to deform $M^{c+\varepsilon}$ in $M^{c-\varepsilon}$, by means of the flow of the vector field

$$X(x) := -\frac{\nabla f(x)}{|\nabla f(x)|^2}, \qquad \text{per} \quad x \in M^{c+\varepsilon} \setminus M^{c-\varepsilon},$$

which could be regularly extended to all M to the zero vector field outside of a neighborhood of $M^{c+\varepsilon} \setminus M^{c-\varepsilon}$. One can easily check that if φ_t is the flow of $X(x)$, then

$$f(\varphi_t(x)) = f(x) + t.$$

By means of this deformation, α and $(\varphi_{2\varepsilon})^\star \alpha$ are homotopically equivalent, *i.e.,* are in the same class, thus it cannot be $\alpha\Big|_{M^{c+\varepsilon}} \neq 0$ and $\alpha\Big|_{M^{c-\varepsilon}} = 0$. □

Note that different cohomology classes could correspond to the same critical level.

3.2 Quasi–quadratic functions

At first, there were the Generating Functions Quadratic at Infinity or *FGQI*[1], which have been defined and developed by Marc Chaperon, Jean-Claude Sikorav and Claude Viterbo, in their works on hamiltonian dynamical systems. (see [Cha84, Cha91, Sik86, Sik87, Thé99, Vit90]).

Definition 3.3. *FGQI are differentiable functions, $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that $F(x) = \langle Qx, x\rangle$, for all $x \notin C$, for a prescribed $C \Subset \mathbb{R}^n$ and a non degenerate quadratic form Q.*

The main features of *FGQI* are the Palais-Smale property and, a consequence, the equivalence at infinity of the cohomological groups of F and Q, *i.e.,*

$$H^\star(F^{-c}, F^c) \cong H^\star(Q^{-c}, Q^c),$$

for a sufficiently large $c > 0$.

Claude Viterbo and David Théret defined also a slighter general class of functions: the *Generating Functions Quasi-Quadratic at Infinity*, *FGQQI* or *FGQ*^{2}*I*. For this class of functions still hold the Palais-Smale property and the cohomological groups equivalence. In fact, as proven in [Thé99], the sub-level sets $S^c := \{x : S(x) < c\}$ of a FGQQI function S are diffeomorphic to the sub-level sets of a suitable *FGQI* function F. As a result, it is clear that for $c > 0$ sufficiently large,

$$S^{\pm c} \cong F^{\pm c} = Q^{\pm c}.$$

In this section, we will directly prove, with great detail, the equivalence:

$$S^{\pm c} \cong Q^{\pm c}.$$

Definition 3.4. *We say that a function $S : \mathbb{R}^n \to \mathbb{R}$ is* quasi-quadratic *if there exists a non-degenerate quadratic form $\langle Qu, u\rangle$ and a constant $K > 0$ such that*

$$\|S(u) - \langle Qu, u\rangle\|_{C^1} = \|S(u) - \langle Qu, u\rangle\| + \left\|S'(u) - 2Qu\right\| \leqslant K$$

Proposition 2. *All the critical points of S are in a compact neighborhood of the origin.*

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[1] in french, *fonctions génératrices quadratiques à l'infiní*

Proof. Indeed,

$$\frac{\partial S}{\partial \mu}(\bar{\mu}) = 0, \quad \Rightarrow \quad |Q\bar{\mu}| \leqslant K, \quad \Rightarrow \quad \bar{\mu} \in B\left(0, \frac{K}{\min \operatorname{Spec} Q}\right).$$

□

PROPOSITION 3. *FGQ^2I are Palais-Smale.*

Proof. Consider a P-S sequence $\{i \mapsto x_i\}$, *i.e.,* $S(x_i)$ bounded, $S'(x_i) \to 0$. For i sufficiently large we have $|S'(x_i)| < K$, and by quasi–quadraticity, $|2Ax_i| < 2K$, then $x_i \in B\left(0, \frac{K}{\min \operatorname{Spec} A}\right)$. From a certain i onwards, the sequence is confined in a compact set, then there exists a converging subsequence. □

