

Esercizi di Fisica Matematica

A.A. 2004-2005

1 febbraio 2005

Cinematica del punto

Lunghezza di curve

$$[t_0, t_1] \ni t \mapsto r(t) = \begin{pmatrix} r_x(t) \\ r_y(t) \\ r_z(t) \end{pmatrix} \in \mathbb{R}^3,$$

$$l(r(\cdot)) := \int_{t_0}^{t_1} |\dot{r}(t)| dt = \int_{t_0}^{t_1} \sqrt{(r'_x(t))^2 + (r'_y(t))^2 + (r'_z(t))^2} dt.$$

1. $r(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle, \quad -10 \leq t \leq 10.$ [??] **R:**

2. $r(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle, \quad 0 \leq t \leq \pi.$ [??] **R:**

3. $r(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} - e^t\mathbf{k}, \quad 0 \leq t \leq 1.$ [??] **R:**

4. $r(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\mathbf{k}, \quad 0 \leq t \leq e.$ [??] **R:**

Curve a velocità unitaria

Riparametrizzare rispetto al parametro d'arco:

$$s = \text{arc}(t) := \int_{t_0}^t |\dot{r}(\tau)| d\tau \quad t(s) := \text{arc}^{-1}(s) \quad \tilde{r}(s) := r \circ t(s).$$

Esempio Svolto

Consideriamo la curva:

$$\mathbb{R} \ni t \xrightarrow{\varphi} a(\cos t + t \sin t, \sin t - t \cos t) \subseteq \mathbb{R}^2$$

$$\begin{aligned}\dot{\varphi} &= at(\cos t, \sin t), \\ |\dot{\varphi}| &= at, \\ s(t) &= \int_0^t at dt = a \frac{t^2}{2}, \\ t(s) &= \sqrt{\frac{2s}{a}}, \quad \tilde{\varphi}(s) = \varphi \circ t(s) = \dots\end{aligned}$$

Verificare anche che $\left| \frac{d\tilde{\varphi}}{ds} \right| = 1$.

Altro Esempio Svolto

Consideriamo la curva:

$$\mathbb{R} \ni t \xrightarrow{\varphi} a(\cos^3 t, \sin^3 t) \subseteq \mathbb{R}^2$$

$$\begin{aligned}\dot{\varphi} &= 3a \sin t \cos t (-\cos t, \sin t), \\ |\dot{\varphi}| &= \frac{3a}{2} |\sin 2t| \\ s(t) &= \frac{3a}{2} \int_0^t |\sin 2t| dt = (\text{ se } 0 \leq t \leq \pi/2) = \frac{3a}{4}(1 - \cos 2t), \\ t(s) &= \frac{1}{2} \arccos(1 - \frac{4}{3a}s) \quad \tilde{\varphi}(s) = \varphi \circ t(s) = \dots\end{aligned}$$

Verificare anche che $\left| \frac{d\tilde{\varphi}}{ds} \right| = 1$.

1. $t \mapsto a(2t, t^2)$
2. $r(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j}, \quad 0 \leq t.$ [??] **R:**
3. $r(t) = (1 + 2t)\mathbf{i} + (3 + t)\mathbf{j} + -5t\mathbf{k}, \quad 0 \leq t.$ [??] **R:**
4. $r(t) = 3 \sin t \mathbf{i} + 4t\mathbf{j} + +3 \cos t \mathbf{k}, \quad 0 \leq t.$ [??] **R:**

Triedro di Frenet - Curvatura

Trovare i versori tangente e normale T ed N . Trovare la curvatura k .

$$T := \frac{r'}{|r'|} \quad N := \frac{T'}{|T'|} \quad B := T \wedge N = \frac{r' \wedge r''}{|r' \wedge r''|} \Rightarrow N = B \wedge T.$$

$$k := \left| \frac{dT}{ds} \right| = \left| \frac{dT}{dt} \cdot \frac{dt}{ds} \right| = \left| \frac{dT}{dt} \right| / \left| \frac{ds}{dt} \right| = \frac{|T'|}{|r'|}.$$

Esempio Svolto

Consideriamo la curva:

$$\mathbb{R} \ni s \mapsto \beta(s) = \left(a \cos \frac{s}{\sqrt{a^2 + b^2}}, a \sin \frac{s}{\sqrt{a^2 + b^2}}, \frac{bs}{\sqrt{a^2 + b^2}} \right)$$

$$\beta' = \frac{1}{\sqrt{a^2 + b^2}} \left(-a \sin \frac{s}{\sqrt{a^2 + b^2}}, a \cos \frac{s}{\sqrt{a^2 + b^2}}, b \right),$$

$$|\beta'| = \frac{1}{\sqrt{a^2 + b^2}} \sqrt{a^2 \sin^2 + a^2 \cos^2 + b^2} = 1$$

$T = \beta'$, s proprio il parametro d'arco,

$$k(s) = \left| \frac{dT}{ds} \right| = \left| \frac{d^2\beta}{ds^2} \right|,$$

$$\beta''(s) = \frac{1}{a^2 + b^2} \left(-a \cos \frac{s}{\sqrt{a^2 + b^2}}, -a \sin \frac{s}{\sqrt{a^2 + b^2}}, 0 \right),$$

$$|\beta''(s)| = \frac{a}{a^2 + b^2} = k(s) = \frac{1}{\rho},$$

$$\rho = \frac{a^2 + b^2}{a} = \text{raggio del cerchio osculatore.}$$

$$N(s) = \frac{\beta''(s)}{|\beta''(s)|} = \left(-\cos \frac{s}{\sqrt{a^2 + b^2}}, -\sin \frac{s}{\sqrt{a^2 + b^2}}, 0 \right).$$

$$B(s) = T(s) \wedge N(s) = \dots$$

1. $r(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle$. [??] **R:**

2. $r(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle$. [??] **R:**

3. $r(t) = \langle \frac{1}{3}t^3, t^2, 2t \rangle.$ [??] **R:**

4. $r(t) = \langle t^2, 2t, \ln t \rangle.$ [??] **R:**

Trovare T, N, B per le curve assegnate:

1. $r(t) = \langle t^2, \frac{2}{3}t^3, t \rangle,$ in $(1, \frac{2}{3}, 1)$ [??] **R:**

2. $r(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle,$ in $(1, 0, 1).$ [??] **R:**