THEOREM 3. *For $c > 0$ sufficiently large,*

$$S^{\pm c} \overset{\text{diffeo}}{\cong} Q^{\pm c}.$$

Proof. Indeed,

$$\cdots \subsetneqq S^c \subsetneqq Q^{c+K} \subsetneqq S^{c+2K} \subsetneqq Q^{c+3K} \subsetneqq \cdots$$

and certainly if c is large enough to overtake every critical value of S,

$$S^c \cong S^{c+K} \cong \ldots \cong S^{c+M} \qquad \text{and} \qquad Q^c \cong Q^{c+K} \cong \ldots \cong Q^{c+M}.$$

So we only need to prove

$$S^c \cong Q^{c+M}$$

for some $c, M > 0$.

It would suffice to find a differentiable function f such that $f^c = S^c$ e $f^{c+M} = Q^{c+M}$. For this purpose we need a differentiable Urysohn function $0 \leqslant \varphi \leqslant 1$ such that

$$\varphi(x) = \begin{cases} 0 \text{ if } S(x) \leqslant c, \\ 1 \text{ if } \ c + M \leqslant Q(x). \end{cases}$$

In such a case we could set

$$f = (1 - \varphi)S + \varphi Q,$$

nevertheless we must exclude f to have intermediate critical points between c and $c + M$.

If we try to set up an estimate we find

$$\begin{aligned}|f'| &= \big|\big[(1-\varphi)S' + \varphi Q'\big] - \big[\varphi'(Q-S)\big]\big| \geqslant \\ &\geqslant \underbrace{\big|(1-\varphi)S' + \varphi Q'\big|}_{\cong|Q'|\pm K} - \underbrace{|\varphi'|}_{?}\,\underbrace{|Q-S|}_{\pm K}.\end{aligned}$$

This derivative is certainly non vanishing if $|\varphi'| \ll \frac{|Q'|}{K}$.

In the following lemmas, we will explicitly build a Urysohn function φ and also prove that $|Q'|$ increases proportionally to $\sqrt{c}$. $\square$

Being $S^c \subsetneqq Q^{c+K}$, and assuming $M \gg K$, we can limit ourselves to:

Lemma 3. *There exists a continuous Urysohn function $\tilde{\varphi}$ which is 0 on S^c and 1 on the complement of Q^{c+M}.*

Proof. By little manipulation of Q we get:

$$\tilde{\varphi}(x) = \begin{cases} 0 & \text{if } Q(x) \leqslant c, \\ \frac{Q(x)-c}{M} & \text{if } c \leqslant Q(x) \leqslant c+M, \\ 1 & \text{if } c+M \leqslant Q(x). \end{cases}$$

Notice that, where the derivative exists, $|\varphi'| = \frac{|Q'(x)|}{M} \ll \frac{|Q'(x)|}{K}$. $\square$

The differentiable version of $\tilde{\varphi}$ is obtained by this slightly more general result.

Lemma 4. *Let $Q(x)$ a differentiable function without critical values between c and $c+M$. Then for every $\varepsilon > 0$ there always exists a differentiable function φ such that*

$$\varphi(x) = \begin{cases} 0 & \textit{if } Q(x) \leqslant c, \\ 1 & \textit{if } c+M \leqslant Q(x), \end{cases}$$

and furthermore that $|\varphi'(x)| \leqslant \frac{|Q'(x)|}{M-\varepsilon}$ for every x.

Proof. We will manipulate the behavior of Q by means of a composition with a suitable $g : \mathbb{R} \to \mathbb{R}$, in order to maintain the same (sub–)level sets, to smoothen the angles and at the same time to dominate the derivative. Indeed, the gradient of $\varphi := g \circ Q$ describes the same trajectories of the gradient of Q (if $g' > 0$). Furthermore, the same sub–level sets of $\tilde{\varphi}$ above are generated.

All the conditions on φ are fulfilled when g behaves as follows:

$$g(t) = \begin{cases} g_1 \equiv 0 & \text{if } \ t \leqslant c, \\ g_3 \equiv \frac{t-c-\varepsilon}{M-2\varepsilon} & \text{if } c+2\varepsilon < t < c+M-2\varepsilon, \\ g_5 \equiv 1 & \text{if } c+M \leqslant t. \end{cases}$$

These are the conditions on g_2, those for g_4 are analogous.

$$\begin{cases} g_2(c) = 0, \\ g_2(c+2\varepsilon) = \frac{\varepsilon}{M-2\varepsilon}, \\ g_2'(c) = 0, \\ g_2'(c+2\varepsilon) = \frac{1}{M-2\varepsilon}. \end{cases} \tag{15}$$

We easily find:

$$g_2(t) := \frac{1}{4\varepsilon(M-2\varepsilon)}(t-c)^2,$$

$$g_4 := c+M - \frac{1}{4\varepsilon(M-2\varepsilon)}(t-c-M)^2.$$

The computation of the derivatives gives:

$$\varphi'(x) = \begin{cases} g_1' \cdot Q' \equiv 0 & \text{if } \ Q(x) \leqslant c, \\ g_2' \cdot Q' = \frac{Q-c}{2\varepsilon(M-2\varepsilon)} \cdot Q' & \text{if } c < Q(x) < c+2\varepsilon, \\ g_3' \cdot Q' = \frac{Q'}{M-2\varepsilon} & \text{if } c+2\varepsilon < Q(x) < c+M-2\varepsilon, \\ g_4' \cdot Q' = \frac{c+M-Q}{2\varepsilon(M-2\varepsilon)} \cdot Q' & \text{if } c+M-2\varepsilon < Q(x) < c+M, \\ g_5' \cdot Q' \equiv 0 & \text{if } c+M \leqslant Q(x). \end{cases}$$

thus

$$\left|\varphi'(x)\right| = \left|g'(Q(x))\right| \cdot \left|Q'(x)\right| \leqslant \frac{1}{(M-2\varepsilon)} \cdot \left|Q'(x)\right|,$$

as wanted. □

Proof of Theorem 3 completed. Finally, we show that $|Q'(x)|$ becomes arbitrarily large, for sufficiently large c.

In fact, if the quadratic form is undefined, $|Q'|$ is not bounded from above in a level set $Q = c$. (e.g. if $Q(x_1, x_2) = -x_1^2 + x_2^2 = c$, then $\forall M > 0, Q(M, \sqrt{c+M^2}) = c$ and $|Q'| \geqslant M$).

On the other hand, $|Q'|$ assumes its minimum in a compact subset of the level set (if $|x| > M$, then $|Q'(x)| > \min|\operatorname{Spec} Q| \cdot M$). It is easy to check that the critical values are assumed on the eigenvectors of Q, thus the minimum of $|Q'|$ is reached by an eigenvector of the smallest in magnitude eigenvalue, say $\min|Q'| = |Q'(w_k)| = |\lambda_k w_k|$. But if $Q(w_k) = \lambda_k |v_k|^2 = c$, then $\min|Q'| = \sqrt{c \cdot \lambda_k}$, *i.e.*, $\min_{Q=c}|Q'| \propto \sqrt{c}$. □

3.3 Existence of a critical point for S

Let us lastly emphasize the existence of a critical point. Trivially, Q has a Morse critical point (0!) with Morse index equal to $q = \operatorname{sgn} Q$. It is not difficult to prove that

$$H^k(Q^c, Q^{-c}) = \begin{cases} \mathbb{R}, & \text{if } k = q, \\ 0, & \text{if } k \neq q. \end{cases} \tag{16}$$

(see, *e.g.*, , [God69], page 188.) Thus, if c is sufficiently large,

$$H^\star(S^c, S^{-c}) \cong H^\star(Q^c, Q^{-c}) \neq 0.$$

So, there must be at least a critical level for S between $-c$ and c. Otherwise, by deformation lemma, we would obtain $S^{-c} \cong S^c$, and thus $H^\star(S^{-c}, S^c) = 0$, a contradiction.

4 Quasi-quadratic reduced energy functional

In this section we will prove that under suitable conditions, the reduced variational principle associated to our Dirichlet BPV is a FGQQI, so there will exist at least one solution.

Consider the nonlinearly perturbed elliptic operator

$$N(u) = -Lu - \lambda u - \varepsilon F(u).$$

Consider $u \in H = H_0^1(\Omega)$ and the following scalar products

$$(u, v) := \int_\Omega uv dx, \qquad \langle u, v \rangle := (-Lu, v) = \int_\Omega (-Lu) \cdot v dx.$$

We recall the variational formulation (Volterra-Vainberg) associated to our problem (see section 2):

$$\begin{aligned} E(u) &= \int_0^1 (N(tu), u)\, dt = \\ &= -\left(\int_0^1 (Ltu, u)\, dt + \lambda \int_0^1 (tu, u)\, dt \right) - \int_0^1 (\varepsilon F(tu), u)\, dt = \\ &\qquad\qquad =: E^{L+\lambda}(u) + E^F(u). \end{aligned}$$

We have seen that u can be represented by means of the eigenfunctions of the elliptic operator L:

$$u = \sum_{j=1}^\infty \hat{u}_j \langle u, u_j \rangle = g(v) = \sum_{j=1}^\infty \frac{1}{\lambda_j} v_j \hat{u}_j = \sum_{j=1}^m \frac{1}{\lambda_j} \mu_j \hat{u}_j + \sum_{j=m+1}^\infty \frac{1}{\lambda_j} \eta_j \hat{u}_j.$$

When the Dirichlet problem collapses to a finite dimensional problem by means of the fixed point technique, η (and by consequence u) becomes a function of the first m parameters μ_j:

$$u(m,\mu) = g(v) = g(\mu + \tilde{\eta}(m,\mu))$$

A reduced functional can readily be obtained,

$$\widetilde{E}(\mu) := E(u(m,\mu)),$$

and $\widetilde{E}$ is equivalent to E in the sense explained in section 2. The reduced functional splits analogously to E,

$$\widetilde{E}(\mu) = E(u(m,\mu)) = E^{L+\lambda}(u(\mu)) + E^{F}(u(\mu)) = \widetilde{E}^{L+\lambda}(\mu) + \widetilde{E}^{F}(\mu),$$

and, furthermore, a quadratic form in the μ_j's can be isolated:

$$\begin{aligned} E^{L+\lambda}(u) = E^{L+\lambda}(gv) &= \sum_{j=1}^{\infty} \frac{1}{\lambda_j} v_j^2 - \lambda \left(\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} v_j^2 \right) = \\ &= \sum_{j=1}^{m} \frac{\mu_j^2}{\lambda_j} + \sum_{j=m+1}^{\infty} \frac{\tilde{\eta}_j^2(\mu)}{\lambda_j} - \lambda \left(\sum_{j=1}^{m} \frac{\mu_j^2}{\lambda_j^2} + \sum_{j=m+1}^{\infty} \frac{\tilde{\eta}_j^2(\mu)}{\lambda_j^2} \right) = \\ &= \sum_{j=1}^{m} \frac{\lambda_j - \lambda}{\lambda_j^2} \mu_j^2 + \sum_{j=m+1}^{\infty} \frac{\lambda_j - \lambda}{\lambda_j^2} \tilde{\eta}_j^2(\mu). \widetilde{E}_q^{L+\lambda}(\mu) + \widetilde{E}_{nq}^{L+\lambda}(\mu). \end{aligned}$$

Thus we can set:

$$\begin{aligned} \widetilde{E}_q^{L+\lambda}(\mu) &:= \textstyle\sum_{j=1}^{m} \frac{\lambda_j - \lambda}{\lambda_j^2} \mu_j^2 \qquad (=: \langle Q\mu, \mu \rangle), \\ \widetilde{E}_{nq}^{L+\lambda}(\mu) &:= \textstyle\sum_{j=m+1}^{\infty} \frac{\lambda_j - \lambda}{\lambda_j^2} \tilde{\eta}_j^2(\mu). \end{aligned}$$

In the general case, there is no reason for $\widetilde{E}_{nq}^{L+\lambda}(\mu)$ and $\widetilde{E}^{F}(\mu)$ to be quadratic.

It is clear that $\widetilde{E}_q^{L+\lambda}(\mu)$ is a non degenerate quadratic form. In what follows, we will show that under some boundedness hypothesis on F, we will obtain $\widetilde{E}_{nq}^{L+\lambda}(\mu)$ and $\widetilde{E}^{F}(\mu)$ to be bounded together with their first derivatives. In such a case, the reduced functional will result in a quasi quadratic function, and we will obtain the existence of at least one critical point, on the basis of the topological considerations exposed in the previous section.

THEOREM 4. *Assume that the nonlinear functional $F : H \to H$ is a Nemitski operator, $F(u) = f \circ u$, associated to $f : \mathbb{R} \longrightarrow \mathbb{R}$. We will* *assume f, its derivative f' and one of its primitives $\bar{f}$ being bounded,* i.e.,

$$\exists K, C > 0, \forall s \in \mathbb{R}, \qquad |f(s)|, \left|\bar{f}(s)\right| \leqslant K, \qquad \left|f'\right| \leqslant C.$$

Then, the reduced functional $\widetilde{E}(\mu)$ is quasi-*quadratic in μ,* i.e.,

$$\left\|\widetilde{E}(\mu) - \langle Q\mu, \mu\rangle\right\|_{C^1} = \left\|\widetilde{E}(\mu) - \widetilde{E}_q^{L+\lambda}(\mu)\right\|_{C^1} =$$
$$= \left\|\widetilde{E}_{nq}^{L+\lambda}(\mu) + \widetilde{E}^F(\mu)\right\|_{C^1} \leqslant \left(\begin{smallmatrix}\textit{Something}\\ \textit{depending}\\ \textit{only on}\end{smallmatrix}\right)(\varepsilon, C, m, K, \mathrm{meas}(\Omega), \lambda_1, \lambda_{m+1}),$$

where

$$Q := \begin{pmatrix} \frac{\lambda-\lambda_1}{\lambda_1^2} & & & \mathbb{O} \\ & \frac{\lambda-\lambda_2}{\lambda_2^2} & & \\ & & \ddots & \\ \mathbb{O} & & & \frac{\lambda-\lambda_m}{\lambda_m^2} \end{pmatrix}.$$

As a result, the problem (1) admits at least a solution.

Example. *An example of such an f could be* $\sin(x)$ *for instance, or an arbitrary $f \in C_0^\infty(\mathbb{R})$, or $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that* $\sup|f'| < +\infty$.

Proof of Theorem 4 . The proof is split in several lemmas. First, we show that $\widetilde{E}_{nq}^{L+\lambda}(\mu)$ and $\widetilde{E}^F(\mu)$ are bounded.

Lemma 5.

$$\left|\widetilde{E}^F(\mu)\right| \leqslant \varepsilon K \ \mathrm{meas}(\Omega).$$

Proof. Whenever $u(x) \neq 0$,

$$\int_0^1 f(tu(x))dt = \frac{1}{u(x)} \int_0^{u(x)} f(\tau)\, d\tau,$$

thus, denoting by $\Omega' = \left\{x \in \Omega \middle| \ u(x) \neq 0\right\}$,

$$\left|\widetilde{E}^F\right| = \left|\left(\int_0^1 \varepsilon F(tu)dt, u\right)\right| = \left|\int_{\Omega'} \varepsilon \left(\frac{1}{u(x)} \int_0^{u(x)} f(\tau)d\tau \cdot u\right) dx\right| =$$
$$= \varepsilon \left|\int_{\Omega'} \left(\int_0^{u(x)} f(\tau)d\tau\right) dx\right| = \varepsilon \left|\int_{\Omega'} \bar{f}(u(x))dx\right| \leqslant$$
$$\leqslant \varepsilon K \ \mathrm{meas}(\Omega).$$

□

Lemma 6.
$$\left|\widetilde{E}^{L+\lambda}_{nq}(\mu)\right| \leqslant \varepsilon^2 K^2 \frac{\lambda_{m+1}}{(\lambda_{m+1}-\lambda)^2}.$$

Proof.
$$\widetilde{E}^{L+\lambda}_{nq}(\mu) = \sum_{j=m+1}^{\infty} \frac{\lambda_j - \lambda}{\lambda_j^2} \tilde{\eta}_j(m,\mu)^2$$

Being λ fixed, while m can be chosen arbitrarily large, it is non restrictive to suppose $\lambda \ll \lambda_j$, $\forall j > m$.

$$\left|\widetilde{E}^{L+\lambda}_{nq}(\mu)\right| \leqslant \frac{1}{\lambda_{m+1}} \sum_{j=m+1}^{\infty} \underbrace{\left|\frac{\lambda_j - \lambda}{\lambda_j}\right|}_{<1} |\tilde{\eta}_j(m,\mu)|^2 \leqslant \frac{1}{\lambda_{m+1}} \left\|\tilde{\eta}\right\|^2 .$$

On the other hand, being $\tilde{\eta}$ obtained by means of the fixed point technique, it satisfies

$$\tilde{\eta} = (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F g(\mu + \tilde{\eta}).$$

We prove that $\left\|(\mathbb{I} - \lambda g)^{-1}\right\| < \frac{\lambda_{m+1}}{\lambda_{m+1}-\lambda}$. Indeed,

$$(\mathbb{I} - \lambda g)x = \left\{\left(\mathbb{I} - \frac{\lambda}{\lambda_j} x_j\right)\right\}_{j=m+1}^{\infty}, \qquad \forall x \in \mathbb{Q}_m H.,$$
$$(\mathbb{I} - \lambda g)^{-1}x = \left\{\left(1 - \frac{\lambda}{\lambda_j}\right)^{-1} x_j\right\}_{j=m+1}^{\infty} = \left\{\frac{\lambda_j}{\lambda_j - \lambda} x_j\right\}_{j=m+1}^{\infty},$$

and if we suppose $\lambda \ll \lambda_m \leqslant \lambda_j$,

$$\frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \geqslant \frac{\lambda_j}{\lambda_j - \lambda} \xrightarrow[j\to\infty]{} 1.$$

Thus,

$$\|\tilde{\eta}\| \leqslant \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \varepsilon \left\|F(g(\mu + \tilde{\eta}))\right\| \leqslant \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \varepsilon K,$$

as claimed. □

Now it is necessary to check if also the difference between the first derivatives of $\widetilde{E}$ and the ones of the quadratic form $\widetilde{E}^{L+\lambda}_q$ is bounded by a constant.

First we look for an estimate for $\tilde{\eta}'(\mu)$ and $u'(\mu)$. From the fixed point equation defining $\tilde{\eta}$ we have:

$$\frac{\partial \tilde{\eta}}{\partial \mu_h} = (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F'(u(\mu)) \cdot \left(g\left(\frac{\partial \mu}{\partial \mu_h} + \frac{\partial \tilde{\eta}}{\partial \mu_h}\right)\right), \qquad h = 1, \ldots, m.$$

$$\begin{aligned}\left\|\frac{\partial\tilde\eta}{\partial\mu_h}\right\| &= \left\|(\mathbb{I}-\lambda g)^{-1}\varepsilon\mathbb{Q}_m F'(u(\mu))\cdot\left(g\left(\frac{\partial\mu}{\partial\mu_h}+\frac{\partial\tilde\eta}{\partial\mu_h}\right)\right)\right\| \leqslant \\ &\leqslant \left\|(\mathbb{I}-\lambda g)^{-1}\right\|\varepsilon\left\|F'\right\|\cdot\left(\frac{1}{\lambda_h}+\frac{1}{\lambda_{m+1}}\left\|\frac{\partial\tilde\eta}{\partial\mu_h}\right\|\right)\leqslant \\ &\leqslant \varepsilon C\frac{\lambda_{m+1}}{\lambda_{m+1}-\lambda}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_{m+1}}\left\|\frac{\partial\tilde\eta}{\partial\mu_h}\right\|\right),\end{aligned}$$

thus,

$$\left\|\frac{\partial\tilde\eta}{\partial\mu_h}\right\|\left(1-\frac{\varepsilon C}{\lambda_{m+1}-\lambda}\right)\leqslant\frac{\varepsilon C}{\lambda_1}\frac{\lambda_{m+1}}{\lambda_{m+1}-\lambda},$$

$$\left\|\tilde\eta'\right\|\leqslant\frac{\varepsilon C}{\lambda_1}\cdot\frac{\lambda_{m+1}}{\lambda_{m+1}-\lambda}\cdot\frac{1}{1-\varepsilon C\frac{1}{\lambda_{m+1}-\lambda}}=\frac{\varepsilon C}{\lambda_1}\cdot\frac{\lambda_{m+1}}{\lambda_{m+1}-\lambda-\varepsilon C}.$$

Note that the rightmost fraction is well defined and positive if m is sufficiently large.

On the other hand, we have by consequence an estimate for u',

$$\begin{aligned}u(\mu)=g(\mu+\tilde\eta(\mu))&\Rightarrow u'=g(\mu'+\tilde\eta'),\\ \left\|u'\right\|=\left\|g(\mu'+\tilde\eta')\right\|&\leqslant\|g\|\,(1+\left\|\tilde\eta'\right\|),\\ \left\|u'\right\|&\leqslant\frac{1}{\lambda_1}(1+\left\|\tilde\eta'\right\|)\end{aligned}$$

We are now able to exhibit the estimates on $\frac{\partial}{\partial\mu}\widetilde{E}^{L+\lambda}_{nq}(\mu)$ and $\frac{\partial}{\partial\mu}\widetilde{E}^{F}(\mu)$.

Lemma 7. $\left|\frac{\partial}{\partial\mu}\widetilde{E}^{L+\lambda}_{nq}(\mu)\right|\leqslant$ *constant.*

Proof. By definition we have

$$\frac{\partial\widetilde{E}^{L+\lambda}_{nq}}{\partial\mu}=\frac{\partial}{\partial\mu}\left(\sum_{j=m+1}^{\infty}\frac{\lambda_j-\lambda}{\lambda_j^2}\tilde\eta_j^2(\mu)\right)=2\sum_{j=m+1}^{\infty}\left(\frac{\lambda_j-\lambda}{\lambda_j}\frac{\partial\tilde\eta_j(\mu)}{\partial\mu}\right)\cdot\left(\frac{\tilde\eta_j(\mu)}{\lambda_j}\right),$$

then

$$\begin{aligned}\left\|\frac{\partial}{\partial\mu}\widetilde{E}^{L+\lambda}_{nq}(\mu)\right\|&\leqslant2\left\|\left\{\frac{\lambda_j-\lambda}{\lambda_j}\frac{\partial\tilde\eta_j}{\partial\mu}\right\}_{m+1}^{\infty}\right\|\cdot\frac{\|\tilde\eta\|}{\lambda_{m+1}}\leqslant\\ &\leqslant2\frac{1}{\lambda_{m+1}}\sup_{j>m}\left|\frac{\lambda_j-\lambda}{\lambda_j}\right|\left\|\tilde\eta'\right\|\cdot\|\tilde\eta\|\leqslant\frac{2}{\lambda_{m+1}}\left\|\tilde\eta'\right\|\cdot\|\tilde\eta\|.\end{aligned}$$

□

Lemma 8. $\left|\frac{\partial}{\partial\mu}\widetilde{E}^{F}(\mu)\right|\leqslant$ *constant.*

Proof. First let us recall that

$$\left(\int_0^1 F(tu)dt, u\right) = \int_\Omega \bar{f}(u(x))dx.$$

Next, it is easy to check,

$$\begin{aligned} \left|\frac{\partial}{\partial\mu}\widetilde{E}^F(\mu)\right| &= \left|\frac{\partial}{\partial\mu}\left(\int_0^1 F(tu(\mu))dt, u\right)\right| = \\ &= \left|\frac{\partial}{\partial\mu}\int_\Omega \bar{f}(u(x))dx\right| = \left|\int_\Omega f(u(x))\frac{\partial u}{\partial\mu}(x)dx\right| \leqslant \\ &\leqslant \int_\Omega |f| \left\|u'\right\| dx \leqslant \left\|u'\right\| K \text{ meas}(\Omega). \end{aligned}$$

$\square$

$\square$

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